Raghuram, A., and F. Shahidi. (2008) "On Certain Period Relations for Cusp Forms on GL_n," International Mathematics Research Notices, Vol. 2008, Article ID rnn077, 23 pages.
doi:10.1093/imrn/rnn077

On Certain Period Relations for Cusp Forms on GL_n

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1 Introduction

The main goal of this paper is to analyze certain periods attached to a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ for a number field F. We analyze the behavior of these periods upon twisting π by algebraic Hecke characters. These periods, which were originally defined by Harder when $n = 2$ and later by Mahnkopf when $F = \mathbb{Q}$, are nonzero complex numbers attached to π which are sometimes intimately linked to the special values of the standard L-function of π.

Henceforth, we will assume the cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ to be regular and algebraic, which is a condition only on the infinity component π_∞ of π. This assumption makes π arithmetically interesting; for example, it ensures that the finite part π_f is defined over a number field $\mathbb{Q}(\pi)$. Let S_r be the set of real places of F. We let $\epsilon = (\epsilon_v)_{v\in S_r}$ be a signature indexed by the real places with $\epsilon_v \in \{\pm\}$. In the signature, ϵ_v can be any sign if n is even, and if n is odd then ϵ is uniquely determined by π. To this data, plus the choice of a generator $\mathbf{w}_\infty$ of a continuous cohomology group attached to π_∞, we attach a nonzero complex number $p^\epsilon(\pi_f, \mathbf{w}_\infty)$, which we call a period of π. See Definition 3.3. We will abbreviate $p^\epsilon(\pi_f, \mathbf{w}_\infty)$ simply by $p^\epsilon(\pi_f)$ (see the explanation toward the end of Section 3.4). These periods are defined by comparing a certain canonical $\mathbb{Q}(\pi)$-structure on the Whittaker model of π_f with a $\mathbb{Q}(\pi)$-structure on

Received January 22, 2008; Revised June 5, 2008; Accepted June 13, 2008
Communicated by Prof. Jim Cogdell

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a suitable cohomology space to which π has a nonzero contribution. The comparison map between these two spaces is essentially the inverse of the map giving the Fourier coefficients of cusp forms in the space of π. These periods were first defined by Harder [10] for representations of $\mathrm{GL}_2(\mathbb{A}_F)$, and later were generalized by Mahnkopf [14] in the case of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. In both of these works they are defined to analyze the special values of the standard L-function $L(s, \pi_f)$ attached to π_f.

Concerning special values of L-functions, often times, it is interesting to know how these values change under functorial operations on the representation at hand. For example, one can ask for the behavior of $L(m, \pi_f \otimes \xi_f)$ as a function of the Hecke character ξ_f. An application of such a question is to the subject of p-adic L-functions. In many, but not all, examples, this question about the relationship of L-values translates into a question about the behavior of the periods attached to π_f upon twisting π by Hecke characters. In fact, in all cases, independent of the connection with values of L-functions, it is meaningful to ask how the periods $p^{\epsilon}(\pi_f)$ transform under twisting of π. In Section 4 we prove, as our main result, the following.

Theorem 1.1. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let ξ be an algebraic Hecke character of F. We attach a signature ϵ_ξ to ξ. We let $\gamma(\xi_f)$ be the Gauss sum attached to ξ. Then

$$p^{\epsilon\cdot\epsilon_\xi}(\pi_f \otimes \xi_f) \sim_{\mathbb{Q}(\pi,\xi)} \gamma(\xi_f)^{-n(n-1)/2}\, p^{\epsilon}(\pi_f)$$

for any permissible signature ϵ for π (which is an issue only when n is odd). By $\sim_{\mathbb{Q}(\pi,\xi)}$ we mean up to an element of the number field $\mathbb{Q}(\pi, \xi)$. Moreover, the quotient $p^{\epsilon\cdot\epsilon_\xi}(\pi_f \otimes \xi_f)/(\gamma(\xi_f)^{-n(n-1)/2} p^{\epsilon}(\pi_f))$ is equivariant under the action of the automorphism group of complex numbers. □

The proof of the above theorem is a little too involved to explain in this introduction; however, we ask the reader to look at the diagram of maps (4.1). The proof comes out of an analysis of that diagram; the period relation may be construed as the obstruction to commutativity of that diagram.

In Section 4.2, we mention two variations of the above theorem. The first is to state it as a reciprocity law as in Blasius [1], and in this formulation the Gauss sums do not appear. The second is to draw attention to periods, and identical-looking relations amongst such periods, obtained by considering cohomology in degrees other than the degree considered in Definition 3.3.

A special case of our theorem is when F is a real quadratic field, and π corresponds to a Hilbert modular form of CM type, then our period relations are formally the same period relations proved by Murty and Ramakrishnan in [15, Theorem A]. The reader should also compare our Theorem 4.1 with the theorems and conjectures of Blasius [2] and Panchishkin [16] on the behavior of Deligne's periods attached to a motive upon twisting it by Artin motives. See also [17, Conjecture 7.1]. We end the introduction by noting that the main motivation to prove the above theorem is in its application to the special values of L-functions. Transcribing the above period relation into a relation between the special values of $L(s, \pi_f)$ and those of the twisted L-function $L(s, \pi_f \otimes \xi_f)$ is work in progress. We hope to report on it in another paper.

2 Notation and Some Preliminaries

For a number field F, we let $\mathbb{A}_F$ stand for its adèle ring, and $\mathbb{I}_F = \mathbb{A}_F^\times$ be its group of idèles. We let $\| \|_F : \mathbb{I}_F \to \mathbb{R}_{>0}$ be the adèlic norm defined by $\|x\|_F = \prod_v |x_v|_v$, with v running over all places of F, and the local absolute values all being the normalized ones. For a finite v, we let ord_v stand for the additive valuation on the completion F_v of F at v, $\mathcal{O}_v$ for the ring of integers of F_v, and $\mathcal{O}_v^\times$ for the group of units of $\mathcal{O}_v$. When there is no confusion about the base field F, we will drop the subscript F from $\mathbb{A}_F$, $\mathbb{I}_F$, and $\| \|_F$. For any finite set S of places of F we use a superscript S to denote a product outside S, and a subscript S to denote a product within S. For example, if S_∞ stands for the set of all infinite places of F, then the ring of finite adèles is $\mathbb{A}_F^{S_\infty}$ and will be denoted by $\mathbb{A}_{F,f}$ or simply by $\mathbb{A}_f$. We let S_r stand for the set of real places and so $S_c := S_\infty - S_r$ is the set of complex places. Let r_1 (respectively, r_2) denote the number of real (respectively, complex) places of F.

Let $G = GL_n$, and let $Z = Z_n$ be the center of G, both regarded as F-groups. Let $G_\infty = G(F \otimes \mathbb{R}) = G(\mathbb{R})^{r_1} \times G(\mathbb{C})^{r_2}$. Following Borel–Jacquet [5, Section 4.6], we say an irreducible representation of $G(\mathbb{A})$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G(\mathbb{A})$ on its space of automorphic forms. We say an automorphic representation is cuspidal if it is a subrepresentation of the representation of $G(\mathbb{A})$ on the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$. For an automorphic representation π of $G(\mathbb{A})$, we have $\pi = \pi_\infty \otimes \pi_f$, where $\pi_\infty = \otimes_{v \in S_\infty} \pi_v$ is a representation of G_∞, and $\pi_f = \otimes_{v \notin S_\infty} \pi_v$ is a representation of $G(\mathbb{A}_f)$.

3 Definition of the Periods

The purpose of this section is to define certain periods attached to a regular algebraic cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$. This definition is due to Harder [10] for GL_2, and is due to Mahnkopf [14] in the case $F = \mathbb{Q}$. (See Clozel [7] for the definitions of a cuspidal representation being regular and algebraic.)

Before we get into the details of the definition, we very roughly indicate the ingredients needed in making the definition. We will have a number field E. We will have two $\mathbb{C}$-vector spaces V_1 and V_2 with E-structures V_1^0 and V_2^0, respectively. (By V_i^0 being an E-structure for V_i, we mean an E-subspace such that the canonical map $V_i^0 \otimes_E \mathbb{C} \to V_i$ is an isomorphism.) In our situation, the spaces V_i will be representation spaces, and not merely vector spaces, and the E-structures will be unique up to homotheties. Finally, we will have a comparison isomorphism $\phi : V_1 \to V_2$. The period attached to ϕ, denoted $p(\phi)$, is a nonzero complex number such that $\phi(V_1^0) = p(\phi)V_2^0$. Observe that $p(\phi)$ is a well-defined element in $\mathbb{C}^*/E^*$. This way of defining periods is due to Borel [3] where he called such numbers regulators. For us, the number field E will be the rationality field of π, the space V_1 will be the Whittaker model of π_f, and the space V_2 will be a certain cohomology space (to which π will have nonzero contribution), and the comparison isomorphism ϕ will be related to taking the Fourier coefficient of a cusp form in the space of π. We now proceed to make all this precise.

