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Inventiones mathematicae

The L-function $L_3(s, \pi_\Delta)$ is entire

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Let

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24},$$

$q = \exp(2\pi i z)$, be the Ramanujan modular form. Then the associated Dirichlet series has the following Euler product (for $\mathrm{Re}(s) > 1$)

$$L(s, \pi_\Delta) = G_{\mathbf{R}}(s+11/2)\, G_{\mathbf{R}}(s+13/2) \prod_p (1-\alpha_p p^{-s})^{-1} (1-\bar{\alpha}_p p^{-s})^{-1},$$

$G_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, where π_Δ denotes the corresponding automorphic form.

In connection with the Sato-Tate conjecture, for every positive integer m, Serre [10, 11] introduced an Euler product $L_m(s, \pi_\Delta)$, denoted by $L(s, \mathrm{Sym}^m(\rho_2), \pi_\Delta)$ in the context of Langlands L-functions, whose local factor at a rational prime p is simply given by

$$\prod_{0 \leqq j \leqq m} (1-\alpha_p^j \bar{\alpha}_p^{m-j} p^{-s})^{-1}.$$

The absolute convergence of this Euler product for $\mathrm{Re}(s) > 1$ is then an immediate consequence of the validity of Ramanujan's conjecture for π_Δ.

For $m \leqq 2$, it has been shown that $L_m(s, \pi_\Delta)$ extends to an entire function of s satisfying an appropriate functional equation ($m=1$ is due to Hecke while $m=2$ was proved by Shimura [15], also see [2] for non-holomorphic forms). For $m = 3$, 4, and 5, while the meromorphic continuation and functional equation have been established in each case [6, 12, 13] (all consequences of the Langlands' theory of Eisenstein series [7]), our knowledge of the regularity of $L_m(s, \pi_\Delta)$ reduces only to the closed half plane $\mathrm{Re}(s) \geqq 1$ (with at most a simple pole at $s=1$, due to Serre, if $m=5$). The purpose of this paper is to establish the holomorphicity of

$$L_3(s, \pi_\Delta) = G_{\mathbf{C}}(s+33/2) \cdot G_{\mathbf{C}}(s+11/2) \prod_{p<\infty} \prod_{0 \leqq j \leqq 3} (1-\alpha_p^j \bar{\alpha}_p^{3-j} p^{-s})^{-1},$$

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where $G_{\mathbf{C}}(s)=G_{\mathbf{R}}(s)\,G_{\mathbf{R}}(s+1)$ (for the local factors at infinity see [9, 10]). More precisely, we shall show

Theorem. *The L-function* $L_3(s,\pi_\Delta)$, *originally defined for* $Re(s)>1$, *extends to an entire function of s on* $\mathbf{C}$ *satisfying*

$$L_3(1-s,\pi_\Delta)=L_3(s,\pi_\Delta) \tag{1}$$

We need several lemmas. For the sake of simplicity let $\pi=\pi_\Delta$. Then $\pi=\pi_\infty\otimes\bigotimes_{p<\infty}\pi_p$, as a restricted tensor product, where we consider π as a cusp form on $PGL_2(\mathbf{A})$, $\mathbf{A}$ being the ring of adeles of $\mathbf{Q}$. Let Π be the Gelbart-Jacquet lift of π (cf. [2]). It is a cusp form on $PGL_3(\mathbf{A})$. Moreover if we write $\Pi=\Pi_\infty\otimes\bigotimes_{p<\infty}\Pi_p$, Π_∞ is tempered while every Π_p, $p<\infty$, is unramified (also tempered). Suppose $p<\infty$ and let $L(s,\pi_p\times\Pi_p)$ be the corresponding local Ranking-Selberg L-function (cf. relation (3.2.1) of [4]). Now, let $p=\infty$ and denote by $W_{\mathbf{R}}$, the Weil group of $\mathbf{C}/\mathbf{R}$. Moreover let $\sigma\colon W_{\mathbf{R}}\to GL_2(\mathbf{C})$ be the corresponding representation of $W_{\mathbf{R}}$ which is attached to π_∞ by Langlands reciprocity at infinity [8]. Then Π_∞ corresponds to the three dimensional representation $\mathrm{Sym}^2(\sigma)$ of $W_{\mathbf{R}}$. Now, using the results in [5], we define

$$L(s,\pi_\infty\times\Pi_\infty)=L(s,\sigma\otimes\mathrm{Sym}^2(\sigma)),$$

where the L-function on the right is a local Artin L-function and for each positive integer m, $\mathrm{Sym}^m(\sigma)=\mathrm{Sym}^m(\rho_2)\cdot\sigma$. Here ρ_2 is the standard representation of $SL_2(\mathbf{C})$ (the L-group of PGL_2).

