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\$GL(n)\$**



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# FOURIER TRANSFORMS OF INTERTWINING OPERATORS AND PLANCHEREL MEASURES FOR $GL(n)$

By FREYDOON SHAHIDI\*

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**Introduction.** In this paper we prove the equality of two local coefficients; one defined in the context of  $L$ -functions attached to the pairs of representations of general linear groups by H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika [8, 11], and another one, defined by means of intertwining operators in [19]. As a consequence, we obtain an interesting formula for the Plancherel measures for  $GL(r)$  over a nonarchimedean field.

More precisely, let  $F$  be a nonarchimedean local field and denote by  $O$  its ring of integers. Fix two positive integers  $m$  and  $n$ , and set  $r = m + n$ . Let  $\pi_1$  and  $\pi_2$  be two irreducible admissible non-degenerate (cf. section 0) representations of  $GL_m(F)$  and  $GL_n(F)$ , respectively. Denote by  $\pi = \pi_1 \otimes \pi_2$  the corresponding representation of  $GL_m(F) \times GL_n(F)$ , and for a complex number  $s$ , let  $I(s, \pi)$  be the space of the representation induced from  $\pi$ , i.e. the space of all  $GL_r(O)$ -finite functions from  $GL_r(F)$  into the space of  $\pi$  satisfying

$$f \left[ \begin{bmatrix} g_1 & u \\ 0 & g_2 \end{bmatrix} g \right] = |\det g_1|^{1/2(s+n/2)} / |\det g_2|^{1/2(s+m/2)} \pi_1(g_1) \otimes \pi_2(g_2) f(g),$$

where  $g_1 \in GL_m(F)$ ,  $g_2 \in GL_n(F)$ ,  $u \in M_{m \times n}(F)$ , and  $g \in GL_r(F)$ .

Let  $A(s, \pi, w_{m,n})$  be the intertwining operator defined by relation (0.1) of section 0, where  $w_{m,n}$  is the permutation matrix defined there.

Fix a non-trivial additive character  $\psi$  of  $F$ . We assume that the largest ideal of  $F$  on which  $\psi$  is trivial is  $O$ . Let  $N_r$  be the subgroup of unipotent elements of  $GL_r$ , and define

$$\chi(n) = \psi(n_{12} + \cdots + n_{r-1,r})$$

for  $n \in N_r(F)$ .

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Finally, let  $\lambda_\pi^s$  be the Whittaker functional on  $I(s, \pi)$  defined by relation (0.2). Let  $\pi' = \pi_2 \otimes \pi_1$ . Then from the uniqueness of Whittaker functionals for induced representations it follows that there exists (cf. [19]) a complex number  $C_\chi(s, \pi)$  such that

$$\lambda_{\pi'}^{-s}(A(s, \pi, w_{m,n})f) = C_\chi^{-1}(s, \pi)\lambda_\pi^s(f)$$

for all  $f \in I(s, \pi)$ .

The factor  $C_\chi(s, \pi_1 \otimes \pi_2)$  has an important arithmetical significance. For example, when  $m = n = 1$  (in which case  $\pi_1$  and  $\pi_2$  are quasi-characters of  $F^*$ ),  $C_\chi^{-1}(s, \pi)$  is the local Tate factor (cf. [17])

$$\rho(\pi_1\pi_2^{-1}\alpha^s) = \epsilon^{-1}(s, \pi_1\pi_2^{-1}, \psi)L(s, \pi_1\pi_2^{-1})/(1 - s, \pi_1^{-1}\pi_2).$$

More generally, it is proved in [19] that, when  $\pi_1$  and  $\pi_2$  are, respectively, local components of two cusp forms  $\sigma_1$  and  $\sigma_2$  on  $GL_m(\mathbf{A})$  and  $GL_n(\mathbf{A})$ , then  $C_\chi(s, \pi_1 \otimes \tilde{\pi}_2)$  becomes the local factor appearing in the functional equation satisfied by the  $L$ -function  $L(s, \sigma_1 \times \sigma_2)$  (cf. [8, 11]).

Now, let  $\epsilon(s, \pi_1 \times \pi_2, \psi)$  and  $L(s, \pi_1 \times \pi_2)$  be the local Langlands' root number and  $L$ -function [14] attached to the pair  $(\pi_1, \pi_2)$  as in [8, 11], respectively. Set

$$\epsilon'(s, \pi_1 \times \pi_2, \psi) = \epsilon(s, \pi_1 \times \pi_2, \psi)L(1 - s, \tilde{\pi}_1 \times \tilde{\pi}_2)/L(s, \pi_1 \times \pi_2).$$

Again, when  $\pi_1$  and  $\pi_2$  are local components of  $\sigma_1$  and  $\sigma_2$ ,  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  becomes the local factor appearing in the same functional equation [8].

The first result of this paper answers the natural question of equality of these two factors and thus computes  $C_\chi(s, \pi)$  in terms of these other arithmetic factors. More precisely, in Theorem 5.1 of this paper, we prove that with an appropriate normalization of the measures defining  $C_\chi(s, \pi_1 \otimes \pi_2)$

$$C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^m(-1)\epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi),$$

where  $\omega_2$  is the restriction of  $\pi_2$  to the center of  $GL_n(F)$ . It is not hard to show that these normalized measures are in fact independent of the representations  $\pi_1$  and  $\pi_2$ .

An important consequence of this equality is a formula for the Plancherel constant  $\mu(s, \pi_1 \otimes \pi_2)$ , which is defined by the relation

$$A(s, \pi_1 \otimes \pi_2, w_{m,n})A(-s, \pi_2 \otimes \pi_1, w_{m,n}^{-1}) = \mu(s, \pi_1 \otimes \pi_2)^{-1}.$$

In fact, suppose  $\pi_1$  and  $\pi_2$  are both unitary (in particular tempered). Write

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = c(\pi_1 \times \pi_2)q^{-n(\pi_1 \times \pi_2)s}.$$

Then Theorem 6.1 shows that with a suitable normalization of the measures

$$\mu(s, \pi_1 \otimes \pi_2) = q^{n(\tilde{\pi}_1 \times \pi_2)} \frac{L(1+s, \pi_1 \times \tilde{\pi}_2)}{L(s, \pi_1 \times \tilde{\pi}_2)} \cdot \frac{L(1-s, \tilde{\pi}_1 \times \pi_2)}{L(-s, \tilde{\pi}_1 \times \pi_2)}.$$

The standard normalization of intertwining operators is a consequence of this equality. This has already been used by H. Jacquet and J. A. Shalika to determine the residual spectrum of  $GL_n(\mathbf{A})$ .

When  $n = 1$  and  $\pi_2 = 1$ , the integer  $n(\tilde{\pi}_1 \times 1) = n(\tilde{\pi}_1)$  is in fact the conductor of  $\tilde{\pi}_1$  [13]. This is the smallest integer  $r \geq 0$  for which there is a vector (then unique up to a complex multiple) in the space of  $\tilde{\pi}_1$  which is fixed by the subgroup

$$\left\{ \begin{pmatrix} h & v \\ u & x \end{pmatrix} \mid h \in GL_{m-1}(O), u \equiv 0(P^r), x \equiv 1(P^r) \right\}$$

of  $GL_m(O)$ . Here  $P$  is the maximal ideal of  $O$ . The  $L$ -functions are now those of [5].

One hopes that similar interpretation can be made for  $n(\pi_1 \times \pi_2)$  in general.

Now Corollaries 6.1.3 and 6.1.4 enable us to compute  $\mu(s, \pi_1 \otimes \pi_2)$  for any pair of irreducible admissible representations.

Finally, in Theorem 6.2 we obtain an irreducibility criteria which generalizes a similar result in [20] (for supercuspidal representations this is originally due to I. N. Bernstein and A. V. Zelevinskii [1]), where we have used global methods to prove a somewhat weaker result of equality of poles and zeros of these two factors (under the assumption that  $\pi_1$  and  $\pi_2$  are both components of cusp forms).

There is a different way of interpreting these results, and that is from the point of view of Fourier transforms of intertwining operators (cf. section 2). Intertwining operators being convolutions, they amount to Fourier transforms of certain measures. For rank one real groups they have been

studied by G. Schiffmann in [16]. Theorem 2.1 and Proposition 4.5 of this paper compute these Fourier transforms for certain rank one parabolic subgroups of  $GL(r)$  over a  $p$ -adic field. One corollary to them is the location of the poles of intertwining operators (Corollary to Theorem 2.1). This is originally due to G. I. Olšanskii [15]. The reader must observe that part (a) of Theorem 2.1 is a generalization of the formula that one obtains for the local Tate factor by choosing the test function to be the characteristic function of an appropriate subgroup of the units in  $O$ .

Theorem 5.1 is proved in three steps. We first prove the equality in the delicate case  $m = n$ . In fact Proposition 3.1 proves the equality if one of the representations is supercuspidal. This is proved in sections 2 and 3. The important step is the proof of Theorem 2.1. Next we consider the case  $m = n - 1$  and in Proposition 4.2 we prove the equality, assuming that both representations are supercuspidal. The general case (Theorem 5.1) is then proved by combining these two cases, results of [11] and [19] on factorization of two factors, and Jacquet's quotient theorem.

The formulas for the Plancherel measures, i.e. Theorem 6.1 and its corollaries, are proved in section 6. Theorem 6.1 is a consequence of the delicate relation

$$C_\chi(s, \pi_1 \otimes \tilde{\pi}_2) C_\chi(1 - s, \tilde{\pi}_1 \otimes \pi_2) = \omega_1^m \omega_2^n(-1).$$

**0. Notation.** Let  $F$  be a nonarchimedean field. Denote by  $O$  its ring of integers. Let  $P$  be the unique maximal ideal for  $O$ , and let  $q = [O:P]$ . We fix a uniformizing parameter  $\tilde{\omega}$  in  $O$ , i.e. an element satisfying  $|\tilde{\omega}| = q^{-1}$ . Then  $P = (\tilde{\omega})$ .

Given a positive integer  $r$ , let  $G_r = GL_r(F)$  and  $K_r = GL_r(O)$ . Then  $K_r$  is a maximal compact subgroup of  $G_r$ . Let  $B_r$  be the subgroup of upper triangulars in  $G_r$ .

Now, let  $\psi$  be a non-trivial additive character of  $F$ . We assume that the largest ideal of  $F$  on which  $\psi$  is trivial is  $O$ . Given  $r \in \mathbf{Z}^+$ , let  $N_r$  be the nilpotent radical of  $B_r$ , and define a character  $\chi$  of  $N_r$  as follows

$$\chi(n) = \psi(n_{12} + \cdots + n_{r-1,r}),$$

where  $n \in N_r$ . An irreducible admissible representation  $\pi$  of  $G_r$  is called non-degenerate, if there exists a linear functional  $\lambda$  on the space  $V$  of  $\pi$  such that

$$\lambda(\pi(n)v) = \chi(n)\lambda(v) \quad (n \in N_r, v \in V).$$

Then for every diagonal element  $a$ , the same is true if we replace  $\chi$  by  $\chi^a$  defined by  $\chi^a(n) = \chi(ana^{-1})$ . Given  $\chi$ , the space of such functionals (called Whittaker functionals) is at most one dimensional [21]. Now, fix a Whittaker functional  $\lambda$ , and for every  $v \in V$ , define a Whittaker function  $W_v$  by  $W_v(g) = \lambda(\pi(g)v)$ ,  $g \in G_r$ . Then  $W_v(ng) = \chi(n)W_v(g)$ . Let  $W(\chi, \pi)$  be the space of all such functions which we call the  $\chi$ -Whittaker model of  $\pi$ .

Now, suppose  $\pi$  is supercuspidal. Set

$$G_{r-1} = \begin{pmatrix} G_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow G_r \quad \text{and} \quad G'_{r-1} = w_r G_{r-1} w_r^{-1}.$$

Given  $W \in W(\chi, \pi)$ , it is proved in [23], that the map  $W \mapsto W|_{G'_{r-1}}$  is a bijection onto the space of all the locally constant complex functions  $\phi$  on  $G_{r-1}$  which are of compact support modulo  $N_{r-1}$ , and which satisfy

$$\phi(ng) = \chi(n)\phi(g) \quad (n \in N_{r-1}, g \in G_{r-1}).$$

In this way, we obtain the so called Kirillov model  $K(\chi, \pi)$  of  $\pi$ . Similar result is true if we replace  $G'_{r-1}$  by  $G_{r-1}$ .

Next, let  $m$  and  $n$  be two positive integers such that  $m + n = r$ . Consider the standard parabolic subgroup  $P$  of  $G_r$  whose standard Levi factor is  $G_m \times G_n$ . Let  $U$  be the unipotent radical of  $P$ . More precisely

$$U = \{(u_{ij}) \in B_r \mid u_{ii} = 1, 1 \leq i \leq r, u_{ij} = 0, 1 \leq i < j \leq m$$

$$\text{and } m < i < j \leq r\}.$$

Let  $w_r$  be the longest element in the Weyl group of  $G_r$ , i.e.

$$w_r = \begin{pmatrix} & & & & 1 \\ & & & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & 0 \\ & & & & \cdot \\ & & & & \cdot \\ & & & & 1 \end{pmatrix}.$$

Set  $V = w_r U w_r^{-1}$ , and let  $U' = {}^t V$ . Finally, let

$$w_{m,n} = \begin{pmatrix} 0_{m \times n} & 1_m \\ 1_n & 0_{n \times m} \end{pmatrix}.$$

Now, let  $\pi_1$  and  $\pi_2$  be two irreducible admissible representations of  $G_m$  and  $G_n$ , respectively. Denote by  $\pi = \pi_1 \otimes \pi_2$  the corresponding representation of  $G_m \times G_n$ . Finally for a complex number  $s$ , let  $I(s, \pi)$  denote the space of  $K_r$ -finite functions from  $G_r$  into the space of  $\pi$  satisfying

$$f\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} u g\right) = |\det g_1|^{1/2(s+n/2)} / |\det g_2|^{1/2(s+m/2)} \pi_1(g_1) \otimes \pi_2(g_2) f(g),$$

where  $g_1 \in G_m$ ,  $g_2 \in G_n$ ,  $u \in U$ , and  $g \in G_r$ . The intertwining operator which we are interested in is defined by

$$(0.1) \quad A(s, \pi, w_{m,n}) f(g) = \int_{U'} f(w_{m,n} u' g) du'$$

where  $f \in I(s, \pi)$ ,  $g \in G_r$ , and  $\operatorname{Re}(s) \gg 0$  (cf. [15, 19, 22]). As a function of  $s$ , it has a meromorphic continuation to a rational function of  $q^{-s}$  on  $\mathbf{C}$ .

