IMRN International Mathematics Research Notices
Volume 2006, Article ID 59326, Pages 1–7

Holomorphy of the 9th Symmetric Power L-Functions for $\mathrm{Re}(s) > 1$

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We prove the holomorphy of the 9th symmetric power L-functions for arbitrary cusp forms on GL(2) over a number field for $\mathrm{Re}(s) > 1$ which was not available earlier in the case of Maass forms. This complements our earlier results for invertibility and in fact absolute convergence of the twisted symmetric power L-functions of these forms up to degree 8 over the same interval.

Let F be a number field and denote by $\mathbb{A}$ its ring of adeles. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $GL_2(\mathbb{A})$. We normalize π so that $\omega_\pi \mid \mathbb{R}_+^* \equiv 1$, where ω_π is the central character of π and $\mathbb{A}^* = \mathbb{A}_1^* \times \mathbb{R}_+^*$. One of the consequences of the existence of $\mathrm{Sym}^3(\pi)$ and $\mathrm{Sym}^4(\pi)$ as automorphic representations of $GL_4(\mathbb{A})$ and $GL_5(\mathbb{A})$, respectively, recently proved in [2, 7], is a proof of certain analytic properties of $L(s, \pi, \mathrm{Sym}^m \rho_2)$ for $m \leq 9$ (cf. [6, 8]). While we succeeded in proving that when $m \leq 8$, each of these symmetric power L-functions are invertible and even absolutely convergent for $\mathrm{Re}(s) > 1$ (cf. [6, 13]), no such results were obtained for $L(s, \pi, \mathrm{Sym}^9 \rho_2)$. In fact, all that we knew (even though it is not stated explicitly in [6]) was that $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ is holomorphic and nonvanishing for $\mathrm{Re}(s) \geq 1$, except possibly for finitely many poles and zeros on the real axis $1 \leq s \leq 2$.

On the other hand, Gelbart and Lapid [1] have recently established zero free regions for each of the L-functions obtained from Langlands-Shahidi method to the left

Received 13 January 2006; Revised 14 February 2006; Accepted 14 June 2006
Communicated by James W. Cogdell

of the line $\mathrm{Re}(s) = 1$. This, in particular, shows the existence of such regions for $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$. One then expects this L-function to be nonvanishing and holomorphic for $\mathrm{Re}(s) \geq 1$, except for a possible pole at $s = 1$. While we still cannot prove the nonvanishing for $1 \leq s \leq 2$, in this paper we prove the holomorphy of $L(s, \pi, \mathrm{Sym}^9 \rho_2)$ for $1 < s \leq 2$.

The reader must note that this result is much stronger than the general results obtained from the theory of Eisenstein series (cf. [10–12]), where one cannot rule out the possibility of a finite number of poles on the real axis. On the other hand, our method fails when it comes to nonvanishing for $\mathrm{Re}(s) > 1$. In fact, using our method, a nonvanishing result for $\mathrm{Re}(s) > 1$ requires ruling out the poles for the corresponding Eisenstein series for $\mathrm{Re}(s) > 0$ (cf. [10, 11]) which presently is not possible even by means of unitary duals (cf. [5]). One can of course use the fact that the Eisenstein series in this case is holomorphic for $\mathrm{Re}(s) > 1$ as we establish here, but this only proves the nonvanishing for $\mathrm{Re}(s) > 2$ which can be concluded from the absolute convergence of the L-function for $\mathrm{Re}(s) > 2$ which was proved in the full generality of our method in [11, 12].

The proof relies upon a criterion proved in [8, Proposition 4.1] by the authors of this paper. We refer to Lemma 2 here for a reference to that.

We should point out that the fact that one can find such regions to the left of $\mathrm{Re}(s) = 1$ without the knowledge of similar results to the right of this line as in [1], must be of no surprise. In fact, this phenomenon was already noticed in [10], where the nonvanishing for a product of these L-functions at $s = 1$ was proved while similar results to the right of $\mathrm{Re}(s) = 1$ are still not available in general. They are both consequences of the main characteristic of Eisenstein series in determining the continuous spectrum.