3.1 The rationality field of π

The first ingredient we need is the rationality field of π. The definitive reference is Clozel [7, Chapter 3]. Given π, suppose V is the representation space of π_f, any $\sigma \in \mathrm{Aut}(\mathbb{C})$ defines a representation π_f^σ on $V \otimes_{\mathbb{C}} \mathbb{C}_{\sigma^{-1}}$ where $G(\mathbb{A}_f)$ acts on the first factor. Let $\mathcal{S}(\pi_f)$ be the subgroup of $\mathrm{Aut}(\mathbb{C})$ consisting of all σ such that $\pi_f^\sigma \simeq \pi_f$. Define the rationality field $\mathbb{Q}(\pi_f)$ of π_f as the subfield of $\mathbb{C}$ fixed by $\mathcal{S}(\pi_f)$; we denote this as $\mathbb{Q}(\pi_f) = \mathbb{C}^{\mathcal{S}(\pi_f)}$. For example, if χ is a Dirichlet character, also thought of as an idèle class character, then $\mathbb{Q}(\chi_f)$ is the field $\mathbb{Q}(\{\text{Values of } \chi\})$. Similarly, if φ is a primitive holomorphic cusp form on the upper half plane, of even weight $2k$, for the Hecke congruence subgroup $\Gamma_0(N)$, with Fourier expansion $\varphi(z) = \sum_{n=1}^{\infty} a_n q^n$, and if $\pi = \pi(\varphi)$ is the cuspidal automorphic representation associated with φ, then $\mathbb{Q}(\pi_f) = \mathbb{Q}(\{a_n : n \geq 1\})$—the field generated by all the Fourier coefficients of φ. (See [20].) In this example, the weight is assumed to be even to ensure that π is algebraic. If the weight is odd, the same is true with π replaced by $\pi \otimes \|\ \|^{-1/2}$. The main results that we need about the rationality field of π_f is stated in the following theorem. (See Clozel [7, Théorème 3.13] and Waldspurger [20, Chapter I].)

Theorem 3.1 (Eichler, Shimura, Harder, Waldspurger, Clozel). Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then

1. $\mathbb{Q}(\pi_f)$ is a number field;
2. π_f admits a $\mathbb{Q}(\pi_f)$-structure, which is unique up to homotheties; and

3. for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, π_f^σ is the finite part of a cuspidal automorphic representation (which we denote by π^σ).

□

We recall that given $\sigma \in \mathrm{Aut}(\mathbb{C})$, one defines π_∞^σ by defining an action of σ on the infinity types of π_∞, which are indexed by the complex embeddings of F. (See Clozel [7, Section 3.3] for the precise definition.) We denote $\mathcal{S}(\pi_\infty) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) : \pi_\infty^\sigma = \pi_\infty\}$, and define $\mathbb{Q}(\pi_\infty) = \mathbb{C}^{\mathcal{S}(\pi_\infty)}$ as the subfield of $\mathbb{C}$ fixed by $\mathcal{S}(\pi_\infty)$. We define the rationality field $\mathbb{Q}(\pi)$ of π as the composite field $\mathbb{Q}(\pi_f)\mathbb{Q}(\pi_\infty)$; this is a number field.

3.2 Rational structure on the Whittaker model of π

The next ingredient we need is the Whittaker model of π_f and a semilinear action of $\mathrm{Aut}(\mathbb{C})$ on this space that commutes with the action of $\mathrm{GL}_n(\mathbb{A}_f)$. Toward this, we fix a nontrivial character ψ of $F\backslash\mathbb{A}_F$. We can write $\psi = \psi_\infty \otimes \psi_f$ (the notation being the obvious one). We let $W(\pi,\psi)$ be the Whittaker model of π, and this factors as $W(\pi,\psi) = W(\pi_\infty,\psi_\infty) \otimes W(\pi_f,\psi_f)$. There is a semilinear action of $\mathrm{Aut}(\mathbb{C})$ on $W(\pi_f,\psi_f)$, which is defined as follows. (See [10, pp. 79–80] or [14, p. 594].) That the values of ψ are all roots of unity suggests that we consider the cyclotomic character

$$\begin{array}{ccccccccc} \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) & \to & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) & \to & \widehat{\mathbb{Z}}^\times \simeq \prod_p \mathbb{Z}_p^\times & \subset & \prod_p \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}^\times, \\ \sigma & \mapsto & \sigma|_{\overline{\mathbb{Q}}} & \mapsto & \sigma|_{\mathbb{Q}(\mu_\infty)} & \mapsto & t_\sigma & \mapsto & t_\sigma \end{array}$$

where the last inclusion is the one induced by the diagonal embedding of $\mathbb{Z}_p^\times$ into $\prod_{\mathfrak{p}|p}\mathcal{O}_\mathfrak{p}^\times$. (Here $\mathfrak{p}$ is a prime of F above p, and $\mathcal{O}_\mathfrak{p}$ is the ring of integers of the completion $F_\mathfrak{p}$ of F at $\mathfrak{p}$.) The element t_σ at the end can be thought of as an element of $\mathbb{A}_f^\times = \mathbb{I}_f$. Let $t_{\sigma,n}$ denote the diagonal matrix $\mathrm{diag}(t_\sigma^{-(n-1)}, t_\sigma^{-(n-2)}, \ldots, 1)$ regarded as an element of $\mathrm{GL}_n(\mathbb{A}_f)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $w \in W(\pi_f,\psi_f)$, define the function w^σ by

$$w^\sigma(g_f) = \sigma(w(t_{\sigma,n}g_f))$$

for all $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$. Note that this action makes sense locally, by replacing t_σ by $t_{\sigma,\mathfrak{p}}$. Further, if π_v is unramified, then a spherical vector is mapped to a spherical vector under σ. If we normalize the spherical vector to take the value 1 on identity, then σ fixes this vector. This makes the local and global actions of σ compatible.

We add that the above definition of the action of $\mathrm{Aut}(\mathbb{C})$ is a direct generalization of the classical action on holomorphic cusp forms. If φ is a holomorphic cusp form with

q-expansion $\varphi(z) = \sum_{n=1}^{\infty} a_n q^n$, then one defines $\varphi^\sigma(z) = \sum_{n=1}^{\infty} \sigma(a_n) q^n$. The twisting by t_σ in the above definition ensures that the additive character that we integrate against for the nth Fourier coefficient is not changed, since $\sigma(\psi_f(t_\sigma^{-1}u)) = \psi_f(u)$ for all $u \in \mathbb{A}_f$.

Lemma 3.2. With notation as above, $w \mapsto w^\sigma$ is a σ-linear $\mathrm{GL}_n(\mathbb{A}_f)$-equivariant isomorphism from $W(\pi_f, \psi_f)$ onto $W(\pi_f^\sigma, \psi_f)$. For any finite extension $E/\mathbb{Q}(\pi_f)$, we have an E-structure on $W(\pi_f, \psi_f)$ by taking invariants:

$$W(\pi_f, \psi_f)_E = W(\pi_f, \psi_f)^{\mathrm{Aut}(\mathbb{C}/E)}.$$

□

Proof. The first part of the lemma about $w \mapsto w^\sigma$ is easy to see and we leave it to the reader. It is the assertion about rational structures that needs a proof. We add that this lemma is well known. (See Harder [10, p. 80] and Mahnkopf [14, p. 594].)

For brevity, let $E_0 = \mathbb{Q}(\pi_f)$. We begin by showing that there is a nonzero vector in $W(\pi_f, \psi_f)$ that is fixed by $\mathrm{Aut}(\mathbb{C}/E_0)$. Consider the vector $w^0 = \otimes_v w_v^0$, where w_v^0, also denoted $w_{\pi_v}^0$ below, is a new vector (also called essential vector as in [12]) in $W(\pi_v)$. Note that w_v^0 is well defined up to scalars. We can choose a normalization of the local new vectors such that they are well behaved under $\mathrm{Aut}(\mathbb{C})$: $(w_{\pi_v}^0)^\sigma = w_{\pi_v^\sigma}^0$. (See [14, 3.4.1].) In particular, any $\sigma \in \mathrm{Aut}(\mathbb{C}/E_0)$ fixes w_v^0, and hence fixes the global new vector w^0. Now let $W(\pi_f, \psi_f)_{E_0}$ be the E_0-span of the $G(\mathbb{A}_f)$-orbit of w^0. Then the canonical map $W(\pi_f, \psi_f)_{E_0} \otimes_{E_0} \mathbb{C} \to W(\pi_f, \psi_f)$ is an isomorphism; surjectivity follows from irreducibility, and injectivity follows exactly as in the proof of [20, Lemme I.1.1]. The action of $\mathrm{Aut}(\mathbb{C}/E_0)$ on $W(\pi_f, \psi_f)$ may then be identified with the action of $\mathrm{Aut}(\mathbb{C}/E_0)$ on $W(\pi_f, \psi_f)_{E_0} \otimes_{E_0} \mathbb{C}$ where it acts on the second factor. We deduce that $W(\pi_f, \psi_f)_{E_0} = W(\pi_f, \psi_f)^{\mathrm{Aut}(\mathbb{C}/E_0)}$. Now if E is a finite extension of $E_0 = \mathbb{Q}(\pi_f)$ then, in the above isomorphism, one can identify $W(\pi_f, \psi_f)_{E_0} \otimes_{E_0} E$ with $W(\pi_f, \psi_f)^{\mathrm{Aut}(\mathbb{C}/E)}$. ■

As a matter of notation, given a $\mathbb{C}$-vector space V, and given a subfield $E \subset \mathbb{C}$, we will let V_E stand for an E-structure on V (if there is one at hand). Fixing an E-structure gives an action of $\mathrm{Aut}(\mathbb{C}/E)$ on V, by making it act on the second factor in $V = V_E \otimes_E \mathbb{C}$. Having fixed an E-structure, for any extension E'/E, we have a canonical E'-structure by letting $V_{E'} = V_E \otimes_E E'$. Further, as a notational convenience, when we talk of Whittaker models, we will henceforth suppress the additive character ψ, since that has been fixed once and for all; for example, $W(\pi_f)$ will denote $W(\pi_f, \psi_f)$. Also, we will denote the map $w \mapsto w^\sigma$ simply by σ.

