Vol. 47, No. 1 DUKE MATHEMATICAL JOURNAL © March 1980

WHITTAKER MODELS FOR REAL GROUPS

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Introduction. Whittaker functions were first introduced for the principal series representations of Chevalley groups by H. Jacquet [3]. Later, they were pursued by G. Schiffmann for algebraic groups of real rank one [12]. They played a very important role in the development of the Hecke theory for GL_n through the work of H. Jacquet, R. P. Langlands, I. I. Piatetski-Shapiro, and J. A. Shalika [4, 5]. More precisely, they were the main tools for the definitions of local and global L-functions and ϵ-factors. They also appeared quite useful in the development of the Hecke theory for other groups (cf. [10]), as well as in the definition of the local γ-factors of certain functional equations [13, 14], particularly in their factorization. There seems to be other evidence of interest, especially in the work of W. Casselman, B. Kostant [7], and G. Zuckerman.

The analytic behavior of these functions is much simpler when the ground field is non-archimedean; a good account of their analytic properties and some interesting formulas for certain class of such functions may be found in a recent paper of W. Casselman and J. A. Shalika [2]. But when the ground field is archimedean, these functions were believed to behave in a rather complicated manner. In fact, this has been one of the main obstacles in the development of the Hecke theory for number fields.

To make a more precise statement of the problem, we let G be a split reductive algebraic group over $\mathbf{R}$. We fix a maximal torus T of G and we let B be a fixed Borel subgroup of G containing T. We write $B = M_0AU$, the Langlands decomposition of B with $T = M_0A$, and fix a non-degenerate (unitary) character χ of U (see section 1).

Now, let π be a continuous representation of G on a Fréchet space V. Denote by (π_∞, V_∞) the corresponding differentiable representation. Topologize V_∞ with the relative topology inherited from $C^\infty(G, V)$. Let V_K be the subspace of K-finite vectors of V, where K is a fixed maximal compact subgroup of G with $G = KB$. We say that the representation (π, V) is non-degenerate, if there exists a continuous linear functional λ on V_∞, called a Whittaker functional, such that

$$\lambda(\pi(u)v) = \chi(u)\lambda(v) \qquad (u \in U, v \in V_\infty).$$

Then for every $v \in V_\infty$, the Whittaker function W_v is defined to be

$$W_v(g) = \lambda(\pi(g^{-1})v).$$

Received December 20, 1978. Revision received October 15, 1979. Supported by NSF Grant MCS 77-02181.

The corresponding embedding of V into the left regular representation of G is called a Whittaker model for π. When π is unitary and irreducible, it follows from [5] that this model is unique.

One may also consider the same concept for the representations of Lie algebras. They may be called "algebraically non-degenerate representations" as we do so in section 3 of the present paper, and they have been investigated by B. Kostant in a very interesting paper [7]. (The same results were obtained independently by W. Casselman and G. Zuckerman for the group $\mathrm{GL}_n(\mathsf{R})$.)

Now, let $\mathfrak{a}_{\mathbf{C}}^*$ be the complex dual of $\mathfrak{a}$, the real Lie algebra of A, and denote by $\hat{M}_0$ the set of all the (unitary) characters of M_0. Fix $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, $\eta \in \hat{M}_0$ and define the principal series representation $I(\nu, \eta)$ of G as in section 1. In [3] Jacquet proved that the Whittaker integrals (see proposition 1.1 here), which are originally defined only inside the positive Weyl chamber, can be continued to holomorphic functions of ν on the whole $\mathfrak{a}_{\mathbf{C}}^*$. His results are restricted to K-finite vectors and unfortunately do not extend easily to the smooth vectors.

The main results of this paper, Theorems 2.1 and 2.2, carry out this extension and show that these analytic continuations do in fact define a Whittaker functional for $I(\nu, \eta)$.

The proof follows the same ideas as those of Jacquet [3]. In section 2.1, we prove a result (corollary 2.1.1) similar to lemma 1.11 of [3]. This is done by means of a result of G. Schiffmann [12].

Intertwining integrals are discussed in section 2.2. They are necessary for the results proved in section 2.3 and are based on the results of [12].

Finally in section 2.3, we prove the main lemma, lemma 2.3.1, which replaces lemma 3.2 of [3]. The proofs of proposition 2.3.1 and theorem 2.1 now follow almost the same line as those of proposition 3.3 and theorem 3.4 of [3].

The proof of lemma 2.3.1 requires introduction of certain convergence factors (mainly Γ-functions) to replace the nice existing formulas for K-finite functions on $\mathrm{SL}_2(R)$ which no longer exist for the C^∞-functions on an arbitrary algebraic group of rank one. They seem to be related to the normalizing factors of intertwining integrals (see lemma 2.3.3).