This paper was written to answer the question posed by Erez Lapid that, whether in view of the results in [1] discussed earlier, one can show the holomorphy of $L(s, \pi, \mathrm{Sym}^9 \rho_2)$ for $\mathrm{Re}(s) > 1$. We like to thank him for that.

We will now state and prove our theorem. In it S denotes a set of places of F including the archimedean ones such that π_v is spherical for every $v \notin S$.

Each π_v with $v \notin S$ is parametrized by a diagonal element $\mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbb{C})$. For each positive integer m, let

$$L^S\left(s, \pi, \mathrm{Sym}^m \rho_2\right) = \prod_{v \notin S} \prod_{j=0}^{m} \left(1 - \alpha_v^j \beta_v^{m-j} q_v^{-s}\right)^{-1}. \tag{1}$$

If $m \leq 9$, we can define a local factor $L(s, \pi_v, \mathrm{Sym}^m \rho_2)$ for $v \in S$ inductively (cf. [12]). For $m \leq 8$, we use the Rankin-Selberg L-functions (see [6]). For example, we define

$L(s, \pi_v, \mathrm{Sym}^5 \rho_2)$ using the relation

$$\begin{aligned} & L\big(s, \mathrm{Sym}^2\,(\pi_v) \times \mathrm{Sym}^3\,(\pi_v)\big) \\ & \quad = L\big(s, \pi_v, \mathrm{Sym}^5\,\rho_2\big) L\big(s, \mathrm{Sym}^3\,(\pi_v) \otimes \omega_{\pi_v}\big) L\big(s, \pi_v \otimes \omega_{\pi_v}^2\big). \end{aligned} \tag{2}$$

Similarly, $L(s, \pi_v, \mathrm{Sym}^8 \rho_2)$ comes from $L(s, \mathrm{Sym}^4(\pi_v) \times \mathrm{Sym}^4(\pi_v))$. For $m = 9$, we use the relation (6) below. Hence if $m \leq 9$, we can define the completed L-functions $L(s, \pi, \mathrm{Sym}^m \rho_2)$. We refer to [6, 8] for any unexplained notation. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, π is of Galois type, and we know that $\mathrm{Sym}^m(\pi)$ is an automorphic representation of $GL_{m+1}(\mathbb{A})$ for all m (see [3]). Hence $L(s, \pi, \mathrm{Sym}^m \rho_2)$ is holomorphic for all $s \in \mathbb{C}$, except possibly at $s = 1$. It is also nonvanishing for $\mathrm{Re}(s) \geq 1$. So we assume that $\mathrm{Sym}^4(\pi)$ is cuspidal.

Theorem 1. (a) Let π be a cuspidal representation of $GL_2(\mathbb{A})$ such that $\mathrm{Sym}^4(\pi)$ is cuspidal. Then the 9th symmetric power L-function $L(s, \pi, \mathrm{Sym}^9 \rho_2)$ is holomorphic for $\mathrm{Re}(s) \geq 1$, except possibly for a simple pole at $s = 1$. It is nonvanishing for $\mathrm{Re}(s) \geq 1$, except possibly for finitely many simple zeros on the real axis $1 \leq s \leq 2$.

(b) Both statements are valid for $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$. □

Proof. Let $\mathrm{Sym}^3(\pi)$ and $\mathrm{Sym}^4(\pi)$ be the symmetric cube and fourth of π which are automorphic representations of $GL_4(\mathbb{A})$ and $GL_5(\mathbb{A})$, respectively. Let $A^3(\pi) = \mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, where ω_π is the central character of π. Let us recall how we obtained $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ in [6].