3.3 Rational structure on the $(\mathfrak{g}, K)$-cohomology of π

As mentioned earlier, the periods come via a comparison of $W(\pi_f)_E$ with a rational structure on a suitable cohomology space. We now describe this cohomology space. Recall that $G = G_n = \mathrm{GL}_n$ and the center of G is denoted by Z_n or Z. Let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞. Let $K_\infty = \otimes_{v\in S_\infty} K_v$ where $K_v = Z_n(\mathbb{R})O_n(\mathbb{R})$ if v is real, and $K_v = Z_n(\mathbb{C})U_n(\mathbb{C})$ if v is complex. Let $K^0 = K^0_\infty$ be the topological connected component of K_∞. Note that $K_\infty/K^0_\infty \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1}$. Let $b^{\mathbb{R}}_n$ be $n^2/4$ if n is even, and $(n^2-1)/4$ if n is odd. We also let $b^{\mathbb{C}}_n$ be $n(n-1)/2$. Now we define $b = r_1 b^{\mathbb{R}}_n + r_2 b^{\mathbb{C}}_n$. The integer b depends only on the base field F and the rank n of GL_n. It is the bottom degree of the so-called cuspidal range for GL_n as an F-group. The next ingredient we need in defining the period is relative Lie algebra cohomology of π in degree b. For a $(\mathfrak{g}_\infty, K^0_\infty)$-module U, we let $H^*(\mathfrak{g}_\infty, K^0_\infty; U)$ stand for relative Lie algebra cohomology of U, for the definition and properties of which we refer the reader to Borel and Wallach's book [6]. Given a representation τ of G_∞, by $H^*(\mathfrak{g}_\infty, K^0_\infty; \tau)$, we will mean the cohomology of the $(\mathfrak{g}_\infty, K^0_\infty)$-module consisting of smooth K_∞-finite vectors of τ.

Let $T = T_n$ denote the maximal torus of GL_n consisting of diagonal matrices. We regard T as an F-group, and let $T_\infty = T(F \otimes \mathbb{R}) = T(\mathbb{R})^{r_1} \times T(\mathbb{C})^{r_2}$. We let $B = B_n$ stand for the Borel subgroup of G of upper triangular matrices. This defines B_∞. We let $X(T_\infty)$ stand for the group of all algebraic characters of T_∞. We let $X^+(T_\infty)$ stand for the subset of $X(T_\infty)$ consisting of all those characters that are dominant with respect to B_∞. A weight $\mu \in X^+(T_\infty)$ may be described as follows: Let $\mu = (\mu_v)_{v\in S_\infty}$, with $\mu_v \in X(T_v)$. If $v \in S_r$, then $\mu_v = (p_1, \ldots, p_n)$, $p_i \in \mathbb{Z}$, $p_1 \geq p_2 \geq \cdots \geq p_n$, and the character is: if $t = \mathrm{diag}(t_1, \ldots, t_n) \in T(F_v) = T(\mathbb{R})$, then $t \mapsto \prod_i t_i^{p_i}$. If $v \in S_c$, then let $\{\iota_v, \bar{\iota}_v\}$ be the corresponding complex embeddings of F. Identify F_v with $\mathbb{C}$ via ι_v (say). In this case, μ_v is a pair of sequences $(\mu_{\iota_v}, \mu_{\bar{\iota}_v})$, with $\mu_{\iota_v} = (p_1, \ldots, p_n)$, $p_i \in \mathbb{Z}$, $p_1 \geq p_2 \geq \cdots \geq p_n$; likewise $\mu_{\bar{\iota}_v} = (q_1, \ldots, q_n)$ with similar conditions on the q_i's; the character μ_v is: if $t = \mathrm{diag}(z_1, \ldots, z_n) \in T(F_v) = T(\mathbb{C})$, then $t \mapsto \prod_i z_i^{p_i} \bar{z}_i^{q_i}$. (Here $\bar{z}_i$ is the complex conjugate of z_i.) For such a character μ, we define a finite-dimensional representation (ρ_μ, M_μ) of G_∞ as follows. For $v \in S_r$, let $(\rho_{\mu_v}, M_{\mu_v})$ be the irreducible representation of $G(F_v) = G(\mathbb{R})$ with highest weight μ_v. For $v \in S_c$, let $(\rho_{\mu_v}, M_{\mu_v})$ be the representation of $G(F_v) = G(\mathbb{C})$ defined as $\rho_{\mu_v} = \rho_{\mu_{\iota_v}} \otimes \rho_{\mu_{\bar{\iota}_v}}$, where $\rho_{\mu_{\iota_v}}$ is the irreducible representation with highest weight μ_{ι_v}, and similarly $\rho_{\mu_{\bar{\iota}_v}}$. Now we let $\rho_\mu = \otimes_{v\in S_\infty} \rho_{\mu_v}$. Since π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, we have from the proof of [7, Théorème 3.13] that there is a dominant algebraic weight $\mu \in X^+(T_\infty)$ such that $H^*(\mathfrak{g}_\infty, K^0_\infty; \pi_\infty \otimes \rho^\vee_\mu) \neq 0$. In defining the periods, we will be looking at $H^b(\mathfrak{g}_\infty, K^0_\infty; \pi_\infty \otimes \rho^\vee_\mu)$.

The group $K_\infty/K_\infty^0 \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1}$ acts on $H^b(\mathfrak{g}_\infty, K_\infty^0; \pi_\infty \otimes M_\mu^\vee)$. We consider certain isotypic components for this action. Consider an r_1 tuple of signs indexed by the set S_r of real places in S_∞. Let $\epsilon = (\epsilon_v)_{v\in S_r} \in \{1\!\!1, \mathrm{sgn}\}^{r_1} = (K_\infty/K_\infty^0)\widehat{\ }$. If n is even then there are no restrictions on ϵ; however, if n is odd then π uniquely determines an ϵ, in that we let $\epsilon_v = \omega_{\pi_v}|_{\pm 1} \cdot \mathrm{sgn}^{\mathrm{wt}(\mu_v)/2}$, where the weight $\mathrm{wt}(\mu_v)$ of μ_v is defined in [14, (3.1)]. (See also [7, Lemme de pureté 4.9].) If n is odd, then for $v \in S_r$, ϵ_v is simply the central character of $\pi_v \otimes \rho_{\mu_v}^\vee$ restricted to $\{\pm 1\}$, since n being odd means that $K_v/K_v^0 \simeq \{\pm 1\} \subset Z_n(F_v)$; and we call such an ϵ *permissible* for π. Let $H^b(\mathfrak{g}_\infty, K_\infty^0; \pi_\infty \otimes M_\mu^\vee)(\epsilon)$ be the corresponding isotypic component. This isotypic component is one dimensional. This can be seen, by using [14, (3.2)] for the real places, [7, Lemme 3.14] for the (real and) complex places, and the Künneth theorem for Lie algebra cohomology [6, I.1.3] as follows:

$$H^b\big(\mathfrak{g}_\infty, K_\infty^0; \pi_\infty \otimes M_\mu^\vee\big)(\epsilon) = \bigoplus_{\sum a_v = b} \left(\bigotimes_{v\in S_r} H^{a_v}\big(\mathfrak{g}_v, K_v^0; \pi_v \otimes M_{\mu_v}^\vee\big)(\epsilon_v) \bigotimes_{v\in S_c} H^{a_v}\big(\mathfrak{g}_v, K_v^0; \pi_v \otimes M_{\mu_v}^\vee\big)\right)$$
$$= \bigotimes_{v\in S_r} H^{b_n^{\mathbb{R}}}\big(\mathfrak{g}_v, K_v^0; \pi_v \otimes M_{\mu_v}^\vee\big)(\epsilon_v) \bigotimes_{v\in S_c} H^{b_n^{\mathbb{C}}}\big(\mathfrak{g}_v, K_v^0; \pi_v \otimes M_{\mu_v}^\vee\big).$$

In the summation, only one term survives, because for all other summands, at least one of the a_v has to be less than $b_n^{\mathbb{R}}$ or $b_n^{\mathbb{C}}$, and by [7, Lemme 3.14] the corresponding factor vanishes. We fix a generator $\mathbf{w}_\infty = \mathbf{w}(\pi_\infty, \epsilon)$ for this one-dimensional space $H^b(\mathfrak{g}_\infty, K_\infty^0; \pi_\infty \otimes M_\mu^\vee)(\epsilon)$.

We have the following comparison isomorphism of the Whittaker model $W(\pi_f)$ with a global version of the above cohomology space. We let $\mathcal{F}_{\pi_f,\epsilon,\mathbf{w}_\infty}$ denote the composition of the three isomorphisms:

$$\begin{aligned} W(\pi_f) &\longrightarrow W(\pi_f) \otimes H^b\big(\mathfrak{g}_\infty, K_\infty^0; W(\pi_\infty) \otimes M_\mu^\vee\big)(\epsilon) \\ &\longrightarrow H^b\big(\mathfrak{g}_\infty, K_\infty^0; W(\pi) \otimes M_\mu^\vee\big)(\epsilon) \\ &\longrightarrow H^b\big(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee\big)(\epsilon), \end{aligned}$$

where the first map is $w_f \mapsto w_f \otimes \mathbf{w}_\infty$; the second map is the obvious one; and the third map is the map induced in cohomology by the inverse of the map, which gives the Fourier coefficient of a cusp form in V_π–the space of functions in $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$ which realizes π.

We now describe a rational structure on $H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)$, by relating relative Lie algebra cohomology with the cohomology of locally symmetric spaces.