Finally suppose  $\pi_1$  and  $\pi_2$  are both non-degenerate. Fix two  $\chi$ -Whittaker functionals  $\lambda_1$  and  $\lambda_2$  for  $\pi_1$  and  $\pi_2$ , respectively. Then for  $f \in I(s, \pi)$

$$(0.2) \quad \lambda_\pi^s(f) = \int_{U'} \langle f(w_{m,n} u'), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u')} du'$$

defines a Whittaker functional on  $I(s, \pi)$ . Finally from [19], it follows that there exists a complex number  $C_\chi(s, \pi)$  such that

$$(0.3) \quad \lambda_\pi^{-s}(A(s, \pi, w_{m,n}) f) = C_\chi^{-1}(s, \pi) \lambda_\pi^s(f)$$

for all  $f \in I(s, \pi)$ , where  $\pi' = \pi_2 \otimes \pi_1$ . Here  $\lambda_{\pi'}^{-s}$  is the Whittaker functional for  $I(-s, \pi')$  defined by

$$\lambda_{\pi'}^{-s}(f) = \int_U \langle f(w_{m,n}^{-1} u), \lambda_2 \otimes \lambda_1 \rangle \overline{\chi(u)} du,$$

where  $f \in I(-s, \pi')$ . Observe that  $\lambda_1 \otimes \lambda_2$  denotes the map induced by  $\lambda_1 \times \lambda_2$  on the tensor product of the spaces  $V_1$  and  $V_2$  of  $\pi_1$  and  $\pi_2$ . The local coefficient  $C_\chi(s, \pi)$  clearly depends on the measures defining  $A(s, \pi, w_{m,n})$ ,  $\lambda_\pi^s$ , and  $\lambda_{\pi'}^{-s}$ .

The purpose of the next several sections is to show that with a suitable normalization of measures,  $C_\chi(s, \pi)$  is equal to another factor defined in a completely different manner by H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika [8, 11]. The formula for the Plancherel measure (cf. Introduction and section 6) is a consequence of this equality.

**1. The case  $GL_n \times GL_n$ .** In this section we start the study of the different and more difficult case when  $m = n$ . Thus in the next three sections  $m = n$  and  $r = 2n$ . We use  $w_0$  to denote  $w_{n,n}$ . For simplicity, in the next three sections, we shall eliminate the index  $n$  from  $G_n$  and its subgroups. Thus we use  $G, N, B, w \dots$ , instead of  $G_n, N_n, B_n, w_n, \dots$ .

We also assume that  $\pi_1$  and  $\pi_2$  are both unitary, and furthermore  $\pi_2$  is supercuspidal. Let  $\langle, \rangle$  be a Hermitian pairing on the space  $V_2$  of  $\pi_2$ . Then as a function of  $g, \langle \pi_2(g)v_2, \tilde{v}_2 \rangle, v_2 \in V_2, \tilde{v}_2 \in V_2$ , is of compact support modulo the center of  $G$ . We start with the following lemma:

**LEMMA 1.1.** *There exists a Hermitian pairing  $\langle, \rangle$  on the space  $V_2$  of  $\pi_2$  such that*

$$\int_N \langle v_2, \pi_2(n)\tilde{v}_2 \rangle \chi(n) dn = \lambda_2(v_2) \overline{\lambda_2(\tilde{v}_2)}$$

for all  $v_2$  and  $\tilde{v}_2$  in  $V_2$ .

*Proof.* Fix a pairing  $\langle, \rangle_0$  on  $V_2$ , and set

$$\kappa(v_2, \tilde{v}_2) = \int_N \langle v_2, \pi_2(n)\tilde{v}_2 \rangle_0 \chi(n) dn.$$

Then

$$\kappa(\pi_2(n)v_2, \tilde{v}_2) = \chi(n)\kappa(v_2, \tilde{v}_2)$$

and

$$\kappa(v_2, \pi_2(n)\tilde{v}_2) = \overline{\chi(n)}\kappa(v_2, \tilde{v}_2).$$

Consequently by uniqueness of Whittaker functionals, there exists a complex constant  $c$  such that

$$\kappa(v_2, \bar{v}_2) = c \lambda_2(v_2) \overline{\lambda_2(\bar{v}_2)}.$$

It remains to show that  $c$  is positive. Normalizing a measure  $d\dot{g}$  on  $Z \backslash G$ , we have

$$(1.1.1) \quad \int_{Z \backslash G} \langle \pi_2(g)v_1, \bar{v}_1 \rangle_0 \overline{\langle \pi_2(g)v_1, v_2 \rangle_0} d\dot{g} = \|v_1\|_0^2 \langle v_2, \bar{v}_1 \rangle_0.$$

where  $v_1, \bar{v}_1$ , and  $v_2$  are all in  $V_2$  with  $\lambda_2(v_2) \neq 0$ . By compactness of the support of  $\langle \pi_2(g)v_1, v_2 \rangle_0$  modulo  $Z$ , it is easy to see that we may in fact replace  $\bar{v}_1$  by  $\lambda_2$  and therefore (1.1.1) can be written as:

$$\begin{aligned} \int_{Z \backslash G} W_{v_1}(g) \left( \int_N \langle v_2, \pi_2(ng)v_1 \rangle_0 \chi(n) dn \right) d\dot{g} \\ = c \lambda_2(v_2) \int_{Z \backslash G} |W_{v_1}(g)|^2 d\dot{g} \end{aligned}$$

which is equal to  $\lambda_2(v_2) \|v_1\|_0^2$ . Here we have used  $W_{v_1}(g)$  to denote  $\lambda_2(\pi_2(g)v_1)$ . Consequently

$$c \int_{Z \backslash G} |W_{v_1}(g)|^2 d\dot{g} = \|v_1\|_0^2$$

which proves  $c > 0$ . Now, define  $\langle, \rangle = c^{-1} \langle, \rangle_0$  to complete the lemma.

The following proposition follows from the irreducibility of the restriction of  $\pi_2$  to the subgroup  $H$  (cf. [2, 10]), defined by

$$H = \{(a_{ij}) \in G \mid a_{11} = 1, a_{i1} = 0, 2 \leq i \leq n\}.$$

**PROPOSITION 1.2.** *There exists an invariant measure  $d'\dot{a}$  on  $N'_{n-1} \backslash G'_{n-1}$  such that for  $v_1$  and  $v_2$  in  $V_2$*

$$\langle v_1, v_2 \rangle = \int_{N'_{n-1} \backslash G'_{n-1}} W_{v_1}(a) \overline{W_{v_2}(a)} d'\dot{a}.$$

Here  $N'_{n-1} = w_n N_{n-1} w_n^{-1}$ , and for every  $v \in V_2$ ,  $W_v(g) = \lambda_2(\pi_2(g)v)$ ,  $g \in G$ .

Now, given a positive integer  $m \geq 1$ , let  $K_{r,m}$  be the subgroup of  $K_r$  defined by  $K_{r,m} \equiv I(\text{mod } P^m)$ . Then every element of  $K_{r,m}$  has diagonal elements in  $1 + P^m$  and the rest in  $P^m$ . We use  $K_{r-1,m}$  also to denote  $K_{r,m} \cap G_{r-1}$ , where  $G_{r-1}$  is embedded in  $G_r$  as previously explained. Set  $K'_{r-1,m} = w_r K_{r-1,m} w_r^{-1}$ .

Let  $\pi$  be an irreducible admissible non-degenerate representation of  $G_r$ . Given  $W \in W(\chi, \pi)$ , let  $\phi_W$  denote the image of  $W$  in the Kirillov model  $K(\chi, \pi)$ . (If  $W = W_\nu$ ,  $\nu \in V(\pi)$ , we use  $\phi_\nu$  to denote  $\phi_{W_\nu}$ .) More precisely, let  $\phi_W$  be the restriction of  $W$  to  $G'_{r-1}$  considered as a function on  $G_{r-1}$ .

Given a set  $S$ , let  $\text{char.}(S)$  denote its characteristic function.

**LEMMA 1.3.** *Fix  $m \geq 1$ , an integer. Then there exists a unique vector  $v_2 \in V_2$  fixed by  $K'_{n-1,m}$  such that*

$$\langle v_1, v_2 \rangle = W_{v_1}(e)$$

for all  $v_1 \in V_2$  fixed by  $K'_{n-1,m}$ . More precisely  $v_2$  is the unique vector of  $V_2$  whose image  $\phi_{v_2}$  in  $K(\chi, \pi_2)$  is given by  $\phi_{v_2} = \text{char.}(K_{n-1,m}) \text{meas.}(K_{n-1,m})^{-1}$  modulo  $N_{n-1}$ . Here  $\text{meas.}(K_{n-1,m})$  denotes the measure of  $K_{n-1,m}$  coming from the restriction of  $d'$ .

*Proof.* For  $v_2$  as above, clearly  $\langle v_1, v_2 \rangle = W_{v_1}(e)$ . To prove the uniqueness, let  $v'_2$  be fixed by  $K'_{n-1,m}$  and  $\langle v_1, v'_2 \rangle = W_{v_1}(e)$ . Set  $\phi' = \phi_{v'_2}$ . Given  $x \in G_{n-1}$ , let  $v_1$  be such that  $\phi_{v_1} = \text{char.}(xK_{n-1,m})(\text{mod } N_{n-1})$ . Then

$$\begin{aligned} \langle v_1, v'_2 \rangle &= \bar{\phi}'(x) \cdot (\text{meas.}(K_{n-1,m})) \\ &= W_{v_1}(e) \end{aligned}$$

which is zero if  $x \notin K_{n-1,m}$ . This completes the lemma.

*Measures.*

1. *Measure on  $G_{n-1}$ .* Let  $U_n \subset N_n$  be the subgroup defined by

$$U_n = \{n \in N_n \mid n_{ij} = 0 \quad 1 \leq i < j \leq n - 1\}.$$

Fix two arbitrary measures  $dn_1$  and  $dn_2$  on  $N_{n-1}$  and  $U_n$ , respectively. Then  $dn_1 dn_2$  is a measure on  $N = N_n$ . Let  $d'a$  be the measure on  $N'_{n-1} \setminus G'_{n-1}$  defined according to Proposition 1.2. Define a measure  $d\hat{a}$  on  $N_{n-1} \setminus G_{n-1}$  by  $d\hat{a} = d'(w' \hat{a}^{-1} w^{-1})$ . Observe that  $d\hat{a}$  depends on both

$dn_1$  and  $dn_2$ , but not on  $\lambda_2$  (Lemma 1.1). Now, let  $d^*a$  be the measure on  $G_{n-1}$  obtained by composing  $d\hat{a}$  and  $dn_1$  in the standard manner.

Observe that  $d^*a$  is independent of  $dn_1$ . Suppose  $dn_2$  is changed to  $cdn_2$ ,  $c > 0$ ; then  $\langle, \rangle$  will change to  $c^{-1}\langle, \rangle$  and hence  $d^*a$  will change to  $c^{-1}d^*a$ . Consequently  $d^*adn_2$  and

$$\int_{K_{n-1} \times O^{n-1}} d^*adn_2$$

are both independent of  $dn_1$  and  $dn_2$ .

2. *Measure on  $Z_n$ .* We fix a measure  $d^*z$  on  $Z = Z_n \cong F^*$ , the center of  $G$ , such that

$$\int_{K_{n-1} \times O^{n-1} \times O^*} d^*adn_2d^*z = 1.$$

Then  $dz = |z|d^*z$  is a measure on  $F$ .

3. *Measure on  $K = K_n$ .* We first fix a self dual measure  $dx_1 \cdots dx_n$  on  $F^n$ . More precisely, if

$$\hat{\phi}(y) = \int_{F^n} \phi(x)\bar{\psi}(x \cdot {}^t y)dx_1 \cdots dx_n \quad (\phi \in \mathcal{S}(F^n)),$$

then  $\hat{\hat{\phi}}(x) = \phi(-x)$ . We further assume that  $dx_1 = dz$  defined in number 2 above. We now normalize the measure on  $K$  so that

$$\int_{\tilde{K}_{n,m}} dk = dx_2(P^m) \cdots dx_n(P^m)$$

for some  $m \geq 1$  (and therefore all), where

$$\tilde{K}_{n,m} = \left\{ \begin{pmatrix} k & y \\ x & a \end{pmatrix} \mid k \in K_{n-1}, x \in M_{1 \times (n-1)}(P^m), y \in M_{(n-1) \times 1}(O), a \in O^* \right\}$$

and  $x = (x_2, \dots, x_n) \in F^{n-1}$ .

4. *Measure on  $G = G_n$ .* We define the measure  $d^*g$  on  $G$ , according to Iwasawa decomposition, by  $d^*g = |\det a|^{-1}dn_2d^*ad^*zdk$ , where

$dn_2, d^*a, d^*z,$  and  $dk,$  are respectively the measures (as specified above) on  $U_n, G_{n-1}, Z_n,$  and  $K_n.$  Observe that this is possible since

$$\int_{G_{n-1}U_nZ_n \cap K} dk' = 1$$

( $dk'$  is the restriction of  $dk$  to  $G_{n-1}U_nZ_n \cap K$ ) which is equal to

$$\int_{G_{n-1}U_nZ_n \cap K} d^*adn_2d^*z,$$

according to number 2 above. As a consequence

$$\int_K d^*g = \int_K dk.$$

As we argued in number 1, the measure  $d^*g$  is independent of both  $dn_1$  and  $dn_2,$  as well as the functional  $\lambda_2.$  From now on, we shall fix this measure as the measure on  $G_n.$

5. *Measure on  $M = M_n.$*  We now define a measure  $da$  on  $M_n$  by  $da = |\det a|^n d^*a,$  where  $d^*a$  is defined by number 4 above. Now, if  $a = (a_{ij}),$  write

$$da = \prod_{i,j=1}^n da_{ij} \quad (\text{not uniquely}).$$

Redefining  $da_{ij},$  if necessary, we may, as we in fact do, assume that  $da_{11} = dx_1 = dz,$  and  $da_{i1} = dx_i, 2 \leq i \leq n,$  where  $dx_1, \dots, dx_n$  are defined as in number 3.

