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The Fourth Moment of Ramanujan τ-Function

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The purpose of this note is to give a proof of the following result for the Ramanujan arithmetical function $\tau(n)=\tau_0(n)n^{11/2}$.

Main Theorem. *The Dirichlet series*

$$F(s)=\sum_{n=1}^{\infty}\tau_0(n)^4 n^{-s}$$

has a meromorphic continuation to the half plane $\mathrm{Re}(s)\geqq\frac{1}{2}$ *and in the region* $\mathrm{Re}(s)\geqq 1$ *it is holomorphic except for a double pole at* $s=1$. *In particular for a positive real number x we have*

$$\sum_{n\leqq x}\tau_0(n)^4\sim cx\log x\,,$$

where c is a positive constant.

The interest in establishing a result of this nature comes from some recent work of Rankin concerning non-trivial estimates for the higher moments of $\tau_0(n)$ and other arithmetical functions [4], and specially from his applications to a result of Elliott about mean values of multiplicative arithmetic functions [1]. The proof we give is based on the work of the second author on the L-function $L(s,\pi,\mathrm{Sym}^4\varrho)$ applied to the automorphic representation $\pi=\pi_\Delta$ corresponding to the Ramanujan modular form

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n\,,$$

and especially the non-vanishing of these L-functions on the line $\mathrm{Re}(s)=1$. These ideas are in turn an outgrowth of the important work of Jacquet, Piatetski-Shapiro and Shalika on GL_n.

In Sect. 1 we relate the study of the Dirichlet series $F(s)$ to the functions $L(s,\pi_\Delta,\mathrm{Sym}^2\varrho)$ and $L(s,\pi_\Delta,\mathrm{Sym}^4\varrho)$. In Sect. 2 we establish the analytic properties of $F(s)$ and show how the non-vanishing of $L(1+it,\pi_\Delta,\mathrm{Sym}^4\varrho)$ implies the presence of a double pole for $F(s)$ at $s=1$.

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1. The Euler Product of $F(s)$

In accordance with Deligne's theorem, $|\tau(p)|<2p^{11/2}$, the roots of the equation $1-\tau(p)p^{-11/2}T+T^2=1-\tau_0(p)T+T^2$ are complex conjugate and hence we can put for each prime p

$$\tau_0(p)=\zeta_p+\zeta_p^{-1},$$

where ζ_p is a complex number of absolute value 1. We let $\varrho:\mathrm{GL}_2(\mathbb{C})\to\mathrm{GL}_2(\mathbb{C})$ denote the standard two dimensional representation. In the notation of [7], with $\pi_\Delta=\pi_\infty\otimes\left(\bigotimes_p\pi_p\right)$ being the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ which corresponds to the Ramanujan modular form, we put

$$L(s,\pi_\Delta,\mathrm{Sym}^1\varrho)=\Gamma(s,\pi_\infty,\mathrm{Sym}^1\varrho)\zeta_1(s),$$
$$L(s,\pi_\Delta,\mathrm{Sym}^2\varrho)=\Gamma(s,\pi_\infty,\mathrm{Sym}^2\varrho)\zeta_2(s),$$
$$L(s,\pi_\Delta,\mathrm{Sym}^4\varrho)=\Gamma(s,\pi_\infty,\mathrm{Sym}^4\varrho)\zeta_4(s),$$

where $\Gamma(s,\pi_\infty,\mathrm{Sym}^i\varrho)$ is a product of Γ-functions which we need not specify and where

$$\zeta_1(s)=\prod_p\{(1-\zeta_p p^{-s})(1-\zeta_p^{-1}p^{-s})\}^{-1},$$
$$\zeta_2(s)=\prod_p\{(1-\zeta_p^2p^{-s})(1-p^{-s})(1-\zeta_p^{-2}p^{-s})\}^{-1},$$

and

$$\zeta_4(s)=\prod_p\{(1-\zeta_p^4p^{-s})(1-\zeta_p^2p^{-2})(1-p^{-s})(1-\zeta_p^{-2}p^{-s})(1-\zeta_p^{-4}p^{-s})\}^{-1}.$$

In particular we have

$$L(s-\tfrac{11}{2},\pi_\Delta,\mathrm{Sym}^1\varrho)=(2\pi)^{-s}\Gamma(s)\prod_p(1-\tau(p)p^{-s}+p^{11-2s})^{-1}$$
$$=(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\tau(n)n^{-s}.$$

Since $\tau_0(n)=\tau(n)n^{-11/2}$ is bounded in absolute value by $d(n)$, the number of positive divisors of n, we can obtain by elementary means fairly precise information about $F(s)$ within the region of absolute convergence.