Consider the case $E_8 - 2$ of [11] (cf. [4]). Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{P}$, the maximal parabolic subgroup obtained by deleting α_5, and denote by $\mathbf{A}$ its center which is connected since $\mathbf{G}$ is adjoint. Since $\mathbf{G} = E_8$ is also simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected as well, and hence $\mathbf{M}_D = SL_4 \times SL_5$. Thus

$$\mathbf{M} = \big(\, GL_1 \times SL_4 \times SL_5\big) / \big(\mathbf{A} \cap \mathbf{M}_D\big). \tag{3}$$

■

As in [2, 4, 7] there is a natural map $\phi : \mathbf{M} \to GL_4 \times GL_5$. Given $i = 1, 2$, let π_i be cuspidal representations of $GL_4(\mathbb{A})$ and $GL_5(\mathbb{A})$ with central characters ω_i, $i = 1, 2$, respectively. Let Σ be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by ϕ and π_1, π_2. Then the central character of Σ is $\omega_\Sigma = \omega_1^5 \omega_2^8$. We then get the L-function $L^S(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5)$ as our first L-function for the triple $(\mathbf{G}, \mathbf{M}, \Sigma)$ (cf. [11, 12]). In fact, there are five

L-functions in the constant term of the corresponding Eisenstein series; namely

$$
\begin{aligned}
L^S(s,\Sigma,r_1) &= L^S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5);\\
L^S(s,\Sigma,r_2) &= L^S(s,\pi_1\otimes(\tilde\pi_2\otimes\omega_2),\wedge^2\rho_4\otimes\rho_5);\\
L^S(s,\Sigma,r_3) &= L^S(s,\tilde\pi_1\times(\pi_2\otimes\omega_1\omega_2));\\
L^S(s,\Sigma,r_4) &= L^S(s,\tilde\pi_2,\wedge^2\rho_5\otimes\omega_1\omega_2^2);\\
L^S(s,\Sigma,r_5) &= L^S(s,\pi_1\otimes\omega_1\omega_2^2).
\end{aligned} \tag{4}
$$

More precisely, let $M(s,\Sigma)$ be the main part of the constant term for the Eisenstein series attached to $(\mathbf{G},\mathbf{M},\Sigma)$. Then

$$
M_S(s,\Sigma) = \prod_{i=1}^{5} \frac{L^S(is,\Sigma,\tilde r_i)}{L^S(1+is,\Sigma,\tilde r_i)} \otimes_{v\in S} A(s,\Sigma_v), \tag{5}
$$

where $M_S(s,\Sigma)$ is the restriction of $M(s,\Sigma)$ to functions in the induced space $I(s,\Sigma)$ of the form $f = f_S\otimes\otimes_{v\notin S} f_v^0$, where $f_S\in\otimes_{v\in S} I(s,\Sigma_v)$, while each f_v^0 is the $\mathbf{G}(O_v)$-invariant (spherical) function in $I(s,\Sigma_v)$ whose value at e_v equals to $f_v^0(e_v) = x_v$, $v\notin S$. Here the vectors $\{x_v\in\mathcal{H}(\Sigma_v)\mid v\notin S\}$ are the fixed $\mathbf{M}(O_v)$-invariant vectors which were used to decompose Σ to $\Sigma = \otimes_v\Sigma_v$. The operators $A(s,\Sigma_v)$ are the standard local intertwining operators originating from $I(s,\Sigma_v)$.

We apply the above to $\pi_1 = A^3(\pi)$ and $\pi_2 = \mathrm{Sym}^4(\pi)$. By standard calculations, we have

$$
\begin{aligned}
&L^S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)\\
&\quad = L^S(s,\pi,\mathrm{Sym}^9\rho_2)L^S(s,\pi,\mathrm{Sym}^7\rho_2\otimes\omega_\pi)L^S(s,\pi,\mathrm{Sym}^5\rho_2\otimes\omega_\pi^2)^2\\
&\quad\quad \times L^S(s,\mathrm{Sym}^3(\pi)\otimes\omega_\pi^3)^2 L^S(s,\pi\otimes\omega_\pi^4).
\end{aligned} \tag{6}
$$

Here $L^S(s,\pi,\mathrm{Sym}^7\rho_2\otimes\omega_\pi)L^S(s,\pi,\mathrm{Sym}^5\rho_2\otimes\omega_\pi^2)^2L^S(s,\mathrm{Sym}^3(\pi)\otimes\omega_\pi^3)^2L^S(s,\pi\otimes\omega_\pi^4)$ is invertible for $\mathrm{Re}(s)>1$ by [6, Propositions 4.2 and 4.5]. So it is enough to prove that $L^S(s,\Sigma,r_1)$ is holomorphic for $\mathrm{Re}(s)>1$. To proceed we need the following.