(See [7, pp. 128–29], [14, Section 3.2].) Let K_f be any open compact subgroup of $G(\mathbb{A}_f)$. Consider the manifold

$$S(K_f) = G(F)\backslash G(\mathbb{A})/K_\infty^0 K_f.$$

This is typically a finite disjoint union of manifolds like $\Gamma\backslash G_\infty/K_\infty^0$. For a dominant algebraic weight $\mu \in X^+(T_\infty)$, and $\sigma \in \mathrm{Aut}(\mathbb{C})$, one defines $\mu^\sigma \in X^+(T_\infty)$ via the action of σ on the complex embeddings of F (exactly like the definition of π_∞^σ). We denote the corresponding rationality field by $\mathbb{Q}(\mu)$. Fix a $\mathbb{Q}(\mu)$-structure $M_{\mu,\mathbb{Q}(\mu)}$ on M_μ, which gives a canonical E-structure for any extension $E/\mathbb{Q}(\mu)$ by $M_{\mu,E} = M_{\mu,\mathbb{Q}(\mu)} \otimes_{\mathbb{Q}(\mu)} E$. Let $\mathcal{M}_{\mu,E}$ be the associated locally constant sheaf on $S(K_f)$. For brevity, we also let $M_\mu = M_{\mu,\mathbb{C}}$ and similarly $\mathcal{M}_\mu = \mathcal{M}_{\mu,\mathbb{C}}$.

We consider the direct limit of various cohomology groups

$$H_?^\bullet\big(\tilde{S}, \mathcal{M}_\mu^\vee\big) = \varinjlim H_?^\bullet\big(S(K_f), \mathcal{M}_\mu^\vee\big),$$

where the direct limit is taken over all open compact subgroups K_f of $G(\mathbb{A}_f)$, and $? \in \{B, dR, c, \mathrm{cusp}\}$ meaning singular (Betti) cohomology, or de Rham cohomology, or cohomology with compact supports, or cuspidal cohomology, respectively. Cuspidal cohomology injects into cohomology with compact supports $H_{\mathrm{cusp}}^\bullet(\tilde{S}, \mathcal{M}_\mu^\vee) \hookrightarrow H_c^\bullet(\tilde{S}, \mathcal{M}_\mu^\vee)$ and the latter is canonically a module for $\mathrm{Aut}(\mathbb{C}) \times G(\mathbb{A}_f)$ with commuting actions for the different groups. The image of cuspidal cohomology is defined over $\mathbb{Q}(\mu)$. Also, cuspidal cohomology decomposes into a direct sum over cuspidal cohomological representations, and for a subfield E of $\mathbb{C}$ containing $\mathbb{Q}(\mu)$, an E-structure on any $\mathrm{Aut}(\mathbb{C}/E)$-stable summand is obtained by intersecting that summand with an E-structure of the ambient space. (As a general reference for all these assertions, see Clozel [7, Section 3.5].)

More precisely, by definition of cuspidal cohomology, we have

$$H_{\mathrm{cusp}}^b\big(\tilde{S}, \mathcal{M}_\mu^\vee\big) \simeq H^b\big(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee\big). \tag{3.1}$$

From the decomposition of $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$ into cuspidal automorphic representations, we deduce that the right-hand side decomposes into a direct sum

$$H^b\big(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee\big) \simeq \bigoplus_{\pi\in\mathrm{Coh}(G,\mu^\vee)} H^b\big(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee\big).$$

This also defines the notation $\mathrm{Coh}(G,\mu^\vee)$ as the set consisting of all π having a nonzero contribution on the right-hand side. Now consider the action of K_∞/K_∞^0, and further decompose each summand into its isotypic components. Let $\epsilon \in (K_\infty/K_\infty^0)\widehat{}$ be permissible for π, i.e. if n is even then ϵ can be any character, and if n is odd then ϵ is uniquely determined by π. We let $\tilde{\pi} = \pi_f \otimes \epsilon$, and denote the inverse image of $H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)$ across the isomorphism in (3.1) by $H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)(\tilde{\pi})$. We have

$$H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee) \simeq \bigoplus_{\pi\in\mathrm{Coh}(G,\mu^\vee)} \bigoplus_{\epsilon} H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)(\tilde{\pi}),$$

where in the second summation ϵ is any permissible character for π. (See [14, (3.13)].)

We now have the following description of the rational structures. The Betti cohomology spaces $H^b_B(\tilde{S}, \mathcal{M}_\mu^\vee)$ are naturally defined over $\mathbb{Q}(\mu)$, since the coefficient system admits a $\mathbb{Q}(\mu)$-structure [7, p. 122]. (This will be exploited in the proof of Proposition 4.5.) The $\mathbb{Q}(\mu)$-structure on Betti cohomology induces a $\mathbb{Q}(\mu)$-structure on cohomology with compact support [7, exact triangle on p. 123], which we denote by $H^b_c(\tilde{S}, \mathcal{M}_\mu^\vee)_{\mathbb{Q}(\mu)}$. This in turn induces a $\mathbb{Q}(\mu)$-structure on the full space of cuspidal cohomology ([7, Théorèm 3.19])

$$H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)_{\mathbb{Q}(\mu)} = H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee) \cap H^b_c(\tilde{S}, \mathcal{M}_\mu^\vee)_{\mathbb{Q}(\mu)}. \tag{3.2}$$

We get for each summand of cuspidal cohomology [7, Lemme 3.2.1]

$$H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)(\tilde{\pi})_E = H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)(\tilde{\pi}) \cap H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)_E \tag{3.3}$$

for any extension $E/\mathbb{Q}(\pi)$. We note that $\mathbb{Q}(\mu) = \mathbb{Q}(\pi_\infty) \subset \mathbb{Q}(\pi)$. We can transport the rational structures (3.2) and (3.3) across the identifications with relative Lie algebra cohomology to get rational structures on the latter spaces as:

$$H^b(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee)_{\mathbb{Q}(\mu)} :\simeq H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}_\mu^\vee)_{\mathbb{Q}(\mu)},$$

and for any extension $E/\mathbb{Q}(\pi)$ we have

$$\begin{aligned} H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)_E = {} & H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon) \\ & \cap H^b(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee)_E. \end{aligned}$$

Given $\sigma \in \mathrm{Aut}(\mathbb{C})$ and a permissible character ϵ, one defines ϵ^σ just like the definitions of π_∞^σ or μ^σ; the action being induced by the action of $\mathrm{Aut}(\mathbb{C})$ on the real embeddings on F. Finally, for $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define $\mathbf{w}_\infty^\sigma$ as $\mathbf{w}(\pi_\infty, \epsilon)^\sigma := \mathbf{w}(\pi_\infty^\sigma, \epsilon^\sigma)$. We are now in a position to define the periods attached to π.

3.4 Definition of the periods attached to π

Definition/Proposition 3.3. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $\mu \in X^+(T_\infty)$ be such that $\pi \in \mathrm{Coh}(G, \mu^\vee)$. Let ϵ be a character of K_∞/K_∞^0 that is permissible for π. (If n is even then ϵ is any character, and if n is odd, then π uniquely determines ϵ.) Let $\mathbf{w}_\infty$ be a generator of the one-dimensional vector space $H^b(\mathfrak{g}_\infty, K_\infty^0, \pi_\infty \otimes M_\mu^\vee)(\epsilon)$. To such a datum $(\pi, \epsilon, \mathbf{w}_\infty)$, there is a nonzero complex number $p^\epsilon(\pi_f, \mathbf{w}_\infty)$, such that the normalized map

$$\mathcal{F}^0_{\pi_f,\epsilon,\mathbf{w}_\infty} := p^\epsilon(\pi_f, \mathbf{w}_\infty)^{-1} \mathcal{F}_{\pi_f,\epsilon,\mathbf{w}_\infty}$$

is $\mathrm{Aut}(\mathbb{C})$-equivariant, i.e. the following diagram commutes:

$$\begin{array}{ccc} W(\pi_f) & \xrightarrow{\mathcal{F}^0_{\pi_f,\epsilon,\mathbf{w}_\infty}} & H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon) \\ \downarrow \sigma & & \downarrow \sigma \\ W(\pi_f^\sigma) & \xrightarrow{\mathcal{F}^0_{\pi_f^\sigma,\epsilon^\sigma,\mathbf{w}_\infty^\sigma}} & H^b(\mathfrak{g}_\infty, K_\infty^0; V_{\pi^\sigma} \otimes M_{\mu^\sigma}^\vee)(\epsilon^\sigma) \end{array}$$

The complex number $p^\epsilon(\pi_f, \mathbf{w}_\infty)$, called a period, is well defined only up to multiplication by elements of $\mathbb{Q}(\pi)^*$. □

Proof. We begin by noting that both the spaces $W(\pi_f)$ and $H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)$ are irreducible $G(\mathbb{A}_f)$-modules, and the map $\mathcal{F}_{\pi_f,\epsilon,\mathbf{w}_\infty}$ is a $G(\mathbb{A}_f)$-equivariant isomorphism. Next, we note that the theory of new vectors [12] says that there is an open compact subgroup K_f of $G(\mathbb{A}_f)$ such that the space of K_f-fixed vectors in either of these two spaces is one dimensional. Applying [20, Lemme I.1.1] we get that these representation spaces have $\mathbb{Q}(\pi)$-rational structures that are unique up to homotheties. We have already fixed rational structures on either side in Sections 3.2 and 3.3; this defines the period $p^\epsilon(\pi_f, \mathbf{w}_\infty)$ as that nonzero complex number needed to modify $\mathcal{F}_{\pi_f,\epsilon,\mathbf{w}_\infty}$ so that the normalized map preserves these rational structures. Note that $p^\epsilon(\pi_f, \mathbf{w}_\infty)$ is well defined only up to multiplication by elements of $\mathbb{Q}(\pi)^*$.