6. *Measure on  $V.$*  We choose this measure arbitrary but the same for the definitions of both  $\lambda_\pi^s$  and  $\lambda_\pi^{-s}$  (replacing  $f$  by  $R_{w_0^{-1}}f$ ).

7. *Measure on  $U.$*  Writing  $u \in U$  as

$$u = \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix},$$

we let  $du = da,$  where  $da$  is the measure on  $M_n$  fixed in number 5 above.

*Further remarks on intertwining operators.* Replacing  $f$  by  $R_{w_0^{-1}}f$  in (0.2) and (0.3), from now on we may consider the integral

$$(1.1) \quad \int_V \langle f(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv$$

which we still denote by  $\lambda_\pi^s(f)$ , where  $\chi'(v) = \chi(w^{-1}vw)$ . Then  $C_\chi(s, \pi)$  satisfies the same equality as (0.3). Given  $f \in I(s, \pi)$ , there are functions  $f_1$  and  $f_2$  in  $I(s, \pi)$ , with  $f_1$  of compact support modulo  $P$  in  $PV$  and  $f_2$  satisfying

$$\int_{V_0} \langle f_2(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv = 0,$$

where  $V_0$  is any compact open subgroup of  $V$  which is larger than a fixed one depending only on  $f$ , such that  $f = f_1 + f_2$  (cf. [4]). In particular the integral over  $V$  in (1.1) may be considered to be only over a sufficiently large open compact subgroup of  $V$ .

Now for every  $f \in I(s, \pi)$ , write  $f = f_1 + f_2$  as above. Then

$$(1.2) \quad \begin{aligned} \lambda_\pi^{-s}(A(s, \pi, w_0)f) &= C_\chi^{-1}(s, \pi)\lambda_\pi^s(f_1) \\ &= \lambda_\pi^{-s}(A(s, \pi, w_0)f_1) \end{aligned}$$

Next, given  $f \in I(s, \pi)$ , choose an integer  $m \geq 1$  such that  $f$  is fixed by the compact subgroup

$$K_{n,m} \cong \begin{pmatrix} K_{n,m} & 0 \\ 0 & 1_n \end{pmatrix} \hookrightarrow G_{2n} \quad \text{and} \quad V \cap K_{2n,m}.$$

**LEMMA 1.4.** *Let  $K'_{n,m} = w_0 K_{n,m} w_0^{-1}$ . Then for every  $k \in K'_{n,m}$  and  $v \in V$*

$$(1.3) \quad (\mathbf{1} \otimes \pi_2)(k)A(s, \pi, w_0)f(v) = A(s, \pi, w_0)f(v).$$

*Proof.* By definition (0.1), we only need to show that for every  $k \in K_{n,m} \subset G_{2n}$  and  $v \in V$ ,  $v^{-1}(kvk^{-1})$  is in  $V \cap K_{2n,m}$ . To show this write

$$v = \begin{pmatrix} 1_n & 0 \\ \tilde{v} & 1_n \end{pmatrix}$$

with  $\tilde{v} \in M_n(F)$ . Then  $\tilde{v}^{-1} = -\tilde{v}$  and  $\tilde{v}\tilde{v}' = \tilde{v} + \tilde{v}'$ . Now  $\widetilde{k\nu k}^{-1} = \tilde{v} \cdot k^{-1}$ . But then  $\tilde{v} \cdot k^{-1} - \tilde{v} \equiv 0 \pmod{P^m}$  since  $k^{-1} \equiv I \pmod{P^m}$ . This completes the lemma.

We also need:

**LEMMA 1.5.** *Let  $V$  and  $W$  be two complex vector spaces. Suppose  $\sum_{i=1}^l v_i \otimes w_i = 0$ ,  $v_i \in V$ ,  $w_i \in W$ , where  $v_i$ 's are linearly independent. Then  $w_i = 0$ ,  $1 \leq i \leq l$ .*

**COROLLARY.** *Let  $T \in \text{End}(W)$ , and suppose for  $u \in V \otimes W$ ,  $(1 \otimes T)u = u$ . Write  $u = \sum_i v_i \otimes w_i$  where  $v_i$ 's are all linearly independent. Then  $Tw_i = w_i$  for all  $i$ .*

Finally, we have

**PROPOSITION 1.6.** *Fix  $f \in I(s, \pi)$  and choose an integer  $m \geq 1$  such that  $K_{n,m}$  and  $V \cap K_{2n,m}$  both leave  $f$  invariant. Let  $\tilde{v}_2$  be the unique element in  $V_2$  whose image in the Kirillov model  $K(\chi, \pi_2)$  is given by  $\phi_{\tilde{v}_2} = \text{char.}(K_{n-1,m}) \cdot \text{meas.}(K_{n-1,m})^{-1}$  modulo  $N_{n-1}$ . Then*

$$\begin{aligned} \int_V \langle A(s, \pi, w_0)f(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv \\ = \int_V \langle A(s, \pi, w_0)f(v), \lambda_1 \otimes \tilde{v}_2 \rangle \overline{\chi'(v)} dv. \end{aligned}$$

*Proof.* Choose  $V_0 \subset V$  an open compact subgroup such that

$$\begin{aligned} (1.6.1) \quad \lambda_{\pi}^{-s}(A(s, \pi, w_0)f) &= \int_{V_0} \langle A(s, \pi, w_0)f(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv \\ &= \left\langle \int_{V_0} A(s, \pi, w_0)f(v) \overline{\chi'(v)} dv, \lambda_1 \otimes \lambda_2 \right\rangle. \end{aligned}$$

But now write

$$\int_{V_0} A(s, \pi, w_0)f(v) \overline{\chi'(v)} dv = \sum_i v_{1i} \otimes v_{2i}.$$

Then by Lemma 1.4,

$$(\mathbf{1} \otimes \pi_2(k))(\sum_i v_{1i} \otimes v_{2i}) = \sum_i v_{1i} \otimes v_{2i}$$

for all  $k \in K'_{n,m}$ . Next, we may assume that  $v_{1i}$ 's are all linearly independent, and therefore the corollary of Lemma 1.5 implies that  $\pi_2(k)v_{2i} = v_{2i}$  for all  $i$  and all  $k \in K'_{n,m}$ . But then Lemma 1.3 will allow us to replace  $\lambda_2$  by  $\bar{v}_2$  in (1.6.1). Now, taking the limit over  $V_0$  completes the proposition.

*Jacquet-Piatetski-Shapiro-Shalika's local coefficient.* Let  $\pi_1$  and  $\pi_2$  be two irreducible admissible non-degenerate representations of  $G = G_n$ . Let  $u_n = (0, \dots, 0, 1) \in F^n$ , and denote by  $\epsilon_n$  the diagonal matrix defined by  $a_{ij} = \delta_{i,j}(-1)^i$ . Finally for  $W \in W(\chi, \pi_i)$ ,  $i = 1$  or  $2$ , define  $\tilde{W}(g) = W(\epsilon_n w_n {}^t g^{-1})$ .

Now, let  $\Phi$  be a Schwartz-Bruhat function on  $F^n$ . Define its Fourier transform by

$$\hat{\Phi}(y) = \int_{F^n} \Phi(x) \bar{\psi}(x \cdot {}^t y) dx,$$

where  $dx$  is the self dual measure, i.e. it is so that  $\hat{\hat{\Phi}}(y) = \Phi(-y)$ .

Given  $s \in \mathbf{C}$ ,  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ , and  $\Phi$  as above, set

$$\Psi(s, W_1, W_2, \Phi) = \int_{N \backslash G} W_1(g) W_2(\epsilon_n g) \Phi(u_n g) |\det g|^s dg.$$

Then it is an important result of H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika [8, 11] that there exists a complex number  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  depending only on  $s, \pi_1, \pi_2$ , and  $\psi$ , such that

$$(1.4) \quad \Psi(1-s, \tilde{W}_1, \tilde{W}_2, \hat{\Phi}) = \epsilon'(s, \pi_1 \times \pi_2, \psi) \Psi(s, W_1, W_2, \Phi)$$

for all  $W_1 \in W(\chi, \pi_1)$ ,  $W_2 \in W(\chi, \pi_2)$ , and  $\Phi$ .

Now, let us assume  $\pi_2$  is unitary. Take  $W \in W(\chi, \pi_2)$  and define  $j(W)(g) = \overline{W(g)}$ . Then  $j(cW) = \bar{c}j(W)$ . Set  $\overline{W}(\chi, \pi_2) = \{\overline{W} \mid W \in W(\chi, \pi_2)\}$  and define  $\bar{\pi}_2$  on  $\overline{W}(\chi, \pi_2)$  by  $\bar{\pi}_2(g)j(W) = \overline{j(\pi_2(g)W)}$ . Then  $\bar{\pi}_2$  on  $\overline{W}(\chi, \pi_2)$  is isomorphic to  $\bar{\pi}_2$  and therefore  $\overline{W}(\chi, \pi_2) = \overline{W}(\bar{\chi}, \bar{\pi}_2)$ .

For  $\overline{W}_2 \in \overline{W}(\bar{\chi}, \bar{\pi}_2)$ , define  $\check{W}_2(g) = \overline{W}_2(\epsilon_n g)$ . Then  $\check{W}_2 \in W(\chi, \bar{\pi}_2)$ .

Consequently the functional equation (1.4) applied to  $W_1$  and  $\check{W}_2$  can be written as

$$(1.5) \quad \int_{N \backslash G} W_1(g) \overline{W_2(g)} \Phi(u_n g) |\det g|^s dg \\ = \omega_2^{n-1} (-1) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi)^{-1} \int_{N \backslash G} \check{W}_1(g) \overline{\check{W}_2(g)} \hat{\Phi}(u_n g) |\det g|^{1-s} dg$$

Here we have used the identity  $\check{W}(\epsilon_n g) = \omega_2^{n-1} (-1) \overline{\check{W}(g)}$ .

Finally, we assume that the measure  $dx$  is defined by number 3 on measures.

**2. Fourier transform of  $A(s, \pi, w_0)$ .** In this section we shall explicitly compute the left hand side of (0.3). We need some notation.

We assume that both  $\pi_1$  and  $\pi_2$  are unitary. For  $i = 1, 2$ , let  $\omega_i = \pi_i|_{Z_n}$  be the corresponding central characters. For a quasi-character  $\theta \in \hat{F}^*$ , let

$$L(s, \theta) = \begin{cases} (1 - \theta(\bar{\omega})q^{-s})^{-1} & \theta|_{O^*} \equiv 1 \\ 1 & \text{otherwise,} \end{cases}$$

be the corresponding Hecke  $L$ -function. Also, let  $d(\pi_1, \pi_2)$  be the relative formal degree between  $\pi_1$  and  $\pi_2$  which is defined by

$$\int_{Z \backslash G} \langle \pi_1(g)v_1, \tilde{v}_1 \rangle \overline{\langle \pi_2(g)v_2, \tilde{v}_2 \rangle} dg = d(\pi_1, \pi_2) \langle v_1, v_2 \rangle \overline{\langle \tilde{v}_1, \tilde{v}_2 \rangle}$$

if  $\pi_1 \cong \pi_2$ , and  $d(\pi_1, \pi_2) = 0$  otherwise.

Now, given  $f \in I(s, \pi)$ , write  $f = f_1 + f_2$  as in section 1, where  $f_1$  has compact support in  $PV$  modulo  $P$  and  $\lambda_\pi^s(f_2) = 0$ . Then

$$\int_V f_1(v) \overline{\chi'(v)} dv$$

is convergent and denotes an element in  $V_1 \otimes V_2$ .

Let  $m \geq 1$  be a fixed positive integer such that

$$(1) \quad (\pi_1(k) \otimes \mathbf{1}) \int_V f_1(v) \overline{\chi'(v)} dv = \int_V f_1(v) \overline{\chi'(v)} dv$$

and

$$(2) \quad \pi_2(k)\tilde{v}_2 = \tilde{v}_2,$$

both for all  $k$  in  $K_{n,m}$ , where  $\tilde{v}_2$  is as in Proposition 1.6 applied to  $f_1$ .

Set  $\Phi_0 = \text{char.}(K_{n,m}) \cdot \mu(K_{n,m})^{-1}$ , where  $m$  is such that  $K_{n,m}$  satisfies conditions 1 and 2 above, and  $\mu$  denotes the measure on  $G$ .

Finally for a Schwartz-Bruhat function  $\Phi$  on  $M_n(F)$ , let  $\hat{\Phi}$  denote its Fourier transform defined by

$$\hat{\Phi}(g) = \int_{M_n(F)} \Phi(a)\psi(\text{tr}(ga))da \quad (g \in M_n(F)).$$

The measure  $da$  is as in section 1, and may not be self dual (cf. number 5 on measures).

As in section 1, we assume that  $\pi_2$  is supercuspidal, and let the pairing on  $V_2$  be as in Lemma 1.1. The main result of this section is the following:

**THEOREM 2.1.** (a) *Suppose  $\text{Re}(s) \gg 0$ . Then*

$$(2.1) \quad \int_V \langle A(s, \pi, w_0)f(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv \\ = \omega_1(-1) \int_G \left\langle \int_V f_1(v) \overline{\chi'(v)} dv, \pi_1(g^{-1})\lambda_1 \otimes \pi_2(g)\tilde{v}_2 \right\rangle \\ \cdot |\det g|^s \hat{\Phi}_0(t_n g) d^*g$$

where  $\langle v_1, \pi_1(g^{-1})\lambda_1 \rangle = \lambda_1(\pi_1(g)v_1)$  and

$$t_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ 0_{n-1} & & & \cdot \\ & & & \cdot \\ & & & 0 \end{pmatrix} \in M_n(F).$$

Furthermore the integral over  $G$  is absolutely convergent for  $\text{Re}(s) > 0$ .

(b) *The right hand side has an analytic continuation to a rational function of  $q^{-s}$ . It has poles same as  $L(ns, \omega_1 \omega_2^{-1})$  with residue equal to*

$$\omega_1(-1)d(\pi_1, \pi_2 \otimes \mu^s) \left\langle \int_V f_1(v) \overline{\chi'(v)} dv, \lambda_1 \otimes \tilde{v}_2 \right\rangle \hat{\Phi}_0(0) \mu(0^*),$$

where  $\mu = dz$ .