We now set

$$L(s,\pi\times\Pi)=\prod_{p\leqq\infty}L(s,\pi_p\times\Pi_p).$$

Then by Theorem 5.3 of [3], $L(s,\pi\times\Pi)$ is absolutely convergent for $\mathrm{Re}(s)>1$. Moreover the discussion in Paragraph 3.5 of [4] (page 801) implies that the zeta function given by the left hand side of (3.5.1) in [4] is in fact entire. Combining relation (3.5.1) of [4] with the results in [5] will then imply that $L(s,\pi\times\Pi)$ is entire. We now have

Lemma 1. *For all* $s\in\mathbf{C}$, *one has*

$$L_3(s,\pi)=L(s,\pi\times\Pi)/L(s,\pi). \tag{1.1}$$

In particular for $\mathrm{Re}(s)>1$ *the Euler product defining* $L_3(s,\pi)$ *is absolutely convergent and extends to a holomorphic function on* $\mathrm{Re}(s)\geqq 1$.

Proof. Let, as before, ρ_2 be the standard representation of $SL_2(\mathbf{C})$. Then (1.1) follows immediately from

$$\rho_2\otimes\mathrm{Sym}^2(\rho_2)=\rho_2\oplus\mathrm{Sym}^3(\rho_2).$$

The rest of the assertion now follows from the discussion before the lemma and the nonvanishing of $L(1+\sqrt{-1}t,\pi)$, $t\in\mathbf{R}$.

Remark. The idea of using (1.1) to obtain the meromorphic continuation and functional equation for $L_3(s, \pi)$ is due to Deligne.

Functional Eq. (1) has been proved in general in [12] (Theorem 5.9). One has to only observe (cf. [14]) that the local coefficient $\gamma(s, \mathrm{Sym}^3(\rho_2), \pi_\infty, \chi_\infty)$ at infinity (π_∞ is in the discrete series and is the only ramification) is simply equal to

$$L(s, \mathrm{Sym}^3(\sigma))/L(1-s, \mathrm{Sym}^3(\sigma)).$$

The following lemma is crucial.

Lemma 2. *The L-function $L_3(s, \pi)$ extends to a meromorphic function of s on* $\mathbf{C}$ *with only a finite number of simple poles, all lying in the open interval* (0, 1). *Moreover it has no pole at $s = 1/2$.*

Proof. We shall freely use the notation from [6] and [12]. Choose ϕ in the space of π as in Sect. 2 of [12] and extend ϕ to $\tilde{\phi}$ on $G(\mathbf{A})$, G being a group of type G_2 (example (xv) of [6]). Let $E(-s, \tilde{\phi}, g, P)$ be the corresponding Eisenstein series defined by relation (2.4) of [12], where P is the maximal parabolic subgroup of G_2 fixed as in Sect. 2 of [12]. Finally, let $M(-s)\tilde{\phi}$ be the corresponding constant term defined by relation (2.6) of [12] for $\mathrm{Re}(s)$ large. Suppose $\tilde{\phi} = \bigotimes_{p \leqq \infty} \tilde{\phi}_p$, where for all p, $p < \infty$, $\tilde{\phi}_p$ is the unique K_p-fixed vector satisfying $\tilde{\phi}_p(e) = 1$. Then using the computations in [6]

$$\begin{aligned} M(-s/5)\tilde{\phi} &= \zeta(2s) L_3(s, \pi)/\zeta(1+2s) L_3(1+s, \pi) \\ &\quad \cdot \gamma_\infty(s) M_\infty(-s/5)\tilde{\phi}_\infty. \end{aligned} \tag{2.1}$$

where $\zeta(s)$ is just the Riemann zeta function,

$$\begin{aligned} \gamma_\infty(s) = G_{\mathbf{R}}(1+2s) G_{\mathbf{R}}(2s)^{-1} G_{\mathbf{C}}(1+s+33/2) G_{\mathbf{C}}(s+33/2)^{-1} \\ \cdot G_{\mathbf{C}}(1+s+11/2) G_{\mathbf{C}}(s+11/2)^{-1}, \end{aligned}$$