Now let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Applying the above paragraph, we have a period $p^{\epsilon^\sigma}(\pi_f^\sigma, \mathbf{w}_\infty^\sigma)$ well defined up to $\mathbb{Q}(\pi^\sigma)^*$. Now consider the diagram in the statement of the definition/proposition, and for brevity, let $\mathcal{F}_1 := \mathcal{F}^0_{\pi_f^\sigma, \epsilon^\sigma, \mathbf{w}_\infty^\sigma} \circ \sigma$, and $\mathcal{F}_2 := \sigma \circ \mathcal{F}^0_{\pi_f, \epsilon, \mathbf{w}_\infty}$. Then both $\mathcal{F}_1$ and $\mathcal{F}_2$ are $G(\mathbb{A}_f)$-equivariant σ-linear isomorphisms that preserve the above-mentioned choice of rational structures. Let w^0 be the global new vector as in the proof of Lemma 3.2. Then $\mathcal{F}_2(w^0)$ and $\mathcal{F}_1(w^0)$ are both new vectors in the same $\mathbb{Q}(\pi^\sigma)$-rational structure on $H^b(\mathfrak{g}_\infty, K_\infty^0; V_{\pi^\sigma} \otimes M^\vee_{\mu^\sigma})(\epsilon^\sigma)$, and hence differ by an element of $\mathbb{Q}(\pi^\sigma)^*$. We can adjust $p^{\epsilon^\sigma}(\pi_f^\sigma, \mathbf{w}_\infty^\sigma)$ by this element and assume that $\mathcal{F}_2(w^0) = \mathcal{F}_1(w^0)$. Irreducibility implies then that $\mathcal{F}_2 = \mathcal{F}_1$, i.e. the diagram commutes. Note that once we have chosen $p^\epsilon(\pi_f, \mathbf{w}_\infty)$, requiring the commutativity of the above diagram actually pins down $p^{\epsilon^\sigma}(\pi_f^\sigma, \mathbf{w}_\infty^\sigma)$, and further, if we change $p^\epsilon(\pi_f, \mathbf{w}_\infty)$ to $t p^\epsilon(\pi_f, \mathbf{w}_\infty)$ with a $t \in \mathbb{Q}(\pi)^*$ then the period $p^{\epsilon^\sigma}(\pi_f^\sigma, \mathbf{w}_\infty^\sigma)$ changes to $\sigma(t) p^{\epsilon^\sigma}(\pi_f^\sigma, \mathbf{w}_\infty^\sigma)$. ■

It is helpful to simplify our notation a bit. We begin by fixing generators for all the possible one-dimensional relative Lie algebra cohomology spaces for the groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. Having done so, we have therefore fixed generators for the cohomology spaces for the group G_∞. We also ask that this choice be compatible with twisting π_∞ by algebraic characters ξ_∞ of G_∞; this condition, although crucial in the proof of Proposition 4.6, is not a serious constraint. (For example, for $G = \mathrm{GL}_2$, the choice of a generator for H^1 as in Waldspurger [20, p. 130–31] is compatible with twisting.) Further, this choice is compatible with the action of $\mathrm{Aut}(\mathbb{C})$ on automorphic representations at infinity. Henceforth, we abbreviate $\mathcal{F}_{\pi_f, \epsilon, \mathbf{w}_\infty}$ by $\mathcal{F}_{\pi_f, \epsilon}$, similarly for the normalized maps, as well as $p^\epsilon(\pi_f, \mathbf{w}_\infty)$ by $p^\epsilon(\pi_f)$, while keeping in mind that $\mathbf{w}_\infty$ has been chosen already. (For example, in the classical setting of modular forms, this is equivalent to fixing a generator $\mathbf{w}_\infty$ for H^1 of the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight k, and now for all weight k modular forms, we work with this choice of $\mathbf{w}_\infty$.)

In terms of the un-normalized maps, we can describe the above commutative diagram by

$$\sigma \circ \mathcal{F}_{\pi_f, \epsilon} = \left(\frac{\sigma(p^\epsilon(\pi_f))}{p^{\epsilon^\sigma}(\pi_f^\sigma)} \right) \mathcal{F}_{\pi_f^\sigma, \epsilon^\sigma} \circ \sigma. \tag{3.4}$$

4 Behavior under Twisting

Before we state and prove the main theorem, we need some preliminaries on Hecke characters. By a Hecke character ξ of F, we mean a continuous homomorphism $\xi : F^* \backslash \mathbb{I}_F \to \mathbb{C}^*$. By an algebraic Hecke character, we mean a Hecke character ξ whose component at infinity, denoted by ξ_∞, is algebraic in the sense of [7, Section 1.2.3]; these are the Grössencharakters of type A_0 of Weil. We let $\mathbb{Q}(\xi)$ denote the rationality field of ξ. Note that ξ being algebraic implies that $\mathbb{Q}(\xi)$ is a number field.

The *Gauss sum* of ξ is defined as follows: We let $\mathfrak{c}$ stand for the conductor ideal of ξ_f, and let $\mathcal{D}_F$ be the absolute different of F. We fix, once and for all, an additive character ψ of $F\backslash\mathbb{A}_F$, as in Tate's thesis, namely, $\psi(x) = e^{2\pi i\Lambda(x)}$ with the Λ as defined in [19, Section 4.1]. The conductor ideal of ψ is $\mathcal{D}_F^{-1}$. Let $y = (y_v)_{v\notin S_\infty} \in \mathbb{A}_f^\times$ be such that $\mathrm{ord}_v(y_v) = -\mathrm{ord}_v(\mathfrak{c}) - \mathrm{ord}_v(\mathcal{D}_F)$. The Gauss sum of ξ is defined as

$$\gamma(\xi_f, \psi_f, y) = \prod_{v\notin S_\infty} \gamma(\xi_v, \psi_v, y_v),$$

where the local Gauss sum $\gamma(\xi_v, \psi_v, y_v)$ is defined as

$$\gamma(\xi_v, \psi_v, y_v) = \int_{\mathcal{O}_{F_v}^\times} \xi_v(u_v)\psi_v(y_v u_v)\, du_v.$$

For almost all v, where everything in sight is unramified, we have $\gamma(\xi_v, \psi_v, y_v) = 1$, and for all v we have $\gamma(\xi_v, \psi_v, y_v) \neq 0$. (See, for example, Godement [9, Equation 1.22].) Given a ξ, we will arbitrarily pick an element y as above to define $\gamma(\xi_f, \psi_f, y)$; having chosen y for ξ, we will work with the same y for every character of the form ξ^σ, where $\sigma \in \mathrm{Aut}(\mathbb{C})$. This choice will not affect us in any serious way, because we will really be concerned with certain quotients involving Gauss sums, and such quotients will not depend on y. (See Lemma 4.3 below.) In the notation we will therefore suppress the dependence on y; and as with Whittaker models, we will also suppress the dependence on the additive character ψ, and denote the Gauss sum $\gamma(\xi_f, \psi_f, y)$ simply as $\gamma(\xi_f)$.

Given an algebraic Hecke character ξ, we define its signature $\epsilon_\xi = (\epsilon_{\xi,v})_{v\in S_r} \in \{\pm 1\}^{r_1}$ as follows. For $v \in S_r$, if $\xi_v(t) = \mathrm{sgn}(t)^{\eta_v}|t|^{m_v}$ for all $t \in \mathbb{R}^*$, where $\eta_v \in \{0, 1\}$ and $m_v \in \mathbb{Z}$ (since ξ is algebraic), then define $\epsilon_{\xi,v} = (-1)^{\eta_v + m_v}$. We will think of ϵ_ξ as a character of K_∞/K_∞^0. We can now state and prove the main result of this paper. We note that stating period relations, or results about special values of L-functions, in an $\mathrm{Aut}(\mathbb{C})$-equivariant manner is originally due to Shimura. (See, for example, [18].)

4.1 The main theorem

Theorem 4.1. Let F be a number field and π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let μ be a dominant algebraic character of T_∞ such that

$\pi \in \mathrm{Coh}(G, \mu^\vee)$. Let ϵ be a character of K_∞/K_∞^0 as in Section 3, and let $p^\epsilon(\pi_f)$ be the period as in Definition 3.3. Let ξ be an algebraic Hecke character of F with signature ϵ_ξ. Then $\pi \otimes \xi$ is also a regular algebraic cuspidal automorphic representation, and the signature $\epsilon \cdot \epsilon_\xi := \epsilon \otimes \epsilon_\xi$ is a character of K_∞/K_∞^0 that is permissible for $\pi \otimes \xi$ (which is an issue only when n is odd), hence the period $p^{\epsilon\cdot\epsilon_\xi}(\pi_f \otimes \xi_f)$ is defined. We have the following relations:

(1) For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

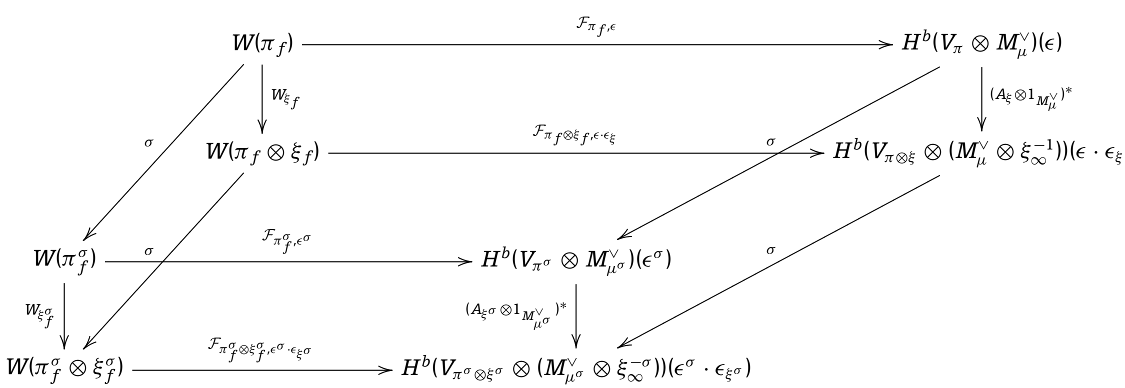
$$\sigma\left(\frac{p^{\epsilon\cdot\epsilon_\xi}(\pi_f \otimes \xi_f)}{\gamma(\xi_f)^{-n(n-1)/2}\, p^\epsilon(\pi_f)}\right) = \left(\frac{p^{\epsilon^\sigma\cdot\epsilon_{\xi^\sigma}}(\pi_f^\sigma \otimes \xi_f^\sigma)}{\gamma(\xi_f^\sigma)^{-n(n-1)/2}\, p^{\epsilon^\sigma}(\pi_f^\sigma)}\right).$$