Lemma 2. Suppose Σ_v is spherical and tempered. Then for each s with $\mathrm{Re}(s)>1$, the induced representation $I(s,\Sigma_v)$ is irreducible, and hence it is not unitary. □

Proof. By [8, Proposition 4.1], we only need to prove that $\prod_{i=1}^5 L(1-is,\Sigma_v,r_i)$ is holomorphic for $\mathrm{Re}(s)>1$. Here $L(s,\Sigma_v,r_i) = \prod_j(1-\alpha_j q_v^{-s})^{-1}$, and since Σ_v is tempered, $|\alpha_j| = 1$ for all j. Our result follows. ■

Ramakrishnan has proved in [9] that $\pi = \otimes_v \pi_v$ has infinitely many tempered spherical components. In fact, he showed that the lower Dirichlet density of the set of places, where π_v is tempered, is greater than $9/10$. (In [6], we improved this to $34/35$.) Choose a π_v which is spherical and tempered. Then Σ_v is spherical and tempered.

Since $I(s, \Sigma_v)$ is not unitary for $\mathrm{Re}(s) > 1$, $M_S(s, \pi)$ is holomorphic at least up to $\mathrm{Re}(s) > 1$. In fact, if $M_S(s, \Sigma)$ has a pole at a point s with $\mathrm{Re}(s) > 1$, its residue will identify a quotient of $I(s, \Sigma)$ as a constituent of the space of automorphic forms. This implies that the corresponding quotient of $I(s, \Sigma_v)$ is unitary for $\mathrm{Re}(s) > 1$ which is a contradiction.

It then follows from (5) that

$$\prod_{i=1}^{5} \frac{L^S(is, \Sigma, \tilde{r}_i)}{L^S(1 + is, \Sigma, \tilde{r}_i)} \tag{7}$$

is holomorphic for $\mathrm{Re}(s) > 1$ since each local intertwining operator $A(s, \Sigma_v)$ is nonvanishing. Since $L^S(s, \Sigma, r_i)$ is invertible for $\mathrm{Re}(s) > 2$ (cf. [11]), we conclude that $L^S(s, \Sigma, \tilde{r}_1)$ is holomorphic for $\mathrm{Re}(s) > 1$.

For $v \in S$, let $L(s, \Sigma_v, \tilde{r}_i)$ be the local L-factors defined in [12], $i = 1, \ldots, 5$. We showed in [4, Proposition 4.9] that the local L-factors $L(s, \Sigma_v, \tilde{r}_i)$ are holomorphic for $\mathrm{Re}(s) \geq 1$. Hence the completed L-function $L(s, \Sigma, \tilde{r}_1)$ is holomorphic for $\mathrm{Re}(s) > 1$. This proves that the completed L-function $L(s, \pi, \mathrm{Sym}^9 \rho_2)$ is holomorphic for $\mathrm{Re}(s) > 1$.