Lemma 1. *For* $\mathrm{Re}(s)=\sigma>1$ *one has*

$$|F(s)|\leqq\frac{\zeta(\sigma)^{16}}{\zeta(2\sigma)^{11}}.$$

Proof. From the multiplicative property of $\tau_0(n)$ and the fact that $d(p^v)=v+1$ we obtain

$$
\begin{aligned}
|F(s)| &= \left|\prod_p \left(\sum_{v=0}^{\infty} \tau_0(p^v)^4 p^{-vs}\right)\right| \\
&\leqq \prod_p \left(\sum_{v=0}^{\infty} |\tau_0(p^v)|^4 p^{-v\sigma}\right) \\
&\leqq \prod_p \left(\sum_{v=0}^{\infty} d(p^v)^4 p^{-v\sigma}\right) \\
&= \prod_p \left(\sum_{v=0}^{\infty} (v+1)^4 p^{-v\sigma}\right).
\end{aligned}
$$

From the identity

$$\sum_{v=0}^{\infty} (v+1)^4 T^v = \frac{1+11T+11T^2+T^3}{(1-T)^5}$$

and the inequality

$$1+11T+11T^2+T^3 \leqq (1+T)^{11} = \frac{(1-T^2)^{11}}{(1-T)^{11}},$$

with $T=p^{-\sigma}$ we get the lemma.

In the following $\zeta_0(s)=\zeta(s)$ will denote the Riemann zeta function.

Lemma 2. *With notations as above and for* $\mathrm{Re}(s)>1$, *one has*

$$\sum_{n=1}^{\infty} \tau_0(n)^4 n^{-s} = \zeta(s)^2 \zeta_2(s)^3 \zeta_4(s) \prod_p L_p(p^{-s}),$$

where $L_p(T)$ *is a polynomial of degree* 14 *whose coefficients are bounded by an absolute constant independent of p and whose constant and highest term equals* 1.

Proof. From the multiplicativity property of $\tau_0(n)^4$ we have the Euler product identity

$$\sum_{n=1}^{\infty} \tau_0(n)^4 n^{-s} = \prod_p \left(\sum_{v=0}^{\infty} \tau_0(p^v)^4 p^{-vs}\right).$$

Therefore, the claim will follow if we prove, with $T=p^{-s}$, that

$$\sum_{v=0}^{\infty} \tau_0(p^v)^4 T^v = \frac{L_p(T)}{(1-T)^2 S_2(T)^3 S_4(T)},$$

where for $g=\begin{pmatrix} \zeta_p & 0 \\ 0 & \zeta_p^{-1} \end{pmatrix}$ we have put

$$
\begin{aligned}
S_2(T) &= \det(I_3 - \mathrm{Sym}^2 \varrho(g) T) \\
&= (1-\zeta_p^2 T)(1-T)(1-\zeta_p^{-2} T), \\
S_4(T) &= \det(I_5 - \mathrm{Sym}^4 \varrho(g) T) \\
&= (1-\zeta_p^4 T)(1-\zeta_p^2 T)(1-T)(1-\zeta_p^{-2} T)(1-\zeta_p^{-4} T),
\end{aligned}
$$

and $L_p(T)$ is a polynomial satisfying the properties of the lemma. To simplify notation we put $\zeta = \zeta_p$. From the basic recurrence relation for the eigenvalues of the Hecke operators, which can be stated in terms of the formal identity

$$\frac{1}{1-\tau_0(p)T+T^2} = \sum_{v=0}^{\infty} \tau_0(p^v)T^v,$$

or equivalently as

$$\frac{1}{(1-\zeta T)(1-\zeta^{-1}T)} = \sum_{v=0}^{\infty} \left(\frac{\zeta^{v+1}-\zeta^{-v-1}}{\zeta-\zeta^{-1}}\right) T^v,$$

we see that our problem is reduced to showing that

$$\sum_{v=0}^{\infty} \left(\frac{\zeta^{v+1}-\zeta^{-v-1}}{\zeta-\zeta^{-1}}\right)^4 T^v = \frac{L_p(T)}{(1-T)^2 S_2(T)^3 S_4(T)}.$$

By a straightforward calculation one shows that

$$\sum_{v=0}^{\infty} \left(\frac{a^{v+1}-1}{a-1}\right)^4 t^v = \frac{1+a(3+5a+3a^2)t+a^3(3+5a+3a^2)t^2+a^6t^3}{(1-a^4t)(1-a^3t)(1-a^2t)(1-at)(1-t)}.$$