Several lemmas are needed. We first explain some more notation.

Given an integer  $m$ , let

$$M_{n,m} = \{a \in M_n(F) \mid a_{ij} \in P^{-m}, 1 \leq i, j \leq n\},$$

where  $P^0 = O$ . Denote by  $\psi_m$ , the characteristic function of  $M_{n,m}$ . Given  $f \in I(s, \pi)$  and  $m \geq 0$ , an integer, set:

$$A_m(s, \pi, w_0)f(g) = \int_G f(w_0 u_a g) \psi_m(a^{-1}) da,$$

where

$$u_a \in \begin{pmatrix} 1_n & a \\ 0_n & 1_n \end{pmatrix} \in U, \quad \text{and } G = G_n.$$

Finally, let  $\| \cdot \|$  be the norm on  $V_1 \otimes V_2$ . In what follows all the limits are with respect to this norm.

LEMMA 2.1. *Suppose  $\text{Re}(s) \gg 0$ , then  $A_m(s, \pi, w_0)$  is convergent and for each  $g \in G$ , and fixed  $s$*

$$\lim_{m \rightarrow \infty} A_m(s, \pi, w_0)f(g) = A(s, \pi, w_0)f(g).$$

*Proof.* Clearly

$$\int_{M_n - G_n} f(w_0 u_a g) da = 0,$$

which implies:

$$\|A(s, \pi, w_0)f(g) - A_m(s, \pi, w_0)f(g)\| \leq \int_{a^{-1} \notin M_{n,m} \cap G_n} \|f(w_0 u_a g)\| da.$$

But this last integral approaches to zero as  $m$  gets large, since  $A(s, \pi, w_0)f(g)$  is absolutely convergent for  $\operatorname{Re}(s) \gg 0$ .

LEMMA 2.2. *Suppose  $\operatorname{Re}(s) \gg 0$ . Then*

$$A(s, \pi, w_0)f(g) = \lim_{m \rightarrow \infty} \int_G \pi_1(-a) \otimes \pi_2(a^{-1}) |\det a|^{s-n} f(v_a g) \psi_m(a) da,$$

where

$$v_a = \begin{pmatrix} 1_n & 0_n \\ a & 1_n \end{pmatrix} \in V,$$

and  $da$  is the measure on  $M_n$  restricted to  $G$ .

*Proof.* By Lemma 2.1, for  $\operatorname{Re}(s) \gg 0$ ,

$$A(s, \pi, w_0)f(g) = \lim_{m \rightarrow \infty} \int_G f(w_0 u_{a^{-1}} g) \psi_m(a) |\det a|^{-n} d^*a.$$

Now, the lemma follows, if we use the decomposition

$$w_0 \begin{pmatrix} 1_n & a^{-1} \\ 0_n & 1_n \end{pmatrix} = \begin{pmatrix} -a & 1_n \\ 0_n & a^{-1} \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ a & 1_n \end{pmatrix}$$

for all  $a \in G$ .

The following lemma is clear.

LEMMA 2.3. *Let  $\Phi_0$  be as before. Then*

$$\begin{aligned} & \int_G \left\langle \pi_1(ag^{-1}) \otimes \pi_2(ga^{-1}) \int_V f_1(v) \overline{\chi'(v)} dv, \lambda_1 \otimes \bar{v}_2 \right\rangle \Phi_0(g) \\ & \quad \cdot |\det a|^s |\det g|^{n-s} d^*g \\ & = \left\langle \pi_1(a) \otimes \pi_2(a^{-1}) \int_V f_1(v) \overline{\chi'(v)} dv, \lambda_1 \otimes \bar{v}_2 \right\rangle |\det a|^s. \end{aligned}$$

LEMMA 2.4. *Let  $C$  be a compact set in  $G$ , and assume  $g \in ZC$ , where  $Z$  is the centre of  $G$ . Suppose  $\hat{\Phi}_0(t_n g) \neq 0$ . Then*

$$\int_G \Phi_0(a) \psi_m(ga) \psi(\text{tr}(t_n ga)) da = \hat{\Phi}_0(t_n g)$$

for all  $m$  sufficiently large, depending only on  $\Phi_0$  and  $C$ .

*Proof.* Up to a constant,  $\Phi_0$  is the characteristic function of  $K_{n,r}$  for some positive integer  $r$ . By compactness of the support of  $\hat{\Phi}_0$  in  $M_n(F)$ , there exists a compact set  $K' \subset M_n(F)$  such that if  $g \in ZC$  and  $\hat{\Phi}_0(t_n g) \neq 0$ , then  $g \in K'$  ( $K'$  depends only on  $\Phi_0$  and  $C$ ). Now, choose  $m$  large enough so that  $K' \cdot K_{n,r} \subset M_{n,m}$ . Then for all such  $g$ ,  $\Phi_0(a) \psi_m(ga) = \Phi_0(a)$ ,  $\forall a \in M_n(F)$ , and the lemma follows.

By relation (1.2), to prove the theorem, we may replace  $f$  by  $f_1$  which is of compact support modulo  $P$ . Consequently we only need to prove the theorem for functions with this property. Furthermore, observe that by Proposition 1.6, we only need to prove that the right hand side of (2.1) is equal to

$$\int_V \langle A(s, \pi, w_0) f_1(v), \lambda_1 \otimes \bar{v}_2 \rangle \overline{\chi'(v)} dv.$$

We choose  $V_0 \subset V$ , a compact open subgroup, such that  $V_0$  contains the support of  $f_1$  modulo  $P$ . Then

$$\int_V f_1(v) \overline{\chi'(v)} dv = \int_{V_0} f_1(v) \overline{\chi'(v)} dv.$$

We denote this vector in  $V_1 \otimes V_2$  by  $w$ .

We now start proving Theorem 2.1 by expanding the right hand side of (2.1). By supercuspidality of  $\pi_2$  and Lemma 2.4, for large  $m$ , the right hand side of (2.1) (without the factor  $\omega_1(-1)$ ) is equal to

$$(2.1.1) \quad \int_G \langle \pi_1(g) \otimes \pi_2(g^{-1}) w, \lambda_1 \otimes \bar{v}_2 \rangle \cdot \left( \int_G \Phi_0(a) \psi_m(ga) \psi(\text{tr}(t_n ga)) \cdot |\det a|^n d^*a \right) |\det g|^s d^*g.$$

To check the convergence of (2.1.1), we observe that (2.1.1) is dominated by

$$\int_G \int_{Z \backslash G} |\langle w, \pi_1(\dot{g}^{-1})\lambda_1 \otimes \pi_2(\dot{g})\bar{v}_2 \rangle| \cdot |\det \dot{g}|^{\operatorname{Re}(s)} \\ \cdot \Phi_0(a) \left( \int_{F^*} \psi_m(z\dot{g}a) |z|^{n \cdot \operatorname{Re}(s)} d^*z \right) |\det a|^n d\dot{g}d^*a.$$

By supercuspidality of  $\pi_2$ , we conclude that  $\dot{g}$  is in a compact subset of  $Z \backslash G$ , and  $z\dot{g}a \in M_{n,m}$  implies the existence of  $K_m > 0$  such that  $|z| < K_m$ . Consequently for  $\operatorname{Re}(s) > 0$

$$\int_{F^*} \psi_m(z\dot{g}a) |z|^{n \cdot \operatorname{Re}(s)} d^*z < \infty,$$

which implies the absolute convergence of (2.1.1) for the same values of  $s$ . We then can use Fubini's theorem to interchange the integrals. Changing  $a$  to  $g^{-1}a$  (2.1.1) is equal to (for  $\operatorname{Re}(s) > 0$ )

$$(2.1.2) \quad \int_G \int_G \langle \pi_1(g) \otimes \pi_2(g^{-1})w, \lambda_1 \otimes \bar{v}_2 \rangle |\det g|^{s-n} \\ \cdot \Phi_0(g^{-1}a) \psi_m(a) \psi(\operatorname{tr}(t_n a)) |\det a|^n d^*g d^*a.$$

Again changing  $g$  to  $g^{-1}$ , we get

$$(2.1.3) \quad (2.1.2) = \int_G \int_G \langle \pi_1(g^{-1}) \otimes \pi_2(g)w, \lambda_1 \otimes \bar{v}_2 \rangle |\det g|^{n-s} \\ \cdot \Phi_0(ga) \psi_m(a) \psi(\operatorname{tr}(t_n a)) |\det a|^n d^*g d^*a.$$

Finally, changing  $g$  to  $ga^{-1}$ , (2.1.3) changes to

$$(2.1.4) \quad \int_G \int_G \langle \pi_1(ag^{-1}) \otimes \pi_2(ga^{-1})w, \lambda_1 \otimes \bar{v}_2 \rangle |\det g|^{n-s} |\det a|^s \\ \cdot \Phi_0(g) \psi_m(a) \psi(\operatorname{tr}(t_n a)) d^*g d^*a.$$

But now Lemma 2.3 implies that for  $m$  large and  $\operatorname{Re}(s) > 0$ , the right hand side of (2.1) is equal to

$$(2.1.5) \quad \omega_1(-1) \int_G \langle \pi_1(a) \otimes \pi_2(a^{-1})w, \lambda_1 \otimes \bar{v}_2 \rangle \\ \cdot |\det a|^s \psi_m(a) \psi(\operatorname{tr}(t_n a)) d^*a.$$

For each such large  $m$ , we choose an open compact subgroup  $V_m$ , such that  $V_m \supset V_0$ , and furthermore  $V_m \supset K_{2n,m} \cap V$ . Then we can clearly use

$$\psi(\text{tr}(t_n a)) \int_{V_0} f_1(v) \overline{\chi'(v)} dv = \int_{V_m} f_1 \left( \begin{pmatrix} 1_n & 0_n \\ a & 1_n \end{pmatrix} v \right) \overline{\chi'(v)} dv,$$

to conclude that (2.1.5) is equal to  $(\text{Re}(s) > 0$  and  $m$  large)

$$(2.1.6) \quad \int_{V_m} \int_G \left\langle \pi_1(a) \otimes \pi_2(a^{-1}) f_1 \left( \begin{pmatrix} 1_n & 0_n \\ a & 1_n \end{pmatrix} v \right), \lambda_1 \otimes \tilde{v}_2 \right\rangle \overline{\chi'(v)} \\ \cdot |\det a|^s \psi_m(a) d^* a dv.$$

Here the interchange of two integrals is justified since  $V_m$  is compact and  $\text{Re}(s) > 0$ .

To complete part (a) of Theorem 2.1, we need

**LEMMA 2.5.** *There exists a large positive integer  $m$  such that*

$$(2.5) \quad \int_V \langle A(s, \pi, w_0) f_1(v), \lambda_1 \otimes \tilde{v}_2 \rangle \overline{\chi'(v)} dv \\ = \omega_1(-1) \int_{V_m} \int_G \left\langle \pi_1(a) \otimes \pi_2(a^{-1}) f_1 \left( \begin{pmatrix} 1_n & 0_n \\ a & 1_n \end{pmatrix} v \right), \lambda_1 \otimes \tilde{v}_2 \right\rangle \\ \cdot \overline{\chi'(v)} \cdot |\det a|^s \psi_m(a) d^* a dv.$$

Here  $V_m$  can be replaced by any larger open compact subgroup of  $V$ .

*Proof.* Fix  $s$  with  $\text{Re}(s) \gg 0$ . Choose a compact open subgroup  $V_1 \subset V$  (depending only on  $f_1$  and  $s$ ) such that

$$(2.5.1) \quad \int_{V_1} \langle A(s, \pi, w_0) f_1(v), \lambda_1 \otimes \tilde{v}_2 \rangle \overline{\chi'(v)} dv \\ = \int_V \langle A(s, \pi, w_0) f_1(v), \lambda_1 \otimes \tilde{v}_2 \rangle \overline{\chi'(v)} dv.$$

Pick  $m$  so large that  $V_m = V_1$ . We first observe that  $\lambda_1$  in (2.5.1) and (2.1.6) can be replaced by a vector in  $V_1$ . More precisely, we choose a compact open subgroup  $K_0 \subset K_n$  such that

$$(\pi_1(k_0) \otimes \mathbf{1})A(s, \pi, w_0)f_1(v) = A(s, \pi, w_0)f_1(v)$$

for all  $k_0 \in K_0$  and all  $v \in V$ . This is possible if one uses a version of Lemma 1.4. Furthermore, write

$$f_1(v) = \sum_i c_i(v)v_{1i} \otimes v_{2i},$$

where  $c_i$ 's are complex functions on  $V$ ,  $v_{1i} \in V_1$ , and  $v_{2i} \in V_2$ . Using compactness of  $\langle v_{2i}, \pi_2(a)\bar{v}_2 \rangle$ ,  $\forall i$ , modulo center, the vectors  $v_{1i}$  can be replaced by a possibly different set  $\{v'_{1j}\}$  (also functions  $c'_j$  and vectors  $v'_{2j}$  with  $\{c'_j\} = \{c_i\}$  and  $\{v'_{2j}\} = \{v_{2i}\}$ ) such that

$$\begin{aligned} & \left\langle \pi_1(a) \otimes \pi_2(a^{-1})f_1\left(\begin{pmatrix} 1_n & 0 \\ a & 1_n \end{pmatrix}v\right), \lambda_1 \otimes \bar{v}_2 \right\rangle \\ &= \sum_j c'_j\left(\begin{pmatrix} 1_n & 0 \\ a & 1_n \end{pmatrix}v\right) \langle v'_{1j}, \lambda_1 \rangle \langle \pi_2(a^{-1})v'_{2j}, \bar{v}_2 \rangle. \end{aligned}$$

Consequently, if we take  $K_0$  so small that it fixes all the  $v'_{1j}$ 's, we then can replace  $\lambda_1$  by  $\check{\pi}_1(\phi)\lambda_1$  denoted by  $\bar{v}_1$ , where  $\phi = \text{char.}(K_0) \cdot \mu(K_0)^{-1}$ ,

$$\langle v_1, \check{\pi}_1(\phi)\lambda_1 \rangle = \langle \pi_1(\phi)v_1, \lambda_1 \rangle,$$

and

$$\pi_1(\phi)v_1 = \int_G \phi(g)\pi_1(g)v_1 d^*g.$$

Observe that  $\bar{v}_1$  depends only on  $f_1$  and  $s$ , but not on  $m$ .