(2) Let $\mathbb{Q}(\pi, \xi)$ be the compositum of the number fields $\mathbb{Q}(\pi)$ and $\mathbb{Q}(\xi)$. We have

$$p^{\epsilon\cdot\epsilon_\xi}(\pi_f \otimes \xi_f) \sim_{\mathbb{Q}(\pi,\xi)} \gamma(\xi_f)^{-n(n-1)/2}\, p^\epsilon(\pi_f).$$

By $\sim_{\mathbb{Q}(\pi,\xi)}$ we mean up to an element of $\mathbb{Q}(\pi, \xi)$. □

Proof. Note that (1) ⇒ (2) follows from the definitions of the rationality field of π and ξ. It is really statement (1) that takes some work to prove; this entails an analysis of the following diagram of maps. *Warning:* This diagram is not commutative! Indeed, the various complex numbers involved in (1) measure the failure of commutativity of this diagram.



(4.1)

We need to explain the undefined and abbreviated notations in the above diagram. We have abbreviated $H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)$ as $H^b(V_\pi \otimes M_\mu^\vee)(\epsilon)$. Same remark applies to three other objects. The maps W_ξ and A_ξ are defined as follows. If w is any Whittaker function for $\mathrm{GL}_n(\mathbb{A})$, then define

$$W_\xi(w)(g) = \xi(\det(g))w(g)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$. It is easy to see that W_ξ maps $W(\pi)$ onto $W(\pi \otimes \xi)$. An identical formula defines W_{ξ_f} and W_{ξ_∞}. Similarly, we define $A_\xi(\phi)$ for any automorphic form ϕ on $\mathrm{GL}_n(\mathbb{A})$ by

$$A_\xi(\phi)(g) = \xi(\det(g))\phi(g)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$. It is easy to see that A_ξ maps V_π onto $V_{\pi\otimes\xi}$. The identity map on the vector space $M_\mu^\vee$ is denoted $1_{M_\mu^\vee}$. Observe that $A_\xi \otimes 1_{M_\mu^\vee}$ is a G_∞-equivariant isomorphism from $V_\pi \otimes M_\mu^\vee$ onto $V_{\pi\otimes\xi} \otimes (M_\mu^\vee \otimes \xi_\infty^{-1})$, and we denote $(A_\xi \otimes 1_{M_\mu^\vee})^*$ the map induced by $A_\xi \otimes 1_{M_\mu^\vee}$ in cohomology.

Analyzing the diagram involves an analysis of certain subdiagrams. Some of these are independently interesting, and so we delineate them in the following propositions. The three propositions below give information about (non-)commutativity of some of the faces of (4.1).

Proposition 4.2. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, and ξ be an algebraic Hecke character of F. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma \circ W_{\xi_f} = \sigma(\xi_f(t_\sigma^{-n(n-1)/2}))\, W_{\xi_f^\sigma} \circ \sigma.$$

□

Proof. Consider the diagram

$$\begin{array}{ccc} W(\pi_f) & \xrightarrow{\sigma} & W(\pi_f^\sigma) \\ \Big\downarrow{\scriptstyle W_{\xi_f}} & & \Big\downarrow{\scriptstyle W_{\xi_f^\sigma}} \\ W(\pi_f \otimes \xi_f) & \xrightarrow{\sigma} & W(\pi_f^\sigma \otimes \xi_f^\sigma) \end{array}$$

We chase an element $w \in W(\pi_f)$ in the above diagram. For $g \in G(\mathbb{A}_f)$, we have $\sigma(W_{\xi_f}(w))(g) = \sigma(\xi_f(t_\sigma^{-n(n-1)/2}))\sigma(\xi_f(\det(g)))\sigma(w(t_{\sigma,n}g))$. On the other hand, we have $W_{\xi_f^\sigma}(\sigma(w))(g) = \sigma(\xi_f(\det(g)))\sigma(w(t_{\sigma,n}g))$. ■

Lemma 4.3.

$$\sigma\big(\xi_f\big(t_\sigma^{-1}\big)\big) = \frac{\sigma(\gamma(\xi_f))}{\gamma(\xi_f^\sigma)}.$$

□

Proof. This is an easy exercise! ■

We remark that Gauss sums are themselves period-like quantities, and the lemma says that these quantities satisfy a reciprocity law. See also the first paragraph of Section 4.2.

Corollary 4.4.

$$\sigma \circ W_{\xi_f} = \left(\frac{\sigma(\gamma(\xi_f))}{\gamma(\xi_f^\sigma)} \right)^{n(n-1)/2} W_{\xi_f^\sigma} \circ \sigma.$$

□

Proof. Follows from Proposition 4.2 and Lemma 4.3. ■

Proposition 4.5. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, and let $\mu \in X^+(T_\infty)$ be such that $\pi \in \mathrm{Coh}(G, \mu^\vee)$. For any algebraic Hecke character ξ we have

$$\sigma \circ \left(A_\xi \otimes 1_{M_\mu^\vee}\right)^* = \left(A_{\xi^\sigma} \otimes 1_{M_{\mu^\sigma}^\vee}\right)^* \circ \sigma.$$

□

Proof. This proof is a little more involved, and to help the reader, we adumbrate it as follows. First go up to a bigger ambient vector space $(H^*_{dR}(\tilde{S}, \mathcal{M}))$ and then use an identification of this bigger space with another space $(H^*_B(\tilde{S}, \mathcal{M}))$ where it will be obvious that a lift of A^*_ξ is Galois equivariant, and hence so is the original A^*_ξ. During the course of the proof, it helps to keep the following scheme of spaces and maps in mind:

$$\begin{array}{ccccc}
H^b(\mathfrak{g}_\infty, K^0_\infty; V_\pi \otimes M^\vee_\mu)(\epsilon) & \simeq & H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}^\vee_\mu)(\tilde{\pi}) & & \\
\downarrow & & \downarrow & & \\
H^b(\mathfrak{g}_\infty, K^0_\infty; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M^\vee_\mu) & \simeq & H^b_{\mathrm{cusp}}(\tilde{S}, \mathcal{M}^\vee_\mu) & & \\
\downarrow & & \downarrow & & \\
H^b(\mathfrak{g}_\infty, K^0_\infty; C^\infty(G(F)\backslash G(\mathbb{A})) \otimes M^\vee_\mu) & \simeq & H^b_{dR}(\tilde{S}, \mathcal{M}^\vee_\mu) & \simeq & H^b_B(\tilde{S}, \mathcal{M}^\vee_\mu)
\end{array}$$

where all the vertical arrows are injections induced by inclusions. Indeed, the rational structures on all the above spaces come from a rational structure on the Betti cohomology space on which it is very easy to describe an action of $\mathrm{Aut}(\mathbb{C})$ (see [7, p. 128]). In the above scheme, we need not (and so did not) refer to cohomology with compact supports because one has [7, p. 129]

$$H^*_{\mathrm{cusp}} \hookrightarrow H^*_c \to H^*_! := \mathrm{Image}\left(H^*_c \to H^*_{\mathrm{dR}}\right)$$

and the composite is an injection, i.e. $H^*_{\mathrm{cusp}} \hookrightarrow H^*_!$, and hence $H^*_{\mathrm{cusp}} \hookrightarrow H^*_{\mathrm{dR}}$.

To begin the proof of Proposition 4.5, observe that the map $(A_\xi \otimes 1_{M_\mu^\vee})^*$ in the statement of the proposition is the restriction to $H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon)$ of the map

$$\left({}_{\mathcal{A}}A_\xi \otimes 1_{M_\mu^\vee}\right)^* : H^b\left(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee\right) \\ \to H^b\left(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes \left(M_\mu^\vee \otimes \xi_\infty^{-1}\right)\right)$$

induced by A_ξ on $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$. From a fundamental theorem of Borel [4], cohomology with coefficients in the space of cusp forms injects into cohomology with coefficients in the space of smooth functions, and the above map $({}_{\mathcal{A}}A_\xi \otimes 1_{M_\mu^\vee})^*$ is the restriction to $H^b(\mathfrak{g}_\infty, K_\infty^0; \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee)$ of the map

$$\left({}_{C^\infty}A_\xi \otimes 1_{M_\mu^\vee}\right)^* : H^b\left(\mathfrak{g}_\infty, K_\infty^0; C^\infty(G(F)\backslash G(\mathbb{A})) \otimes M_\mu^\vee\right) \\ \to H^b\left(\mathfrak{g}_\infty, K_\infty^0; C^\infty(G(F)\backslash G(\mathbb{A})) \otimes \left(M_\mu^\vee \otimes \xi_\infty^{-1}\right)\right)$$

induced by A_ξ on $C^\infty(G(F)\backslash G(\mathbb{A}))$.