Substituting in this identity $a=\zeta^2$ and $t=\frac{T}{\zeta^4}$, and using the fact that

$$\begin{aligned} a(3+5a+3a^2)t &= \zeta^2(3+5\zeta^2+3\zeta^4)\frac{T}{\zeta^4} \\ &= (3\tau_0(p)^2-1)T, \end{aligned}$$

we obtain

$$\sum_{v=0}^{\infty} \tau_0(p^v)^4 T^v = \frac{N(T)}{S_4(T)},$$

where

$$N(T) = 1+(3\tau_0(p)^2-1)T+(3\tau_0(p)^2-1)T^2+T^3.$$

If we make use of the fact that

$$S_2(T) = 1-(\tau_0(p)^2-1)T+(\tau_0(p)^2-1)T^2+T^3,$$

we obtain that

$$\begin{aligned} L_p(T) &= N(T)S_2(T)^3(1-T)^2 \\ &= 1-c_2(p)T^2+\ldots+T^{14}, \end{aligned}$$

is a polynomial of degree 14 whose coefficients are bounded by an absolute constant independent of p, whose constant and highest term equal 1, and whose linear term equals 0. This completes the proof of the lemma.

2. Analytic Properties of the Function $F(s)$

The meromorphic continuation of the Euler product

$$L(s,\pi_\Delta,\mathrm{Sym}^2\varrho)=\Gamma(s,\pi_\infty,\mathrm{Sym}^2\varrho)\zeta_2(s),$$

where $\Gamma(s,\pi_\infty,\mathrm{Sym}^2\varrho)=\Gamma_{\mathbb{R}}(s+11)\Gamma_{\mathbb{R}}(s+12)\Gamma_{\mathbb{R}}(s+1)$, and $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, was established by Rankin [5] who also proved that $\zeta_2(1+it)\neq 0$ for all real t. That $L(s,\pi_\Delta,\mathrm{Sym}^2\varrho)$ is actually an entire function was first proved by Shimura ([8], Theorem 1). This was generalized by Jacquet and Gelbart for all automorphic representations π of GL_2 over any global field [9].

From the convergence of the series $\sum_p p^{-2s}$ for all s with $\mathrm{Re}(s)>\frac{1}{2}$, and the information we have about the polynomial $L_p(T)$ given by Lemma 2, we obtain that the Euler product

$$\prod_p L_p(p^{-s})$$

converges uniformly for any $\varepsilon>0$ in the region $\mathrm{Re}(s)\geqq\frac{1}{2}+\varepsilon$; furthermore in this region it represents a holomorphic function which is free of zeros. To complete the proof of the main theorem we now prove the following result.

Theorem 1. *The function $L(s,\pi_\Delta,\mathrm{Sym}^4\varrho)$ has a meromorphic continuation to the whole s-plane, which is holomorphic in the region* $\mathrm{Re}(s)\geqq 1$ *and satisfies*

(i) $L(s,\pi_\Delta,\mathrm{Sym}^4\varrho)=L(1-s,\pi_\Delta,\mathrm{Sym}^4\varrho)$;

(ii) $L(1+it,\pi_\Delta,\mathrm{Sym}^4\varrho)\neq 0$ *for all real t.*

2. The function

$$L(s,\pi_\Delta,\mathrm{Sym}^0\varrho)L(s,\pi_\Delta,\mathrm{Sym}^2\varrho)L(s,\pi_\Delta,\mathrm{Sym}^4\varrho)$$

is holomorphic in the whole s-plane except for simple poles at $s=1$ and 0, and is free of zeros outside the strip $0<\mathrm{Re}(s)<1$.

Proof. In [7], Theorem 4.1.1 and Theorem 5.3, it has been shown that for any automorphic representation π of PGL(2) which is not monomial, the L-function $L(s,\pi,\mathrm{Sym}^4\varrho)$ has a meromorphic continuation to the whole s-plane, satisfies a functional equation of the type (i) and does not vanish on the line $\mathrm{Re}(s)=1$, except possibly for at most a simple zero at $s=1$. Hence it remains to show that it actually has no zero at $s=1$ and that in the half plane $\mathrm{Re}(s)\geqq 1$ it is free of poles. This will follow from the following auxiliary lemma, where we use the notation of [2] and [7]. Let S be the finite set of ramified primes including the infinite ones and put $L_s(s,\pi)=\prod_{v\notin S}L_v(s,\pi_v)$.