and $M_\infty(s)$ is the standard intertwining operator acting on $\operatorname{Ind}_{P(\mathbf{R})\uparrow G(\mathbf{R})} \pi_\infty \otimes \delta^s_{P,\infty}$. The factor $\gamma_\infty(s)$ is clearly holomorphic and nonzero for $\mathrm{Re}(s) > 0$. Since π_∞ is in the discrete series, Lemma 3.10 of [8] implies that for $\mathrm{Re}(s) > 0$, $M_\infty(-s/5)\tilde{\phi}_\infty$ is holomorphic. Moreover for any given s one may choose $\tilde{\phi}_\infty$ in such a way that $M_\infty(-s/5)\tilde{\phi}_\infty \not\equiv 0$. Consequently (2.1) implies that for $\mathrm{Re}(s) > 0$, the poles of $\zeta(2s) L_3(s, \pi)$ are exactly those of $M(-s/5)$. From the general theory of Eisenstein series [7], it follows that for $\mathrm{Re}(s) \geqq 0$, $M(-s/5)$ is holomorphic except for a finite number of simple poles, all lying on the real axis. But, then for $\mathrm{Re}(s) \geqq 1/2$, $\zeta(2s)^{-1}$ is holomorphic and non-zero except for a simple zero at $s = 1/2$. This implies that for $\mathrm{Re}(s) \geqq 1/2$, $L_3(s, \pi)$ is holomorphic with only a finite number of simple poles all lying in (1/2, 1). The lemma is now a consequence of the functional Eq. (1).

Lemma 3. *The L-function $L(s, \pi)$ has no zeros on* [0, 1].

Proof. The Mellin transform of $\Delta(x+iy)$ along the imaginary axis gives

$$L(s, \pi) = \int_0^\infty y^6 \Delta(iy) y^s \, dy/y.$$

The functional relation $\Delta(-1/z)=z^{12}\Delta(z)$ now implies that the integral from 0 to 1 can be combined with the integral from 1 to ∞ to yield

$$L(s,\pi)=\int_1^\infty y^6\Delta(iy)(y^s+y^{1-s})\,dy/y.$$

Since $\Delta(iy)>0$ for $y\geqq 1$, it is then clear that $L(s,\pi)>0$ for s real. This proves the lemma.

Proof of the theorem. The L-function $L(s,\pi\times\Pi)$ is an entire function of s. By Lemmas 1 and 3, $L_3(s,\pi)$ has no poles on $[0, 1]$. Now the theorem is a consequence of Lemma 2.

Corollary 1. *Let s_0 be a zero of $L(s,\pi)$ of order $N\geqq 0$. Then s_0 is also a zero of $L(s,\pi\times\Pi)$ of order at least N.*

Corollary 2. *Let $E(-s,\tilde{\phi},g,P)$ be the Eisenstein series attached to π* (cf. Lemma 2). *Then for* $\mathrm{Re}(s)\geqq 0$, *$E(-s,\tilde{\phi},g,P)$ is holomorphic except possibly for a simple pole at $s=1/10$. The point $s=1/10$ is a pole if and only if $L_3(1/2,\pi)\neq 0$ or equivalently $L(1/2,\pi\times\Pi)\neq 0$.*

Remark 1. It is clear that the method also applies to other cases where Lemma 3 can be verified directly (e.g. the unique cusp form of weight 2 on $\Gamma_0(11)$). In particular, using the results of [1], our results immediately extend to all the holomorphic cusp forms (with respect to $SL_2(\mathbf{Z})$) of weight $\leqq 50$.

Remark 2. It is remarkable that the theory of Eisenstein series does also hand us the holomorphy of $L_3(s,\pi)$ at $s=1/2$, since in general there are L-functions with zeros at $s=1/2$ and therefore at this point Lemma 3 fails (no other zeros are expected on (0, 1)). In fact this is crucial in extending our results to the examples in [1] for which $k/2$ is odd (k being the weight of π) since then $L(1/2,\pi)=0$. In particular if $k/2$ is odd, $L(1/2,\pi\times\Pi)=0$.

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