On the other hand, relative Lie algebra cohomology can be identified with de Rham cohomology, and we can transport the map $({}_{C^\infty}A_\xi \otimes 1_{M_\mu^\vee})^*$ across to get

$${}_{dR}A_\xi^* : H_{dR}^b\left(\tilde{S}, \mathcal{M}_\mu^\vee\right) \to H_{dR}^b\left(\tilde{S}, \mathcal{M}_\mu^\vee \otimes \xi_\infty^{-1}\right).$$

(By $\mathcal{M}_\mu^\vee \otimes \xi_\infty^{-1}$ we mean the locally constant sheaf induced by the representation $M_\mu^\vee \otimes \xi_\infty^{-1}$.) We can describe the map ${}_{dR}A_\xi^*$ as follows. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ such that $\xi(\det(k)) = 1$ for all $k \in K_f$. Recall the identification ([13, Section 1.1])

$$H_{dR}^b\left(S(K_f), \mathcal{M}_\mu^\vee\right) \simeq H^b\left(\mathfrak{g}_\infty, K_\infty^0; C^\infty(G(F)\backslash G(\mathbb{A}))^{K_f} \otimes M_\mu^\vee\right).$$

The choice of K_f implies that A_ξ stabilizes $C^\infty(G(F)\backslash G(\mathbb{A}))^{K_f}$ and so induces a map $({}_{C^\infty}A_{\xi,K_f} \otimes 1_{M_\mu^\vee})^*$ on the right-hand side above. Clearly, $\varinjlim_{K_f} ({}_{C^\infty}A_{\xi,K_f} \otimes 1_{M_\mu^\vee})^* = ({}_{C^\infty}A_\xi \otimes 1_{M_\mu^\vee})^*$. Moving across to de Rham cohomology, we can describe the map ${}_{dR}A_{\xi,K_f}^*$ as acting on differential forms by *pointwise multiplication by* ξ, i.e. if ω is a $\mathcal{M}_\mu^\vee$-valued (closed) differential form of degree b on $S(K_f)$ then

$${}_{dR}A_{\xi,K_f}^*(\omega)_{\underline{g}} = \xi(\det(g))\omega_{\underline{g}}$$

for any $g \in \mathrm{GL}_n(\mathbb{A})$, where $\underline{g}$ is the image of g in $S(K_f)$. (For any $x \in S(K_f)$, by ω_x we mean the value at x of ω, which is a section of the bth exterior of the cotangent bundle twisted by $\mathcal{M}_\mu^\vee$ over the manifold $S(K_f)$.) Observe that the above equation is well defined. Passing to the limit, we get the map ${}_{dR}A_\xi^* = \varinjlim_{K_f} {}_{dR}A_{\xi,K_f}^*$.

Now we move across to Betti cohomology via the de Rham isomorphism, and get a map,

$$ {}_B A_\xi^* : H_B^b(\tilde{S}, \mathcal{M}_\mu^\vee) \to H_B^b(\tilde{S}, \mathcal{M}_\mu^\vee \otimes \xi_\infty^{-1}). $$

The point of going to Betti cohomology is because the action of $\mathrm{Aut}(\mathbb{C})$ is especially simple to describe—it acts by acting on the coefficient system. (See [7, page 128].) Namely, if $\sigma \in \mathrm{Aut}(\mathbb{C})$ then we have a σ linear isomorphism

$$ H_B^*(S(K_f), \mathcal{M}_\mu^\vee) \to H_B^*(S(K_f), \mathcal{M}_{\mu^\sigma}^\vee). $$

This isomorphism is the one induced in cohomology by the following map on the singular cochain complex

$$ \Delta^*(S(K_f), \mathcal{M}_\mu^\vee) \to \Delta^*(S(K_f), \mathcal{M}_{\mu^\sigma}^\vee), $$

which is simply $\tau \mapsto l_\sigma \circ \tau$, if l_σ is the σ-linear isomorphism from $M_\mu^\vee$ to $M_{\mu^\sigma}^\vee$. (Recall that $M_\mu^\vee$ is defined over $\mathbb{Q}(\mu)$, and the definition of μ^σ is via the embeddings of F into $\mathbb{C}$.) The action of $\mathrm{Aut}(\mathbb{C})$ on $H_B^*(S(K_f), \mathcal{M}_\mu^\vee)$ can be transported to an action on $H_{dR}^*(S(K_f), \mathcal{M}_\mu^\vee)$, and after passing to the limit, induces an action of $\mathrm{Aut}(\mathbb{C})$ on each of the spaces

$$ H^*(\mathfrak{g}_\infty, K_\infty^0, V_\pi \otimes M_\mu^\vee)(\epsilon) \subset H_{\mathrm{cusp}}^*(\tilde{S}, \mathcal{M}_\mu^\vee) \subset H_{dR}^*(\tilde{S}, \mathcal{M}_\mu^\vee). $$

The statement in the proposition may be phrased as that the map $(A_\xi \otimes 1_{M_\mu^\vee})^*$ is $\mathrm{Aut}(\mathbb{C})$-equivariant. From the above description of the action of $\mathrm{Aut}(\mathbb{C})$, we can see that the $(A_\xi \otimes 1_{M_\mu^\vee})^*$ is $\mathrm{Aut}(\mathbb{C})$-equivariant if and only if any of the maps $({}_{\mathcal{A}}A_\xi \otimes 1_{M_\mu^\vee})^*$, $({}_{C^\infty}A_\xi \otimes 1_{M_\mu^\vee})^*$, ${}_{dR}A_\xi^*$, or ${}_B A_\xi^*$ is $\mathrm{Aut}(\mathbb{C})$-equivariant.

It is easy to see that the map ${}_{dR}A_\xi^*$ is $\mathrm{Aut}(\mathbb{C})$-equivariant, since $\mathrm{Aut}(\mathbb{C})$ acts on the coefficient system, and ${}_{dR}A_\xi^*$ is intrinsic to the manifold. More precisely, consider the de Rham map $\Omega^*(S(K_f), \mathcal{M}_\mu^\vee) \to \Delta^*(S(K_f), \mathcal{M}_\mu^\vee)$ from the space of differential forms to the space of singular cochains, given by integration. (This induces the de Rham isomorphism in cohomology.) We can describe the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on $\omega \in \Omega^b(S(K_f), \mathcal{M}_\mu^\vee)$, by

$\sigma(\omega)_x = l_\sigma \circ \omega_x$ for $x \in S(K_f)$. For any $g \in \mathrm{GL}_n(\mathbb{A})$, if $\underline{g}$ denotes the image of g in $S(K_f)$, we have

$$\begin{aligned}\sigma({}_{dR}A_{\xi,K_f}(\omega))_{\underline{g}} &= l_\sigma \circ {}_{dR}A_{\xi,K_f}(\omega)_{\underline{g}} = l_\sigma \circ \xi(\det(g))\omega_{\underline{g}} \\ &= \sigma(\xi(\det(g)))l_\sigma \circ \omega_{\underline{g}} = {}_{dR}A_{\xi^\sigma,K_f}(\sigma(\omega))_{\underline{g}}.\end{aligned}$$

In other words, $\sigma \circ {}_{dR}A_{\xi,K_f} = {}_{dR}A_{\xi^\sigma,K_f} \circ \sigma$. Passing to the limit over all K_f, we get $\sigma \circ {}_{dR}A_\xi = {}_{dR}A_{\xi^\sigma} \circ \sigma$, which induces the desired equality of maps in cohomology. ■

Proposition 4.6. The diagram

$$\begin{array}{ccc} W(\pi_f) & \xrightarrow{\mathcal{F}_{\pi_f,\epsilon}} & H^b(\mathfrak{g}_\infty, K_\infty^0; V_\pi \otimes M_\mu^\vee)(\epsilon) \\ \downarrow{\scriptstyle W_{\xi_f}} & & \downarrow{\scriptstyle (A_\xi \otimes 1_{M_\mu^\vee})^*} \\ W(\pi_f \otimes \xi_f) & \xrightarrow{\mathcal{F}_{\pi_f\otimes\xi_f,\epsilon\cdot\epsilon_\xi}} & H^b(\mathfrak{g}_\infty, K_\infty^0; V_{\pi\otimes\xi} \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) \end{array}$$

commutes. (The horizontal maps are the un-normalized maps.) □

Proof. That this diagram commutes can be seen by observing that the following three diagrams commute, since the horizontal maps are both the composition of three maps. (For brevity we denote $H^b(-)$ for $H^b(\mathfrak{g}_\infty, K_\infty^0; -)$.)

$$\begin{array}{ccc} W(\pi_f) & \longrightarrow & W(\pi_f) \otimes H^b(W(\pi_\infty) \otimes M_\mu^\vee)(\epsilon) \\ \downarrow{\scriptstyle W_{\xi_f}} & & \downarrow{\scriptstyle W_{\xi_f}\otimes(W_{\xi_\infty}\otimes 1_{M_\mu^\vee})^*} \\ W(\pi_f \otimes \xi_f) & \longrightarrow & W(\pi_f \otimes \xi_f) \otimes H^b(W(\pi_\infty \otimes \xi_\infty) \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) \end{array} \tag{4.2}$$

$$\begin{array}{ccc} W(\pi_f) \otimes H^b(W(\pi_\infty) \otimes M_\mu^\vee)(\epsilon) & \longrightarrow & H^b(W(\pi) \otimes M_\mu^\vee)(\epsilon) \\ \downarrow{\scriptstyle W_{\xi_f}\otimes(W_{\xi_\infty}\otimes 1_{M_\mu^\vee})^*} & & \downarrow{\scriptstyle (W_\xi\otimes 1_{M_\mu^\vee})^*} \\ W(\pi_f \otimes \xi_f) \otimes H^b(W(\pi_\infty \otimes \xi_\infty) \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) & \longrightarrow & H^b(W(\pi\otimes\xi) \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) \end{array} \tag{4.3}$$

$$\begin{array}{ccc} H^b(W(\pi) \otimes M_\mu^\vee)(\epsilon) & \longrightarrow & H^b(V_\pi \otimes M_\mu^\vee)(\epsilon) \\ \downarrow{\scriptstyle (W_\xi\otimes 1_{M_\mu^\vee})^*} & & \downarrow{\scriptstyle (A_\xi\otimes 1_{M_\mu^\vee})^*} \\ H^b(W(\pi\otimes\xi) \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) & \longrightarrow & H^b(V_{\pi\otimes\xi} \otimes (M_\mu^\vee \otimes \xi_\infty^{-1}))(\epsilon\cdot\epsilon_\xi) \end{array} \tag{4.4}$$