Lemma 3. *Suppose π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_k)$ of a global field K which is not monomial (i.e. $\pi\otimes\eta\cong\pi$ for no Hecke character of $K^\times\backslash\mathbb{A}_K^\times$). Then $L_S(s,\pi,\mathrm{Sym}^4\varrho)$ is holomorphic in the region $\mathrm{Re}(s)\geqq 1$ and does not vanish on the line $\mathrm{Re}(s)=1$.*

Proof of Lemma 3. We shall use the method of Deligne and Gelbart [2]. Let Π be the lift of π to $\mathrm{PGL}_3(\mathbb{A}_K)$ as defined by Gelbart and Jacquet [9]. Then Π is cuspidal and

$$\begin{aligned} L_S(s, \Pi \times \Pi) &= L_S(s, \pi, \mathrm{Sym}^2 \varrho \otimes \mathrm{Sym}^2 \varrho) \\ &= L_S(s, \pi, \mathrm{Sym}^4 \varrho) L_S(s, \pi, \mathrm{Sym}^2 \varrho) L_S(s, \pi, \mathrm{Sym}^0 \varrho), \end{aligned}$$

where $L_S(s, \pi, \mathrm{Sym}^0 \varrho)$ is the partial Hecke L-function attached to S. $L_S(s, \pi, \mathrm{Sym}^0 \varrho)$ has a simple pole at $s=1$ and otherwise is non-zero in the region $\mathrm{Re}(s) \geqq 1$. If $L(s, \Pi \times \Pi)$ denotes the full L-function of Jacquet, Piatetski-Shapiro, and Shalika on $\mathrm{GL}(3) \times \mathrm{GL}(3)$, then we know it has a simple pole at $s=1$ ($\Pi \cong \tilde{\Pi}$). Moreover, the local factors $L(s, \Pi_v \times \Pi_v)$, for $v \in S$, are all holomorphic and non-zero in the region $\mathrm{Re}(s) \geqq 1$ ([3], Proposition 1.5, p. 507 and Proposition 3.17, p. 542). Consequently, $L_S(s, \Pi \times \Pi)$ has a simple pole at $s=1$ and otherwise is non-zero on the line $\mathrm{Re}(s)=1$ by [7]. This completes the proof of the auxiliary lemma and hence also of the previous theorem.

From the basic identity of Lemma 2

$$\sum_{n=1}^{\infty} \tau_0(n)^4 n^{-s} = \zeta(s)^2 \zeta_2(s)^3 \zeta_4(s) \prod_p L_P(p^{-s}),$$

and from the previous theorem applied to $\pi = \pi_\Delta$, we obtain that the expression on the right hand side has a meromorphic continuation to the half plane $\mathrm{Re}(s) \geqq \frac{1}{2}$, at $s=1$ it has a double pole, and otherwise it is a holomorphic function free of zeros in the region $\mathrm{Re}(s) \geqq 1$. By a standard use of the Wiener-Ikehara Theorem we obtain for positive real x sufficiently large

$$\sum_{n \leqq x} \tau_0(n)^4 \sim cx \log x,$$

and

$$\sum_{p \leqq x} \tau_0(p)^4 \log p \sim 2x.$$

This completes the proof of the Main Theorem.

Remarks. The meromorphic continuation of $F(s)$ to the region $\mathrm{Re}(s) \geqq \frac{1}{4}$ can be obtained by a more careful analysis of the Euler product $\prod_p L_p(p^{-s})$ using the fact that

$$L_p(T) = 1 - (7 - 12\tau_0(p)^2 + 6\tau_0(p)^4) T^2 + \ldots + T^{14}.$$

This is of some interest when trying to locate the possible singularities of $F(s)$ to the left of $\mathrm{Re}(s)=1$. Such information would lead to an asymptotic estimate with an error term $O(x^{4/5}(\log x)^4)$, as in Rankin's paper [5]. This would then have applications to the problem of the Petersson-Ramanujan conjecture for the Fourier coefficients of the real analytic cusp forms of Maass, namely to

$|a(p)| \ll p^{0.2+\varepsilon}$ ($\varepsilon > 0$), which is the best estimate presently known for the coefficients of real analytic cusp forms; we do not pursue this line here because it would lengthen the paper dramatically. The best estimate for the Petersson-Ramanujan conjecture at infinity follows more directly from the work of the second author concerning the L-functions $L(s, \pi, \mathrm{Sym}^5 \varrho)$.

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