Now, for  $r \in \mathbf{Z}^+$ ,  $r \geq m$ , and  $v \in V$ , set

$$(2.5.2) \quad F_r(v) = \int_{G_n \cap (M_{n,r} - M_{n,m})} \left\langle \pi_1(a) \otimes \pi_2(a^{-1})f_1\left(\begin{pmatrix} 1_n & 0 \\ a & 1_n \end{pmatrix}v\right), \bar{v}_1 \otimes \bar{v}_2 \right\rangle \overline{\chi'(v)} \cdot |\det a|^s d^*a,$$

Then, the difference between the right hand side of (2.5.1) and that of (2.5) is equal to

$$(2.5.3) \quad \omega_1(-1) \int_{V_1} \lim_r F_r(v) dv$$

using Lemma 2.2 and Schwartz inequality. But by compactness of  $V_1$ , (2.5.3) is equal to

$$\omega_1(-1) \lim_r \int_{V_1} F_r(v) dv.$$

Let  $N_r$  be the double integral

$$N_r = \int_{V_1} \int_{G_n \cap (M_{n,r} - M_{n,m})} \left\langle \pi_1(a) \otimes \pi_2(a^{-1}) f_1 \left( \begin{pmatrix} 1_n & 0 \\ a & 1_n \end{pmatrix} v \right), \bar{v}_1 \otimes \bar{v}_2 \right\rangle \cdot \overline{\chi'(v)} \cdot |\det a|^s d^* a dv.$$

Then since  $V_1 = V_m \supset \text{supp. } f_1 \pmod{P}$ , we conclude that the integral over  $G_n$  is in fact over  $G_n \cap \tilde{V}_1$ , where

$$V_1 = \begin{pmatrix} 1_n & 0_n \\ \tilde{V}_1 & 1_n \end{pmatrix}.$$

Enlarging  $m$  and consequently  $V_1 = V_m$ , if necessary, we may assume  $\tilde{V}_1 = M_{n,m}$ , and therefore for  $r \geq m$ ,  $\tilde{V}_1 \cap (M_{n,r} - M_{n,m}) = \phi$ , and the integral over  $G_n$  vanishes. Consequently for  $r$  large  $N_r = 0$ , which implies vanishing of (2.5.3), and the lemma follows.

To prove part (b), we write

$$\int_V f_1(v) \overline{\chi'(v)} dv = \sum_i v_{1i} \otimes v_{2i}$$

with  $v_{1i} \in V_1$  and  $v_{2i} \in V_2$ . Then by an argument similar to the one given in Lemma 2.5, we may replace  $\lambda_1$  by a vector  $\bar{v}_1 \in V_1$  such that

$$\begin{aligned} & \left\langle \int_V f_1(v) \overline{\chi'(v)} dv, \pi_1(g^{-1}) \lambda_1 \otimes \pi_2(g) \bar{v}_2 \right\rangle \\ &= \omega_1 \omega_2^{-1}(z) \sum_i \langle v_{1i} \otimes v_{2i}, \pi_1(\dot{g}^{-1}) \bar{v}_1 \otimes \pi_2(\dot{g}) \bar{v}_2 \rangle, \end{aligned}$$

and

$$\left\langle \int_V f_1(v) \overline{\chi'(v)} dv, \lambda_1 \otimes \bar{v}_2 \right\rangle = \left\langle \int_V f_1(v) \overline{\chi'(v)} dv, \bar{v}_1 \otimes \bar{v}_2 \right\rangle,$$

where  $g = zg$ , with  $\dot{g}$  in a compact subset of  $Z \backslash G$ ,  $z \in Z$ . Clearly  $\hat{\Phi}_0|_{M_n(O)} = \hat{\Phi}_0(0) \cdot \text{char.}(M_n(O))$ . Now, part (b) follows if we use the Iwasawa decomposition of  $g$  in the right hand side of (2.1), and the definition of relative formal degree for  $\pi_1$  and  $\pi_2 \otimes \mu^s$ .

**COROLLARY.** *The operator  $A(s, \pi, w_0)$  has a pole at  $s = s_0$  if and only if  $\pi_1 \cong \pi_2 \otimes \mu^{s_0}$  (here  $\pi_1$  is also assumed to be supercuspidal).*

*Proof.* From holomorphy of  $\lambda_{\pi'}^{-s}$  it follows that  $A(s, \pi, w_0)f$  has a pole if  $\pi_1 \cong \pi_2 \otimes \mu^s$  (part (b) of Theorem 2.1). Conversely suppose  $A(s, \pi, w_0)f$  has a pole at  $s = s_0$ . Then by [22],  $I(s_0, \pi)$  is irreducible, and from the non-vanishing of  $\lambda_{\pi'}^{-s}$ , it follows that the left hand side of (2.1) must have a pole. But by part (b) of Theorem 2.1, this is only possible if  $\pi_1 \cong \pi_2 \otimes \mu^{s_0}$ . (Here  $\pi_1$  is also assumed to be supercuspidal.)

**3. Relation with  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$ .** In this section we prove:

**PROPOSITION 3.1.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary non-degenerate representations of  $G$ . Assume  $\pi_2$  is supercuspidal. Normalize measures as in section 1. Then*

$$C_{\chi}(s, \pi_1 \otimes \pi_2) = \omega_2^n(-1) \epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi).$$

*Proof.* Fix  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ , such that  $W_i(e) \neq 0$ ,  $e = 1_n$ . Choose  $m \geq 1$ , an integer, such that  $K_{n,m}$  fixes both  $W_1$  and  $W_2$ . Let

$$\Phi_1 = \text{char.}(P \cap K_{2n,m}) \cdot \text{meas.}(P \cap K_{2n,m})^{-1}$$

and

$$\Phi_2 = \text{char.}(V \cap K_{2n,m}) \cdot \text{meas.}(V \cap K_{2n,m})^{-1}.$$

Define a function  $\Phi$  on  $G_{2n}$  by

$$\Phi(g) = \begin{cases} \Phi_1(p)\Phi_2(v) & g = pv \in PV \\ 0 & \text{otherwise.} \end{cases}$$

Set  $\tilde{\Phi}(g) = \Phi(g)v_1 \otimes v_2$ , where  $W_i = W_{v_i}$ ,  $v_i \in V_i$ ,  $i = 1, 2$ , and define:

$$f_{\tilde{\Phi}}(g) = \int_P (\pi \otimes \delta_p^s)^{-1}(p) \tilde{\Phi}(pg) d_p.$$

Then  $f_{\tilde{\Phi}} \in I(s, \pi)$ , and

$$\int_V \langle f_{\tilde{\Phi}}(v), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi'(v)} dv = W_1(e)W_2(e).$$

Finally, let  $\tilde{v}_2$  be as in Proposition 1.6, i.e. the unique element in  $V_2$  whose image in  $K(\chi, \pi_2)$  is  $\text{char.}(K_{n-1,m}) \cdot \text{meas.}(K_{n-1,m})^{-1}$  modulo  $N_{n-1}$ .

Now, for  $\text{Re}(s) > 0$ , by Theorem 2.1, we have

$$\begin{aligned} (3.1.1) \quad & C_{\chi}^{-1}(s, \pi) W_1(e) W_2(e) \\ &= \omega_1(-1) \int_G W_1(g) \langle v_2, \pi_2(g) \tilde{v}_2 \rangle |\det g|^s \hat{\Phi}_0(t_n g) d^*g \\ &= \omega_1(-1) \int_{N \setminus G} W_1(g) \left( \int_N \langle v_2, \pi_2(n g) \tilde{v}_2 \rangle \chi(n) dn \right) \\ &\quad \cdot |\det g|^s \hat{\Phi}_0(t_n g) d\dot{g}. \end{aligned}$$

By Lemma 1.1, (3.1.1) is equal to

$$(3.1.2) \quad \omega_1(-1) W_2(e) \int_{N \setminus G} W_1(g) \overline{W_{\tilde{v}_2}}(g) |\det g|^s \hat{\Phi}_0(t_n g) d\dot{g}$$

Fix  $r \geq 1$ , an integer, such that  $\Phi_0$  in Theorem 2.1 is  $\text{char.}(K_{n,r}) \cdot \text{meas.}(K_{n,r})^{-1}$ . Let  $\phi \in \mathcal{S}(F^n)$  be

$$\phi = (\text{char. } P^r)^{n-1} \times \text{char.}(1 + P^r).$$

Then

$$\hat{\Phi}_0(t_n g) = dx_1(1 + P^r)^{-1} \cdot \prod_{i=2}^n dx_i(P^r)^{-1} \hat{\phi}(-u_n g w),$$

where we refer to number 5 on measures. Consequently (3.1.2) is equal to

$$(3.1.3) \quad \omega_1(-1)cW_2(e) \cdot \omega_1\omega_2(-1) \int_{N \setminus G} W_1(g) \overline{W_{\tilde{v}_2}}(g) |\det g|^s \hat{\phi}(u_n g w) d\dot{g},$$

where

$$c = dx_1(1 + P^r)^{-1} \cdot \prod_{i=2}^n dx_i(P^r)^{-1}.$$

Finally, changing  $g$  to  $gw$  and using functional equation (1.5), applied to  $\pi_1(w)W_1$  and  $\pi_2(w)W_{\tilde{v}_2}$  in (3.1.3), we conclude that:

$$(3.1.4) \quad C_{\tilde{\chi}}^{-1}(s, \pi) W_1(e) W_2(e) \\ = c \omega_1(-1) W_2(e) \omega_2(-1)^{n-1} \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi)^{-1} \\ \cdot \int_{N \setminus G} (\pi_1(\overline{w}) W_1)(g) (\pi_2(\overline{w}) \overline{W_{\tilde{v}_2}})(g) \phi(u_n g) |\det g|^{1-s} d\dot{g}.$$

To complete the proposition, we need the following lemma:

**LEMMA 3.1.** *Let  $\phi = (\text{char. } P^r)^{n-1} \times \text{char.}(1 + P^r)$ . Then:*

$$(3.1.5) \quad \int_{N \setminus G} (\pi_1(\overline{w}) W_1)(g) (\pi_2(\overline{w}) \overline{W_{\tilde{v}_2}})(g) \phi(u_n g) |\det g|^{1-s} d\dot{g} \\ = c^{-1} \omega_1 \omega_2(-1) W_1(e).$$

*Proof.* For simplicity, let  $W'_1 = \pi_1(w)W_1$  and  $W'_2 = \pi_2(w)W_{\tilde{v}_2}$ , and set

$$\tilde{\phi}(g) = \int_{F^*} \phi((0, \dots, 0, z)g) \omega_1^{-1} \omega_2^{-1}(z) |z|^{n(1-s)} d^*z.$$

Then the left hand side of (3.1.5) is equal to

$$(3.1.6) \quad \int_{ZN \setminus G} \tilde{W}'_1(g) \overline{\tilde{W}'_2}(g) \tilde{\phi}(g) |\det g|^{1-s} d\dot{g}.$$

Now, if we write  $\dot{g} \in ZN \setminus G$  as  $\dot{g} = ak$ , with  $a \in N_{n-1} \setminus G_{n-1}$  and  $k \in K$ , and  $d\dot{g} = |\det a|^{-1} da dk$ , (3.1.6) will change to

$$(3.1.7) \quad \int_{N_{n-1} \backslash G_{n-1} \times K} \tilde{W}'_1(ak) \tilde{\overline{W}}'_2(ak) \tilde{\phi}(k) |\det a|^{-s} d^*a dk.$$

It is easy to see that if  $k = (k_{ij})$ , then  $\phi((0, \dots, 0, z)k) \neq 0$  implies that  $k_{in} \in P^r$ ,  $1 \leq i < n$ ,  $k_{nn} \in O^*$ , and  $z \in O^*$ , with  $zk_{nn} \in 1 + P^r$ . Consequently

$$\tilde{\phi}(k) = \omega_1 \omega_2(k_{n,a}) \int_{1+P^r} \omega_1^{-1} \omega_2^{-1}(z) d^*z$$

if  $k_{in} \in P^r$ ,  $1 \leq i < n$ , and  $k_{nn} \in O^*$ , and  $\tilde{\phi}(k) = 0$ , otherwise. But from  $\pi_i(1 + P^r)W_i = W_i$ ,  $i = 1, 2$ , we conclude that  $\omega_i(1 + P^r) = 1$ , and therefore by definition of  $d^*z = d^*x_1$

$$\tilde{\phi}(k) = \begin{cases} \omega_1 \omega_2(k_{nn}) \cdot dx_1(1 + P^r) & \text{if } k \in \tilde{K}_{n,r} \\ 0 & \text{otherwise.} \end{cases}$$

Since  $W'_1$  and  $W'_2$  are both fixed by  $K_{n,r}$ , it then easily follows that (3.1.7) is equal to

$$c^{-1} \int_{N_{n-1} \backslash G_{n-1} \times K_{n-1}} \tilde{W}'_1(ak) \tilde{\overline{W}}'_2(ak) |\det a|^{-s} d^*a dk \cdot \int_{U_n Z_n \cap K} dk''$$

( $dk''$  is the restriction of  $dk$  to  $U_n Z_n \cap K$ )

$$(3.1.8) \quad \begin{aligned} &= c^{-1} \int_{N_{n-1} \backslash G_{n-1}} \tilde{W}'_1(a) \tilde{\overline{W}}'_2(a) |\det a|^{-s} d^*a \\ &= c^{-1} \int_{N_{n-1} \backslash G_{n-1}} W_1(\epsilon_n w^t a^{-1} w^{-1}) \overline{W}_{\tilde{v}_2}(\epsilon_n w^t a^{-1} w^{-1}) |\det a|^{-s} d^*a \\ &= c^{-1} \omega_1 \omega_2(-1) \int_{N'_{n-1} \backslash G'_{n-1}} W_1(a) \overline{W}_{\tilde{v}_2}(a) |\det a|^s d^*a \end{aligned}$$

But, now from the definition of  $\tilde{v}_2$  it is clear that (3.1.8) is equal to  $c^{-1} \omega_1 \omega_2(-1) W_1(e)$ . This completes the lemma and therefore the proposition.