For the commutativity of (4.2), note that the linear map W_{ξ_∞} induces a G_∞-equivariant isomorphism $W_{\xi_\infty} \otimes 1_{M_\mu^\vee} : W(\pi_\infty) \otimes M_\mu^\vee \to W(\pi_\infty \otimes \xi_\infty) \otimes (M_\mu^\vee \otimes \xi_\infty^{-1})$, and hence induces an isomorphism $(W_{\xi_\infty} \otimes 1_{M_\mu^\vee})^*$ in cohomology. From the choice we made on the generators of such one-dimensional cohomology spaces, we have $(W_{\xi_\infty} \otimes 1_{M_\mu^\vee})^*(\mathbf{w}(\pi_\infty, \epsilon)) = \mathbf{w}(\pi_\infty \otimes \xi_\infty, \epsilon \cdot \epsilon_\xi)$. Now it is easy to check that (4.2) commutes. The commutativity of the diagrams in (4.3) and (4.4) are easy to see. ■

We can now finish the proof of Theorem 4.1 as follows. We consider the composite map $(A_{\xi^\sigma} \otimes 1_{M_{\mu^\sigma}^\vee})^* \circ \sigma \circ \mathcal{F}_{\pi_f,\epsilon}$ in the diagram (4.1). On the one hand, using Equation (3.4) and Proposition 4.6, we have

$$\begin{aligned}\left(A_{\xi^\sigma} \otimes 1_{M_{\mu^\sigma}^\vee}\right)^* \circ \sigma \circ \mathcal{F}_{\pi_f,\epsilon} &= \left(A_{\xi^\sigma} \otimes 1_{M_{\mu^\sigma}^\vee}\right)^* \circ \left(\frac{\sigma(p^\epsilon(\pi_f)}{p^{\epsilon^\sigma}(\pi_f^\sigma)}\right) \mathcal{F}_{\pi_f^\sigma,\epsilon^\sigma} \circ \sigma \\ &= \left(\frac{\sigma(p^\epsilon(\pi_f))}{p^{\epsilon^\sigma}(\pi_f^\sigma)}\right) \mathcal{F}_{\pi_f^\sigma \otimes \xi_f^\sigma, \epsilon^\sigma \cdot \epsilon_{\xi^\sigma}} \circ W_{\xi_f^\sigma} \circ \sigma.\end{aligned}$$

On the other hand, using Propositions 4.5 and 4.6, Equation (3.4) and Corollary 4.4 (in that order), we have

$$\begin{aligned}\left(A_{\xi^\sigma} \otimes 1_{M_{\mu^\sigma}^\vee}\right)^* \circ \sigma \circ \mathcal{F}_{\pi_f,\epsilon} &= \sigma \circ \left(A_\xi \otimes 1_{M_\mu^\vee}\right)^* \circ \mathcal{F}_{\pi_f,\epsilon} \\ &= \sigma \circ \mathcal{F}_{\pi_f \otimes \xi_f, \epsilon \cdot \epsilon_\xi} \circ W_{\xi_f} \\ &= \left(\frac{\sigma(p^{\epsilon \cdot \epsilon_\xi}(\pi_f \otimes \xi_f))}{p^{\epsilon^\sigma \cdot \epsilon_{\xi^\sigma}}(\pi_f^\sigma \otimes \xi_f^\sigma)}\right) \mathcal{F}_{\pi_f^\sigma \otimes \xi_f^\sigma, \epsilon^\sigma \cdot \epsilon_{\xi^\sigma}} \circ \sigma \circ W_{\xi_f} \\ &= \left(\frac{\sigma(p^{\epsilon \cdot \epsilon_\xi}(\pi_f \otimes \xi_f))}{p^{\epsilon^\sigma \cdot \epsilon_{\xi^\sigma}}(\pi_f^\sigma \otimes \xi_f^\sigma)}\right) \left(\frac{\sigma(\gamma(\xi_f))}{\gamma(\xi_f^\sigma)}\right)^{n(n-1)/2} \mathcal{F}_{\pi_f^\sigma \otimes \xi_f^\sigma, \epsilon^\sigma \cdot \epsilon_{\xi^\sigma}} \circ W_{\xi_f^\sigma} \circ \sigma.\end{aligned}$$

Putting both together, we have

$$\frac{\sigma(p^\epsilon(\pi_f))}{p^{\epsilon^\sigma}(\pi_f^\sigma)} = \left(\frac{\sigma(p^{\epsilon \cdot \epsilon_\xi}(\pi_f \otimes \xi_f))}{p^{\epsilon^\sigma \cdot \epsilon_{\xi^\sigma}}(\pi_f^\sigma \otimes \xi_f^\sigma)}\right) \left(\frac{\sigma(\gamma(\xi_f))}{\gamma(\xi_f^\sigma)}\right)^{n(n-1)/2}$$

from which the theorem follows. ■

4.2 **Some variations**

As a first variation of the main theorem, we note that the theorem may be rephrased as that a certain quotient of periods satisfies a reciprocity law. This kind of a reciprocity law, over a CM field, is originally due to Blasius. (See [1, Theorems 2.5.1 and 6.2.1] and

[2, Introduction].) With the notation as in Theorem 4.1, we can state the period relation as

$$\sigma\left(\frac{p^{\epsilon\cdot\epsilon_\xi}(\pi_f\otimes\xi_f)}{p^{\epsilon}(\pi_f)}\right)=\sigma(\xi_f(t_\sigma))^{n(n-1)/2}\left(\frac{p^{\epsilon^\sigma\cdot\epsilon_{\xi^\sigma}}\left(\pi_f^\sigma\otimes\xi_f^\sigma\right)}{p^{\epsilon^\sigma}\left(\pi_f^\sigma\right)}\right). \tag{4.5}$$

The quantity $\sigma(\xi_f(t_\sigma))$ is canonically attached to ξ and σ. In contrast, the Gauss sum $\gamma(\xi_f)$, or its conjugate $\sigma(\gamma(\xi_f))$, is not canonical—it depends on a somewhat artificial choice of a finite idèle and an additive character.

A second variation of the main theorem is to look at cohomology in other degrees. We thank Haruzo Hida for pointing out that in Definition 3.3, if instead of looking at bottom degree of cuspidal cohomology, we considered, say, the top degree cohomology, then it is likely that one would get Beilinson-type periods, which are (conjecturally) related to the *noncritical* values of certain L-functions. For example, in [11], Hida proves an algebraicity result for noncritical values of quadratic twists of adjoint L-functions for SL(2) in terms of periods obtained by considering top degree cohomology. Toward this second variation, we define $t_n^{\mathbb{R}}$ as $((n+1)^2-1)/4-1$ if n is even, and $(n+1)^2/4-1$ if n is odd. Define $t_n^{\mathbb{C}}$ as $(n-1)(n+2)/2$. Now let $t=r_1t_n^{\mathbb{R}}+r_2t_n^{\mathbb{C}}$. Then one can check, using [7, Lemme 3.14], that the vector space $H^t(\mathfrak{g}_\infty, K_\infty^0;\pi_\infty\otimes M_\mu^\vee)(\epsilon)$ is of dimension one. In Definition 3.3, one can consider cohomology in degree t, and get certain periods, which we denote as $q^\epsilon(\pi_f)$. Then Theorem 4.1 is true with the periods $p^\epsilon(\pi_f)$ replaced by $q^\epsilon(\pi_f)$, i.e. we have

$$\sigma\left(\frac{q^{\epsilon\cdot\epsilon_\xi}(\pi_f\otimes\xi_f)}{\gamma(\xi_f)^{-n(n-1)/2}\,q^{\epsilon}(\pi_f)}\right)=\left(\frac{q^{\epsilon^\sigma\cdot\epsilon_{\xi^\sigma}}\left(\pi_f^\sigma\otimes\xi_f^\sigma\right)}{\gamma\left(\xi_f^\sigma\right)^{-n(n-1)/2}\,q^{\epsilon^\sigma}\left(\pi_f^\sigma\right)}\right). \tag{4.6}$$

The proof is absolutely identical to the proof of Theorem 4.1, because the only place where the degree of the cohomology group matters is in the definition of the comparison isomorphism $\mathcal{F}_{\pi_f,\epsilon}$. It is a very interesting question, even for $F=\mathbb{Q}$, to find the relation between the periods $q^\epsilon(\pi_f)$ with L-functions attached to π_f.

Acknowledgments

We are grateful to Don Blasius, Laurent Clozel, Paul Garrett, Haruzo Hida, Joachim Mahnkopf, Dinakar Ramakrishnan, and David Vogan for helpful discussions. It is a pleasure to thank the referee for helping us formulate statements in just the right way! The first author thanks the warm hospitality of Purdue University. Both authors would like to thank Steve Kudla, Michael Rapoport, and Joachim Schwermer for the invitation to spend some time in the stimulating atmosphere of the

Erwin Schrödinger Institute in Vienna, where the work took its final form. This work is partially supported by the Vaughn Foundation for A. R., and by NSF grant DMS-0700280 for F. S.

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