All other assertions are implicit in the proof of [6, Proposition 4.7]. For the sake of completeness, we state it here explicitly. Due to the normalization of π, the poles of the Eisenstein series for $\mathrm{Re}(s) > 0$ are on the real axis. So the poles of $\prod_{i=1}^{5} L^S(is, \Sigma, \tilde{r}_i)/L^S(1 + is, \Sigma, \tilde{r}_i)$ for $\mathrm{Re}(s) > 0$ are all real. However, we showed in the proof of [6, Proposition 4.7] that for each $i = 2, 3, 4, 5$, $L^S(s, \Sigma, \tilde{r}_i)$ is invertible for $\mathrm{Re}(s) \geq 1$. Hence the poles of $L^S(s, \Sigma, \tilde{r}_1)/L^S(1 + s, \Sigma, \tilde{r}_1)$ for $\mathrm{Re}(s) \geq 1/2$ are all on the real axis. Inductively, this implies that the poles of $L^S(s, \Sigma, \tilde{r}_1)$ for $\mathrm{Re}(s) \geq 1/2$ are on the real axis. Since $L^S(s, \pi, \mathrm{Sym}^7 \rho_2 \otimes \omega_\pi) L^S(s, \pi, \mathrm{Sym}^5 \rho_2 \otimes \omega_\pi^2)^2 L^S(s, \mathrm{Sym}^3(\pi) \otimes \omega_\pi^3)^2 L^S(s, \pi \otimes \omega_\pi^4)$ is invertible for $\mathrm{Re}(s) \geq 1$, (6) implies that the possible poles of $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ for $\mathrm{Re}(s) \geq 1$ are all real. Hence $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ has no poles for $\mathrm{Re}(s) \geq 1$ except possibly at $s = 1$. We proved in [8, Proposition 7.2] that $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ has at most a simple pole or a simple zero at $s = 1$.

The Eisenstein series is holomorphic for $\mathrm{Re}(s) = 0$, and the possible poles for $\mathrm{Re}(s) > 0$ are on the real axis, and they are simple. Therefore by considering the nonconstant term of the Eisenstein series, we conclude that the same is true for the possible zeros of $\prod_{i=1}^{5} L^S(1 + is, \Sigma, \tilde{r}_i)$ for $\mathrm{Re}(s) > 0$. Consequently, since for each $i = 2, 3, 4, 5$, $L^S(s, \Sigma, \tilde{r}_i)$ is invertible for $\mathrm{Re}(s) \geq 1$, the possible zeros of $L^S(s, \Sigma, \tilde{r}_1)$ for $\mathrm{Re}(s) \geq 1$ are on the real axis, and they are simple. Since $L^S(s, \Sigma, \tilde{r}_1)$ is absolutely convergent for $\mathrm{Re}(s) > 2$, it is nonvanishing there. Using the fact that $L^S(s, \pi, \mathrm{Sym}^7 \rho_2 \otimes \omega_\pi) L^S(s, \pi, \mathrm{Sym}^5 \rho_2 \otimes \omega_\pi^2)^2 L^S(s, \mathrm{Sym}^3(\pi) \otimes \omega_\pi^3)^2 L^S(s, \pi \otimes \omega_\pi^4)$ is invertible for $\mathrm{Re}(s) \geq 1$, from (6), we see that $L^S(s, \pi, \mathrm{Sym}^9 \rho_2)$ is nonvanishing for $\mathrm{Re}(s) \geq 1$, except possibly for finitely many simple zeros on the real axis with $1 \leq s \leq 2$. This completes the proof of Theorem 1.

Remark 3. We refer to [5] for a generalization of our argument and other results for $1/2 < \mathrm{Re}(s) < 1$.

For the sake of completeness, we conclude by recording a result from [13] due to the authors of this paper, proving the absolute convergence of twisted symmetric power L-functions for $m \leq 8$. We refer to [13] for some of its consequences.

Let $\chi = \otimes_v \chi_v$ be an idele class character of $\mathbb{A}^*$ and assume χ_v is unramified for all $v \notin S$. Let

$$L^S\left(s, \pi, \mathrm{Sym}^m \rho_2 \otimes \chi\right) = \prod_{v \notin S} \prod_{j=0}^{m} \left(1 - \alpha_v^j \beta_v^{m-j} \chi_v(\varpi_v) q_v^{-s}\right)^{-1}. \tag{8}$$

Theorem 4 (cf. [13])**.** The partial L-functions $L^S(s, \pi, \mathrm{Sym}^m \rho_2 \otimes \chi)$ are all absolutely convergent for $\mathrm{Re}(s) > 1$ for all $m \leq 8$ and every idele class character of $\mathbb{A}^*$, unramified outside S. □

Acknowledgments

The first author was partially supported by an NSERC grant, and the second author was partially supported by NSF Grant DMS-0200325.

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