**4. The case  $GL(n-1) \times GL(n)$ .** In this section we shall consider the case  $(n-1, n)$ . Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary supercuspidal

representations of  $G_{n-1}$  and  $G_n$ , respectively. Fix  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ , and set

$$\Psi(s, W_1, W_2) = \int_{N_{n-1} \backslash G_{n-1}} W_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) W_1(\epsilon_{n-1} a) |\det a|^{s-1/2} da,$$

where  $\epsilon_{n-1}$  is defined in section 1.

Again, it follows from [8, 11] that there exists a complex number  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  depending only on  $s, \pi_1, \pi_2$ , and  $\psi$  such that

$$(4.1) \quad \Psi(1-s, \bar{W}_1, \bar{W}_2) = \omega_1 \omega_2(-1) \epsilon'(s, \pi_1 \times \pi_2, \psi) \Psi(s, W_1, W_2),$$

where  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ .

We first normalize our measures. Let  $w_0 = w_{n-1, n}$ . More precisely

$$w_0 = \begin{pmatrix} 0 & 1_{n-1} \\ 1_n & 0 \end{pmatrix}.$$

Let

$$U' = \left\{ \begin{pmatrix} 1 & 0 & Y \\ 0 & 1_{n-1} & a \\ 0 & 0 & 1_{n-1} \end{pmatrix} \mid Y \in F^{n-1}, a \in M_{n-1}(F) \right\},$$

where  $Y$  is a row matrix. Also set

$$U = \left\{ \begin{pmatrix} 1_{n-1} & X & b \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-1} \end{pmatrix} \mid X \in F^{n-1}, b \in M_{n-1}(F) \right\}.$$

Then by (0.1), for  $\text{Re}(s) \gg 0$ ,

$$(4.2) \quad A(s, \pi, w_0) f(g) = \int_{Y \in F^{n-1}} \int_{a \in G_{n-1}} f \left( w_0 \begin{pmatrix} 1 & 0 & Y \\ 0 & 1_{n-1} & a \\ 0 & 0 & 1_{n-1} \end{pmatrix} g \right) dY da$$

since  $M_{n-1}(F) - GL_{n-1}(F)$  has measure zero. Again  $\pi = \pi_1 \otimes \pi_2$ .

Also we can rewrite (0.3) as follows:

$$\begin{aligned}
 (4.3) \quad & \int_{X \in F^{n-1}} \int_{b \in M_{n-1}(F)} \left\langle A(s, \pi, w_0) \right. \\
 & \cdot f \left( w_0^{-1} \begin{pmatrix} 1_{n-1} & X & b \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-1} \end{pmatrix} \right), \lambda_1 \otimes \lambda_2 \left. \right\rangle \bar{\psi}(x_{n-1}) dX db \\
 & = C_X^{-1}(s, \pi) \int_{X' \in F^{n-1}} \int_{b' \in M_{n-1}(F)} \\
 & \cdot \left\langle f \left( w_0 \begin{pmatrix} 1_{n-1} & 0 & b' \\ 0 & 1 & X' \\ 0 & 0 & 1_{n-1} \end{pmatrix} \right), \lambda_1 \otimes \lambda_2 \right\rangle \bar{\psi}(x'_1) dX' db',
 \end{aligned}$$

where  ${}^tX = (x_1, \dots, x_{n-1})$  and  $X' = (x'_1, \dots, x'_{n-1})$ . Clearly  $C_X(s, \pi)$  depends on a number of measures, which we shall now start to fix.

1. *Measures  $dX'$ ,  $dY$ ,  $db$ , and  $db'$ .* We normalize these measures only by the relations  $db = db'$  and  $dX' = dY$ . Otherwise they are free to change.

To define the measures  $dX$  and  $da$ , we proceed as follows:

2. *Measure on  $G_{n-2}$ .* Let  $\langle, \rangle$  be the Hermitian pairing on the space  $V_1$  of  $\pi_1$  which satisfies Lemma 1.1 for  $\lambda_1$ . We clearly have:

**PROPOSITION 4.1.** *There exists an invariant measure  $d\dot{g}$  on  $N_{n-2} \setminus G_{n-2}$  such that for all  $v_1$  and  $v_2$  in  $V_1$*

$$\langle v_1, v_2 \rangle = \int_{N_{n-2} \setminus G_{n-2}} W_{v_1}(g) \overline{W_{v_2}(g)} d\dot{g},$$

where for every  $v \in V_1$ ,  $W_v(g) = \lambda_1(\pi_1(g)v)$ ,  $g \in G_{n-1}$ .

Fix two arbitrary measures  $dn_1$  and  $dn_2$  on  $N_{n-2}$  and  $U_{n-1}$ , respectively. Then  $dn_1 dn_2$  is a measure on  $N_{n-1}$ . Now, let  $d^*g$  be the measure on  $G_{n-2}$  obtained by composing  $d\dot{g}$  and  $dn_1$ . Again  $d^*g dn_2$ , as well as

$$\int_{K_{n-2} \times O^{n-2}} d^*g dn_2$$

are both independent of  $dn_1$  and  $dn_2$ .

3. *Measure on  $Z_{n-1}$ .* We fix  $d^*z$  on  $Z_{n-1}$ , such that

$$\int_{K_{n-2} \times O^{n-2} \times O^*} d^*g dn_2 d^*z = 1.$$

Set  $dz = |z| d^*z$ .

4. *Measure  $dX$ .* Let  $dx_1, \dots, dx_{n-1}$  be measures on  $F$  with  $dx_{n-1} = dz$ . We let  $dX = dx_1 \cdots dx_{n-1}$ .

5. *Measure on  $K_{n-1}$ .* We normalize the measure  $dk$  on  $K_{n-1}$  so that

$$\int_{\tilde{K}_{n-1,m}} dk = dx_1(P^m) \cdots dx_{n-2}(P^m)$$

for some integer  $m \geq 1$  (and therefore all), where  $\tilde{K}_{n-1,m}$  is defined as in section 1.

6. *Measure on  $G_{n-1}$ .* Again we define the measure  $d^*a$  on  $G_{n-1}$  by  $d^*a = |\det a|^{-1} dn_2 d^*g d^*z dk$ .

7. *Measure  $da$ .* We now set  $da = |\det a|^{n-1} d^*a$  with  $d^*a$  as in number 6.

Let  $\omega_1 = \pi_1|_{Z_{n-1}}$  and  $\omega_2 = \pi_2|_{Z_n}$ . The main result of this section is as follows.

**PROPOSITION 4.2.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary supercuspidal representations of  $G_{n-1}$  and  $G_n$ , respectively. Normalize measures as above. Then*

$$C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^{n-1}(-1) \epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi)$$

We need some preparation.

First we define our Kirillov model a little differently. For every  $W \in W(\chi, \pi_1)$ , let  $\phi_W$  be the restriction of  $W$  to

$$G_{n-2} \cong \begin{pmatrix} G_{n-2} & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow G_{n-1}.$$

Denote by  $K(\chi, \pi_1)$  the set of these restrictions. The following results can be proved as in section 1.

**LEMMA 4.3.** *Let  $f \in I(s, \pi)$ , and choose an integer  $m \geq 1$  such that  $f$  is fixed by the compact subgroups*

$$K_{n-1,m} \cong \begin{pmatrix} K_{n-1,m} & 0 \\ 0 & 1_n \end{pmatrix} \subset G_{2n-1} \quad \text{and} \quad U \cap K_{2n-1,m}.$$

Then for every  $k \in K_{n-1,m}$  and  $u \in U$

$$(\pi_1 \otimes \mathbf{1})(k)A(s, \pi, w_0)f(w_0^{-1}u) = A(s, \pi, w_0)f(w_0^{-1}u).$$

**PROPOSITION 4.4.** *Fix  $f \in I(s, \pi)$  and choose an integer  $m \geq 1$  as in Lemma 4.3. Let  $\bar{v}_1$  be the unique element in  $V_1$  whose image in  $K(\chi, \pi_1)$  is given by*

$$\phi_{\bar{v}_1} = \text{char.}(K_{n-2,m}) \text{meas.}(K_{n-2,m})^{-1}$$

modulo  $N_{n-2}$ . Then

$$\begin{aligned} \int_U \langle A(s, \pi, w_0)f(w_0^{-1}u), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u)} du \\ = \int_U \langle A(s, \pi, w_0)f(w_0^{-1}u), \bar{v}_1 \otimes \lambda_2 \rangle \overline{\chi(u)} du. \end{aligned}$$

As in the other case we first compute the Fourier transform of  $A(s, \pi, w_0)$ .

Given  $f \in I(s, \pi)$ , write  $f = f_1 + f_2$ , where  $f_1$  has compact support in  $Pw_0U'$  modulo  $P$ , and  $\lambda_\pi^s(f_2) = 0$  (cf. [4]). Fix the pairing on  $\pi_1$  as in Lemma 1.1. We now prove:

**PROPOSITION 4.5.** *Let  $f \in I(s, \pi)$ ,  $\pi = \pi_1 \otimes \pi_2$ , where  $\pi_1$  and  $\pi_2$  are two irreducible unitary supercuspidal representations of  $G_{n-1}$  and  $G_n$ , respectively. Then there exists a positive integer  $m_0$  such that for all  $m \geq m_0$ :*

$$\begin{aligned} (4.4) \quad \int_U \langle A(s, \pi, w_0)f(w_0^{-1}u), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u)} du \\ = \omega_1(-1) \int_{\chi \in (P^{-m})^{n-1}} \int_{a \in G_{n-1}} \left\langle \int_{U'} f_1(w_0u') \overline{\chi(u')} du' \right\rangle, \end{aligned}$$

$$\begin{aligned} & \cdot \pi_1(a)\tilde{v}_1 \otimes \pi_2^{-1} \left( \left( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1_{n-1} \end{pmatrix} w_1 \right) \lambda_2 \right) \\ & \cdot \overline{\psi}(x_{n-1}) |\det a|^{1/2-s} dXd^*a, \end{aligned}$$

where

$${}^tX = (x_1, \dots, x_{n-1}) \quad \text{and} \quad w_1 = \begin{pmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{pmatrix},$$

and where the double integral in the right hand side converges absolutely for all  $s$ .

*Proof.* For a fixed  $m$  the absolute convergence of the right hand side follows from the fact that  $a$  must have compact support modulo  $Z_{n-1}$  and  $N_{n-1}$ .

Now, we fix  $m$  and observe that the right hand side of (4.4) can be written as

$$\begin{aligned} (4.5.1) \quad & \omega_1(-1) \int_{G_{n-1} \times (P^{-m})^{n-1}} \left\langle \int_{M_{n-1}(F) \times F^{n-1}} f_1 \left( \begin{pmatrix} 1_{n-1} & 0 & 0 \\ b & 1_{n-1} & 0 \\ Y & 0 & 1 \end{pmatrix} \right) \right. \\ & \cdot \overline{\psi}(y_1) dbdY, \pi_1(a)\tilde{v}_1 \otimes \pi_2^{-1} \left( \left( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1_{n-1} \end{pmatrix} w_1 \right) \lambda_2 \right) \\ & \left. \cdot \overline{\psi}(x_{n-1}) |\det a|^{1/2-s} d^*adX, \right\rangle \end{aligned}$$

where  $Y = (y_1, \dots, y_{n-1}) \in F^{n-1}$ .

Changing  $Y$  to  $Ya^{-1}$  and  $b$  to  $b + a^{-1}X + XYa^{-1}$ , (4.5.1) can be written as

$$(4.5.2) \quad \omega_1(-1) \int_{G_{n-1} \times (P^{-m})^{n-1}} \left\langle \int_{M_{n-1}(F) \times F^{n-1}}$$

$$\begin{aligned}
 & f_1 \left( \left( \begin{array}{ccc} 1_{n-1} & 0 & 0 \\ b + a^{-1}X + XYa^{-1} & 1_{n-1} & 0 \\ -Ya^{-1} & 0 & 1 \end{array} \right) dbdY, \right. \\
 & \left. \pi_1(a)\bar{v}_1 \otimes \pi_2^{-1} \left( \left( \begin{array}{cc} 1 & Ya^{-1} \\ 0 & 1_{n-1} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ X & 1_{n-1} \end{array} \right) w_1 \right) \lambda_2 \right) \\
 & \cdot \bar{\psi}(x_{n-1}) |\det a|^{-(s+1/2(2n-1))} dadX.
 \end{aligned}$$

Now the proposition follows from replacing  $f$  by  $R_{w_0^{-1}}f$  and the following lemma, together with Proposition 4.4, if we choose  $m_0$  large enough so that

$$\begin{aligned}
 & \int_U \langle A(s, \pi, w_0)f(w_0^{-1}u), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u)} du \\
 & = \int_{U \cap M_{2n-1, m_0}} \langle A(s, \pi, w_0)f(w_0^{-1}u), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u)} du.
 \end{aligned}$$

(The process of going from  $f$  to  $f_1$  is same as in the other case.)

**LEMMA 4.6.** *Let  $a \in G_{n-1}$ ,  $b \in M_{n-1}(F)$ ,  $X \in M_{(n-1) \times 1}(F)$ , and  $Y \in M_{1 \times (n-1)}(F)$ . Then:*

$$\begin{aligned}
 & w_0 \left( \begin{array}{ccc} 1 & 0 & Y \\ 0 & 1_{n-1} & a \\ 0 & 0 & 1_{n-1} \end{array} \right) w_0^{-1} \left( \begin{array}{ccc} 1_{n-1} & X & b \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-1} \end{array} \right) w_0^{-1} \\
 & = \left( \begin{array}{ccc} -a^{-1} & 1_{n-1} & X \\ 0 & Y & 1 + YX \\ 0 & a & aX \end{array} \right) \left( \begin{array}{ccc} 1_{n-1} & 0 & 0 \\ b + a^{-1}X + XYa^{-1} & 1_{n-1} & 0 \\ -Ya^{-1} & 0 & 1 \end{array} \right).
 \end{aligned}$$

*Proof.* Straightforward but tedious.  
We now prove Proposition 4.2.

*Proof of Proposition 4.2.* Fix  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ , such that  $W_1(1_{n-1}) \neq 0$  and  $W_2(1_n) \neq 0$ . Choose  $m \geq 1$ , an integer, such that

$K_{n-1,m}$  and  $K_{n,m}$  fix  $W_1$  and  $W_2$ , respectively, and furthermore  $m \geq m_0$ .  
Let

$$\Phi_1 = \text{char.}(P \cap K_{2n-1,m}) \cdot \text{meas.}(P \cap K_{2n-1,m})^{-1}$$

and

$$\Phi_2 = \text{char.}(U' \cap K_{2n-1,m}) \cdot \text{meas.}(U' \cap K_{2n-1,m})^{-1}.$$

Define a function  $\Phi$  on  $G_{2n-1}$  by

$$\Phi(g) = \begin{cases} \Phi_1(p)\Phi_2(u') & g = pw_0u' \\ 0 & \text{otherwise.} \end{cases}$$

Set  $\tilde{\Phi}(g) = \Phi(g)v_1 \otimes v_2$ , where  $W_i = W_{v_i}$ ,  $v_i \in V_i$ ,  $i = 1, 2$ , and define

$$f_{\tilde{\Phi}}(g) = \int_P (\pi \otimes \delta_p^s)^{-1}(p) \tilde{\Phi}(pg) d_r p.$$

Then  $f_{\tilde{\Phi}} \in I(s, \pi)$ , and

$$\int_{U'} \langle f_{\tilde{\Phi}}(w_0u'), \lambda_1 \otimes \lambda_2 \rangle \overline{\chi(u')} du' = W_1(1_{n-1})W_2(1_n).$$

Now, by Proposition 4.5 and Lemma 1.1 applied to  $\pi_1$

$$\begin{aligned} (4.2.1) \quad & C_X^{-1}(s, \pi) W_1(1_{n-1}) W_2(1_n) \\ &= \omega_1(-1) W_1(1_{n-1}) \int_{N_{n-1} \backslash G_{n-1} \times (P^{-m})^{n-1}} \\ & W_2 \left( \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X & 1_{n-1} \end{pmatrix} w_1 \right) \overline{W_{\tilde{v}_1}(a)} \overline{\psi}(x_{n-1}) \\ & \cdot |\det a|^{1/2-s} d^* a dX. \end{aligned}$$

Using  $w_1 = w_n w_{n-1}$ , we observe that (4.2.1) is equal to

$$(4.2.2) \quad \omega_1(-1)W_1(1_{n-1}) \int_{(P^{-m})^{n-1}} \cdot \left( \int_{N_{n-1} \setminus G_{n-1}} W'_{2,X} \left( w_n \begin{pmatrix} w_{n-1}a & 0 \\ 0 & 1 \end{pmatrix} \right) \overline{W}_{\tilde{v}_1}(a) |\det a|^{1/2-s} d\dot{a} \right) \cdot \bar{\psi}(x_{n-1}) dX,$$

where

$$W'_{2,X} = \pi_2 \begin{pmatrix} 1_{n-1} & X \\ 0 & 1 \end{pmatrix} W_2.$$

To proceed, let  $\check{W}_1(a) = \overline{W}_1(\epsilon_{n-1}a)$ ,  $a \in G_{n-1}$ ; then  $\check{W}_1 \in W(\chi, \tilde{\pi}_1)$ , and the functional equation (4.1) changes to

$$(4.2.3) \quad \int_{N_{n-1} \setminus G_{n-1}} \tilde{W}_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \tilde{\overline{W}}_{\tilde{v}_1}(a) |\det a|^{1/2-s} d\dot{a} \\ = \omega_1^{n-2}(-1)\omega_1\omega_2(-1)\epsilon'(s, \tilde{\pi}_1 \times \pi_2, \psi) \\ \cdot \int_{N_{n-1} \setminus G_{n-1}} W_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \overline{W}_{\tilde{v}_1}(a) |\det a|^{s-1/2} d\dot{a}.$$

We now observe that the inner integral in (4.2.2) can be written as

$$\omega_2(-1) \int_{N_{n-1} \setminus G_{n-1}} \tilde{W}'_{2,X} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \tilde{\overline{W}}_{\tilde{v}_1}(a) |\det a|^{s-1/2} d\dot{a},$$

and therefore using (4.2.3), (4.2.2) is equal to

$$(4.2.4) \quad W_1(1_{n-1})\omega_1^{n-2}(-1)\epsilon'(1-s, \tilde{\pi}_1 \times \pi_2, \psi) \int_{(P^{-m})^{n-1}} \int_{N_{n-1} \setminus G_{n-1}} W_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & X \\ 0 & 1 \end{pmatrix} \right) \overline{W}_{\tilde{v}_1}(a) |\det a|^{1/2-s} \bar{\psi}(x_{n-1}) dXd\dot{a}.$$

It is easy to check that

$$\epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi) \epsilon'(1-s, \tilde{\pi}_1 \times \pi_2, \psi) = \omega_1^n \omega_2^{n-1}(-1),$$

and consequently (4.2.4) is equal to

$$\begin{aligned} (4.2.5) \quad & \omega_2^{n-1}(-1) W_1(1_{n-1}) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi)^{-1} \\ & \cdot \int_{N_{n-1} \backslash G_{n-1}} W_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \overline{W}_{\tilde{v}_1}(a) |\det a|^{1/2-s} d\dot{a} \\ & \cdot \int_{(P^{-m})^{n-1}} \psi(a_{n-1,1}x_1 + \cdots + (a_{n-1,n-1} - 1)x_{n-1}) dX \\ & = \omega_2^{n-1}(-1) W_1(1_{n-1}) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi)^{-1} \\ & \cdot \int_{N_{n-2} \backslash G_{n-2}} W_2 \left( \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \overline{W}_{\tilde{v}_1} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |\det a|^{1/2-s} d\dot{a} \end{aligned}$$

using the orthogonality of characters and the normalization of measures.

Now the proposition is a consequence of the definition of  $\tilde{v}_1$  which reduces the integral in (4.2.5) to  $W_2(1_n)$ .

**5. The general case.** Fix two positive integers  $m$  and  $n$ , and let  $\pi_1$  and  $\pi_2$  be two irreducible admissible non-degenerate representations of  $G_m$  and  $G_n$ , respectively. Let  $\omega_1 = \pi_1|Z_m$  and  $\omega_2 = \pi_2|Z_n$ . When  $m = n$ , let  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  be defined as in section 1. Observe that  $\epsilon'(s, \pi_1 \times \pi_2, \psi) = \epsilon'(s, \pi_2 \times \pi_1, \psi)$ .

Now suppose  $m \leq n - 1$ . Let  $(j, k)$  be two integers such that  $j + k = n - m - 1$ ,  $j \geq 0$  and  $k \geq 0$ . Let  $v_{n,m}$  be the matrix

$$v_{n,m} = \begin{pmatrix} 1_m & 0 \\ 0 & -\epsilon_{n-m} \cdot w_{n-m} \end{pmatrix}.$$

Fix  $W_i \in W(\chi, \pi_i)$ ,  $i = 1, 2$ , and set

$$\Psi(s, W_1, W_2, j) = \int_{g \in N_m \backslash G_m} \int_{x \in M_j \times_m(F)} W_2 \left( \begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{k+1} \end{pmatrix} \right) \cdot W_1(\epsilon_m g) |\det g|^{s-(n-m)/2} dx dg.$$

Again, it follows from [8, 11] that there exists a complex number  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  depending only on  $s, \pi_1, \pi_2$ , and  $\psi$  such that:

$$(5.1) \quad \Psi(1-s, \tilde{W}_1, \tilde{\pi}_2(v_{n,m})\tilde{W}_2, k) = \omega_1 \omega_2 (-1)^{n-m} \epsilon'(s, \pi_1 \times \pi_2, \psi) \Psi(s, W_1, W_2, j)$$

for every pair  $(j, k)$  as above. In particular  $\epsilon'(s, \pi_1 \times \pi_2, \Psi)$  is also independent of  $(j, k)$ . Observe that when  $m = n - 1$ ,  $\epsilon'(s, \pi_1 \times \pi_2, \psi)$  is same as the one defined in section 4.

When  $m > n$ , set

$$\epsilon'(s, \pi_1 \times \pi_2, \psi) = \epsilon'(s, \pi_2 \times \pi_1, \psi),$$

where  $\epsilon'(s, \pi_2 \times \pi_1, \psi)$  in the right hand side is defined by (5.1).

We now prove the first main result of this paper:

**THEOREM 5.1.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible admissible non-degenerate representations of  $G_m$  and  $G_n$ , respectively. Then there exists a normalization of measures defining  $C_\chi(s, \pi_1 \otimes \pi_2)$  such that*

$$(5.2) \quad C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^m (-1) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi)$$

*Reduction to the supercuspidal case.* Suppose that the theorem is proved for every pair of supercuspidal representations. By Jacquet's quotient theorem, choose, respectively, two standard parabolic subgroups  $P_m = M_m U_m$  and  $P_n = M_n U_n$  of  $G_m$  and  $G_n$ , and irreducible supercuspidal representations  $\sigma = (\sigma_1, \dots, \sigma_r)$  and  $\tau = (\tau_1, \dots, \tau_t)$  of  $M_m$  and  $M_n$  such that  $\pi_1 \subset \text{Ind}_{P_m \backslash G_m} \sigma$  and  $\pi_2 \subset \text{Ind}_{P_n \backslash G_n} \tau$  ( $M_m$  and  $M_n$  consist of  $r$  and  $t$  blocks, respectively). Then by Theorem 3.2.1 of [19]

$$C_\chi(s, \pi_1 \otimes \pi_2) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq t}} C_\chi(s, \sigma_i \otimes \tau_j).$$

Also by Theorem 3.1 of [11]

$$\epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq t}} \epsilon'(s, \sigma_i \times \bar{\tau}_j, \psi).$$

Now, equality (5.2) is an immediate consequence of the equalities

$$C_\chi(s, \sigma_i \otimes \tau_j) = \omega_{\tau_j}^{m_i}(-1) \epsilon'(s, \sigma_i \times \bar{\tau}_j, \psi)$$

for all  $i$  and  $j$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq t$ . Here  $m_i$  is the dimension of the block of  $M_m$  attached to  $\sigma_i$ .

To prove Theorem 5.1 for supercuspidal representations we need several lemmas.

**LEMMA 5.1.** (a) *Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary representations of  $G_n$ . Assume  $\pi_1$  is supercuspidal. Then there exists a normalization of measures defining  $C_\chi(s, \pi_1 \otimes \pi_2)$  such that*

$$C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^n(-1) \epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi)$$

(b) *Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary supercuspidal representations of  $G_n$  and  $G_{n-1}$ , respectively. Then there exists a normalization of measures defining  $C_\chi(s, \pi_1 \otimes \pi_2)$  such that*

$$C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^n(-1) \epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi).$$

*Proof.* (a) By Proposition 3.1.3 of [19] and Proposition 3.1 of this paper, there is a normalization of measures such that

$$\begin{aligned} C_\chi(s, \pi_1 \otimes \pi_2) &= \overline{C_\chi(\bar{s}, \pi_2 \otimes \pi_1)} \\ &= \omega_1^n(-1) \overline{\epsilon'(\bar{s}, \pi_2 \times \bar{\pi}_1, \psi)}. \end{aligned}$$

Now, a little manipulation of the functional equation (1.5) shows that

$$\overline{\epsilon'(\bar{s}, \pi_2 \times \bar{\pi}_1, \psi)} = \omega_1^n \omega_2^n (-1) \epsilon'(s, \pi_1 \times \bar{\pi}_2, \psi),$$

which proves part (a). Part (b) can be proved exactly the same way.

**LEMMA 5.2.** *Let  $\rho_1$  be a unitary character of  $F^* \cong G_1$ , and let  $\pi_2$  be an irreducible unitary supercuspidal representation of  $G_n$ . Then there are normalizations of measures such that*

$$C_\chi(s, \rho_1 \otimes \pi_2) = \omega_2(-1) \epsilon'(s, \rho_1 \times \bar{\pi}_2, \psi)$$

and

$$C_\chi(s, \pi_2 \otimes \rho_1) = \rho_1^n (-1) \epsilon'(s, \pi_2 \times \bar{\rho}_1, \psi).$$

*Proof.* We only prove the first equality, since the second one can be proved exactly the same way, using Lemma 5.1.

Let  $P$  be the standard parabolic subgroup of  $G_n$  whose standard Levi factor is of type  $(n - 1, 1)$ . Fix an irreducible unitary supercuspidal representation  $\pi'_1$  of  $G_{n-1}$ , and let  $\pi_1$  be the unique non-degenerate subrepresentation of  $\text{Ind}_{P|G_n} \pi'_1 \otimes \rho_1$ . Then by Theorem 3.2.1 of [19], Theorem 3.1 of [11], and Propositions 3.1 and 4.2 of this paper, there exists a normalization of measures defining  $C_\chi(s, \rho_1 \otimes \pi_2)$  such that

$$\begin{aligned} C_\chi(s, \pi'_1 \otimes \pi_2) C_\chi(s, \rho_1 \otimes \pi_2) &= \omega_2^n (-1) \epsilon'(s, \pi'_1 \times \bar{\pi}_2, \psi) \epsilon'(s, \rho_1 \times \bar{\pi}_2, \psi) \\ &= \omega_2^{n-1} (-1) \epsilon'(s, \pi'_1 \times \bar{\pi}_2, \psi) C_\chi(s, \rho_1 \otimes \pi_2), \end{aligned}$$

which proves the lemma, if one cancels out  $\epsilon'(s, \pi'_1 \times \bar{\pi}_2, \psi)$  from both sides of the last equality.

**PROPOSITION 5.3.** *Theorem 5.1 is true if  $\pi_1$  and  $\pi_2$  are supercuspidal.*

*Proof.* We may assume  $m \leq n$ , since the other case can be proved exactly the same way, using Lemma 5.1 and the second equality of Lemma 5.2. Furthermore, we may assume that  $\pi_1$  and  $\pi_2$  are both unitary. For, if  $\pi_i = \pi'_i \otimes \mu^{s_i}$ ,  $s_i \in \mathbf{C}$ ,  $i = 1, 2$ , with  $\pi'_i$  and  $\pi'_2$  unitary, where  $\mu = |\det|$ , then it is easy to see that

$$C_\chi(s, \pi_1 \otimes \pi_2) = C_\chi(s + s_1 - s_2, \pi'_1 \otimes \pi'_2)$$

and

$$\epsilon(s, \pi_1 \times \bar{\pi}_2, \psi) = \epsilon(s + s_1 - s_2, \pi'_1 \times \bar{\pi}'_2, \psi).$$

Now, let  $P$  be the standard parabolic subgroup of  $G_n$  whose standard Levi factor is of type  $(m, 1, \dots, 1)$ . Fix  $n - m$  unitary characters  $\rho_1, \dots, \rho_{n-m}$  of  $F^* \cong G_1$ , and set

$$\pi'_1 = \text{Ind}_{P|G_n} \pi_1 \otimes \rho_1 \otimes \cdots \otimes \rho_{n-m}$$

(or rather its unique non-degenerate subrepresentation). Then by Theorem 3.2.1 of [19] and Proposition 3.1 of this paper:

$$C_\chi(s, \pi_1 \otimes \pi_2) \prod_{i=1}^{n-m} C_\chi(s, \rho_i \otimes \pi_2) = \omega_2^m(-1) \epsilon'(s, \pi'_1 \times \tilde{\pi}_2, \psi),$$

which is equal to

$$\omega_2^m(-1) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi) \prod_{i=1}^{n-m} \epsilon'(s, \rho_i \times \tilde{\pi}_2, \psi)$$

by Theorem 3.1 of [11].

But now for each  $i$ ,  $1 \leq i \leq n - m$ , Lemma 5.2 states that

$$C_\chi(s, \rho_i \otimes \pi_2) = \omega_2(-1) \epsilon'(s, \rho_i \times \tilde{\pi}_2, \psi),$$

which then implies that

$$C_\chi(s, \pi_1 \otimes \pi_2) = \omega_2^m(-1) \epsilon'(s, \pi_1 \times \tilde{\pi}_2, \psi).$$

This completes the proposition, and consequently, by the observations which were made before, the theorem follows in general.

**6. Plancherel measures.** Let  $\pi_1$  and  $\pi_2$  be two irreducible admissible representations of  $G_m$  and  $G_n$ , respectively. Set  $w_0 = w_{m,n}$ . Then it follows from [19, 22], that there exists a complex constant  $\mu(s, \pi_1 \otimes \pi_2)$ , which we call the Plancherel constant attached  $\pi_1$  and  $\pi_2$ , such that

$$A(s, \pi_1 \otimes \pi_2, w_0) A(-s, \pi_2 \otimes \pi_1, w_0^{-1}) = \mu(s, \pi_1 \otimes \pi_2)^{-1}.$$

It clearly depends on the measures defining the intertwining operators. In this section we shall obtain a formula for  $\mu(s, \pi_1 \otimes \pi_2)$ .

Assume first that  $\pi_1$  and  $\pi_2$  are both non-degenerate. Write

$$\epsilon'(s, \pi_1 \times \pi_2, \psi) = \epsilon(s, \pi_1 \times \pi_2, \psi)L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2)/L(s, \pi_1 \times \pi_2),$$

where the local root number  $\epsilon(s, \pi_1 \times \pi_2, \psi)$  is of the form

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = c(\pi_1 \times \pi_2)q^{-n(\pi_1 \times \pi_2)s}$$

with  $c(\pi_1 \times \pi_2) \in \mathbf{C} - \{0\}$  and  $n(\pi_1 \times \pi_2) \in \mathbf{Z}$ ; and  $L(s, \pi_1 \times \pi_2)$  denotes the corresponding local  $L$ -function. We now prove:

**THEOREM 6.1.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible unitary non-degenerate representations of  $G_m$  and  $G_n$ , respectively. Normalize measures defining the intertwining operators as in Theorem 5.1. Then:*

$$\mu(s, \pi_1 \otimes \pi_2) = q^{n(\tilde{\pi}_1 \times \pi_2)} \frac{L(1+s, \pi_1 \times \tilde{\pi}_2)}{L(s, \pi_1 \times \tilde{\pi}_2)} \cdot \frac{L(1-s, \tilde{\pi}_1 \times \pi_2)}{L(-s, \tilde{\pi}_1 \times \pi_2)}.$$

*Proof.* By Proposition 3.1.1 of [19], we have

$$(6.1.1) \quad C_\chi(s, \pi_1 \otimes \pi_2)C_\chi(-s, \pi_2 \otimes \pi_1) = \mu(s, \pi_1 \otimes \pi_2).$$

But now Theorem 5.1 states that

$$C_\chi(s, \pi_1 \otimes \pi_2) = c(\pi_1 \otimes \pi_2)q^{-n(\pi_1 \otimes \pi_2)s}L(1-s, \tilde{\pi}_1 \times \pi_2)/L(s, \pi_1 \times \tilde{\pi}_2),$$

where

$$c(\pi_1 \otimes \pi_2) = \omega_2^m(-1)c(\pi_1 \times \tilde{\pi}_2)$$

and  $n(\pi_1 \otimes \pi_2) = n(\pi_1 \times \tilde{\pi}_2)$ . The following lemma is crucial.

$$\text{LEMMA 6.1. } |c(\pi_1 \otimes \pi_2)|^2 = q^{n(\tilde{\pi}_1 \times \pi_2)}.$$

*Proof.* From Proposition 3.1.3 of [19], it follows that  $c(\pi_1 \otimes \pi_2) = \overline{c(\pi_2 \otimes \pi_1)}$  and  $n(\pi_1 \otimes \pi_2) = n(\pi_2 \otimes \pi_1)$ . More precisely, one also needs the equality

$$\overline{L(\bar{s}, \pi_1 \times \tilde{\pi}_2)} = L(s, \tilde{\pi}_1 \times \pi_2).$$

Consequently, using  $\epsilon'(s, \pi_1 \times \pi_2, \psi) = \epsilon'(s, \pi_2 \times \pi_1, \psi)$  and Theorem 5.1, we have:

$$\begin{aligned} c(\pi_1 \otimes \tilde{\pi}_2) &= \omega_2^m(-1)c(\pi_2 \times \pi_1) \\ &= \omega_1^n \omega_2^m(-1)c(\pi_2 \otimes \tilde{\pi}_1) \\ &= \omega_1^n \omega_2^m(-1)\overline{c(\tilde{\pi}_1 \otimes \pi_2)}, \end{aligned}$$

as well as the equalities of the integers  $n(\tilde{\pi}_1 \times \tilde{\pi}_2)$ ,  $n(\tilde{\pi}_1 \otimes \pi_2)$ ,  $n(\pi_2 \otimes \tilde{\pi}_1)$ ,  $n(\pi_2 \times \pi_1)$ , and finally  $n(\pi_1 \times \pi_2)$ . But then

$$(6.1.2) \quad c(\pi_1 \times \pi_2) = \omega_2^m(-1)c(\pi_1 \otimes \tilde{\pi}_2)$$

and

$$= \omega_1^n(-1)\overline{c(\tilde{\pi}_1 \otimes \pi_2)}$$

$$(6.1.3) \quad c(\tilde{\pi}_1 \times \tilde{\pi}_2) = \omega_2^m(-1)c(\tilde{\pi}_1 \otimes \pi_2).$$

Now, using functional equations (1.4) and (5.1), it is easy to see that

$$(6.1.4) \quad \epsilon'(s, \pi_1 \times \pi_2, \psi)\epsilon'(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2, \psi) = \omega_1^n \omega_2^m(-1).$$

Substituting (6.1.2) and (6.1.3) in (6.1.4), then simply implies:

$$|c(\tilde{\pi}_1 \otimes \pi_2)|^2 = q^{n(\pi_1 \times \pi_2)},$$

and the lemma follows.

To complete the theorem, one must now only observe that (using (6.1.1))

$$\mu(s, \pi_1 \otimes \pi_2) = |c(\pi_1 \otimes \pi_2)|^2 \frac{L(1+s, \pi_1 \times \tilde{\pi}_2)}{L(s, \pi_1 \times \tilde{\pi}_2)} \cdot \frac{L(1-s, \tilde{\pi}_1 \times \pi_2)}{L(-s, \tilde{\pi}_1 \times \pi_2)}.$$

Now, suppose  $n = 1$  and  $\pi_2$  is the trivial representation  $\mathbf{1}$  of  $G_1$ . Denote by  $n(\pi_1)$  the smallest integer  $r \geq 0$  for which there exists a vector (then unique up to a complex multiple) in the space of  $\pi_1$  which is fixed by the subgroup

$$K'_{m,r} = \left\{ \begin{pmatrix} h & v \\ u & x \end{pmatrix} \mid h \in K_{m-1}, u \equiv 0(P^r), x \equiv 1(P^r) \right\}$$

of  $K_m$ . Set  $K'_{m,0} = K_m$ . The integer  $n(\pi_1)$  is called the conductor of  $\pi_1$  (cf. [3, 13]). Also denote by  $L(s, \pi_1)$  the local Godement-Jacquet  $L$ -function attached to  $\pi_1$  (cf. [5, 12]). We then have:

**COROLLARY 6.1.1.** *Let  $\pi$  be an irreducible unitary non-degenerate representation of  $G_m$ . Let  $n(\tilde{\pi})$  denote the conductor of  $\tilde{\pi}$ . Then :*

$$\mu(s, \pi \otimes \mathbf{1}) = q^{n(\tilde{\pi})} \frac{L(1+s, \pi)}{L(s, \pi)} \cdot \frac{L(1-s, \tilde{\pi})}{L(-s, \tilde{\pi})}.$$

*Proof.* This is a simple consequence of Theorem 6.1 and the main theorem of [13].

**COROLLARY 6.1.2.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible tempered representations of  $G_m$  and  $G_n$ , respectively. Then the statement of Theorem 6.1 is true.*

*Proof.* This is a simple consequence of the fact that for  $GL(n)$  all the tempered representations are non-degenerate [9].

**COROLLARY 6.1.3.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible supercuspidal representations of  $G_m$  and  $G_n$ , respectively.*

(a) *Suppose  $m \neq n$ . Then*

$$\mu(s, \pi_1 \otimes \pi_2) = q^{n(\tilde{\pi}_1 \times \pi_2)}.$$

(b) *Suppose  $m = n$ , but for no  $s \in \mathbf{C}$ ,  $\pi_1 \cong \pi_2 \otimes \mu^s$ . Then*

$$\mu(s, \pi_1 \otimes \pi_2) = q^{n(\tilde{\pi}_1 \times \pi_2)}.$$

(c) *Suppose  $m = n$ , and for some  $s \in \mathbf{C}$ ,  $\pi_1 \cong \pi_2 \otimes \mu^s$ . Then*

$$\begin{aligned} \mu(s, \pi_1 \times \pi_2) &= q^{n(\tilde{\pi}_1 \times \pi_2)} \frac{1 - \omega_1 \omega_2^{-1}(\tilde{\omega}) q^{-ms}}{1 - \omega_1 \omega_2^{-1}(\tilde{\omega}) q^{-m(1+s)}} \\ &\quad \cdot \frac{1 - \omega_1^{-1} \omega_2(\tilde{\omega}) q^{ms}}{1 - \omega_1^{-1} \omega_2(\tilde{\omega}) q^{-m(1-s)}}. \end{aligned}$$

Finally, let  $\pi_1$  and  $\pi_2$  be two irreducible admissible (resp. unitary) (not necessarily non-degenerate) representations of  $G_m$  and  $G_n$ , respectively. Choose, respectively, two standard parabolic subgroups  $P_m =$

$M_m U_m$  and  $P_n = M_n U_n$  of  $G_m$  and  $G_n$ , and irreducible supercuspidal (resp. quasi-tempered) representations  $\sigma = (\sigma_1, \dots, \sigma_r)$  and  $\tau = (\tau_1, \dots, \tau_t)$  of  $M_m$  and  $M_n$  such that  $\pi_1 \subset \text{Ind}_{P_m} \sigma$  and  $\pi_2 \subset \text{Ind}_{P_n} \tau$ . We now have

**COROLLARY 6.1.4.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible admissible (resp. unitary) representations of  $G_m$  and  $G_n$ , respectively. Fix  $\sigma = (\sigma_1, \dots, \sigma_r)$  and  $\tau = (\tau_1, \dots, \tau_t)$  as above. Then*

$$\mu(s, \pi_1 \otimes \pi_2) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq t}} \mu(s, \sigma_i \otimes \tau_j),$$

where for each  $i$  and  $j$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq t$ ,  $\mu(s, \sigma_i \otimes \tau_j)$  is given by Corollary 6.1.3 (resp. Corollary 6.1.2).

*Remark.* The reader must observe that when  $\pi_1$  and  $\pi_2$  are in the discrete series, the function  $\mu(s, \pi_1 \otimes \pi_2)$  defined here differs from the one defined by Harish-Chandra [22], by the positive constant multiple  $\gamma^2(G/P)$  (cf. [22]).

*Irreducibility criteria.* We now state a generalization of Theorem 4.4 of [20]. It is an immediate consequence of the definition of  $C_\chi$  and Theorem 5.1. It takes on an importance since tempered representations are the building blocks for unitary representations.

**THEOREM 6.2.** *Let  $P = MU$  be a standard parabolic subgroup of  $G_m$ . Fix an irreducible admissible non-degenerate (in particular quasi-tempered) representation  $\sigma = (\sigma_1, \dots, \sigma_r)$  of  $M$  ( $M$  has  $r$  blocks). Suppose the representations  $\sigma_1, \dots, \sigma_r$  are so that the product  $\prod_{1 \leq i \neq j \leq r} L(1, \bar{\sigma}_i \times \sigma_j) / L(0, \sigma_i \times \bar{\sigma}_j)$  is infinite. Then the induced representation  $\text{Ind}_{P|G_m} \sigma$  is reducible.*

*Remark.* When  $\sigma$  is supercuspidal, using Corollary 6.1.3, Theorem 6.2 gives a new proof of a theorem of I. N. Bernstein and A. V. Zelevinskii [1] on reducibility of representations induced from supercuspidal ones (Theorem 4.3 of [20]).

The following corollary is an immediate consequence of Proposition 2.1 of [13] and the above theorem.

**COROLLARY 6.2.1.** *Let  $\pi$  be an irreducible unitary non-degenerate (in particular tempered) representation of  $G_m$ . Suppose  $s_0$  is a pole of the local Godement-Jacquet  $L$ -function  $L(1 - s, \bar{\pi})$ . Then the induced representation  $I(s_0, \pi \otimes \mathbf{1})$  is reducible. Here  $\mathbf{1}$  denotes the trivial representation of  $G_1$ .*

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