The proofs are quite general, and we believe that they work for an arbitrary algebraic group. This is examined in the Appendix for a certain class of quasi-split groups (which are the only groups to be studied by [7]), where we have also given new proofs for some of the results of sections 2.2 and 2.3.

Finally in section 3, we make certain observations concerning the Whittaker functionals of a certain class of non-degenerate representations, and prove a result on non-degeneracy of the representations induced from such representations (proposition 3.2). In fact, in light of the results of W. Casselman, B. Kostant [7], D. Vogan [17], and N. Wallach, they are nothing but the functionals whose explicit existence is proved in section 2.

As it was mentioned before, the most important application of these results is the analytic continuation of the zeta functions of the pairs of representations of $\mathrm{GL}_n(\mathsf{R})$ and $\mathrm{GL}_m(\mathsf{R})$. This in turn will lead us to the definition of the corresponding local and global L-functions. This is the subject of a work in progress of H. Jacquet and J. A. Shalika. They may also be used to prove certain non-vanishing theorems for these L-functions (cf. [6]). Finally, they are necessary to establish certain functional equations (cf. [13] and [14]), and in fact this has been the first reason for the author to consider this problem.

I would like to thank Joseph Shalika, William Casselman and Nolan Wallach for their interest and many useful discussions and communications.

1. Preliminaries on non-degenerate representations. Let G be a split reductive algebraic group over $\mathbf{R}$. Fix a maximal split torus T of G over $\mathbf{R}$. We use B to denote a fixed Borel subgroup of G containing T. Put U for its unipotent radical. Let $\mathfrak{g}$ be the Lie algebra of G. We use $\mathfrak{U}(\mathfrak{g})$ to denote the complex universal enveloping algebra of $\mathfrak{g}$.

We write $B = M_0AU$, the Langlands' decomposition of B with $T = M_0A$. Let $\mathfrak{a}$ be the Lie algebras of A. By assumption on G, it is also the Lie algebra of T.

Let ψ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We use Δ, ψ^+, and ψ^- for simple, positive and negative roots, respectively. We have

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \psi} \mathfrak{g}_\alpha$$

with root spaces $\mathfrak{g}_\alpha$.

For $\alpha \in \psi$, we define H_α by

$$\hat{\alpha}(H) = \kappa(H, H_\alpha) \qquad \forall H \in [\mathfrak{a}, \mathfrak{a}],$$

where κ denotes the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ and

$$\hat{\alpha} = \frac{2\alpha}{(\alpha, \alpha)} .$$

We put $\rho = (1/2)\sum_{\alpha \in \psi^+} \alpha$.

Let W be the Weyl group of G. For every $\alpha \in \psi$ let $w_\alpha \in W$ be the corresponding reflection. Put w_l for the longest element of W. W acts on T in usual manner.

Let K be a maximal compact subgroup of G relative to T, so that $G = KB$. We shall use the same notation for the corresponding groups of $\mathbf{R}$-rational points.

We use $\hat{M}_0$ to denote the group of (unitary) characters of M_0. We put $\mathfrak{a}_{\mathbf{C}}^*$ for the complex dual of $\mathfrak{a}$. Then one-dimensional representations of B, trivial on U, are of the form ηe^ν, $\eta \in \hat{M}_0$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ where e^ν is defined by

$$e^\nu(a) = e^{\nu(\log a)}.$$

Here $\log : A \to \mathfrak{a}$ is the inverse of the exponential map $\exp : \mathfrak{a} \to A$.

Using the notation of [12], given w in W let:

$$S(w) = \{\nu \in \mathfrak{a}_{\mathbf{C}}^* \mid \operatorname{Re}(\nu(H_\alpha)) > 0, \forall \alpha \in \Delta(w)\}, \quad \text{where}$$

$$\Delta(w) = \{\alpha \in \psi^+ \mid w(\alpha) \in \psi^-\}.$$

We denote the principal series representation $\operatorname{Ind}_{B\uparrow G} \eta e^\nu$ of G by $I(\nu, \eta)$, $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $\eta \in \hat{M}_0$. More precisely, this is the left regular representation of G on the Hilbert space $V(\nu, \eta)$ of the complex functions f on G satisfying

(1) $f(xm_0au) = \eta(m_0^{-1})e^{-(\nu+\rho)(\log a)}f(x)$ for all $a \in A$, $m_0 \in M_0$, $u \in U$, $x \in G$, and

(2) $\int_K |f(k)|^2 dk < \infty$.

Now, let π be a continuous representation of G on a Fréchet space V. Denote by (π_∞, V_∞) the corresponding differentiable representation of G on the space of differentiable vectors of V. Then $\mathfrak{g}$ and consequently $\mathfrak{U}(\mathfrak{g})$ will act on the space V_K of K-finite vectors of V_∞, and V_K is dense in V_∞ when it is equipped with the relative topology induced from the Schwartz topology of $C^\infty(G, V)$. Consequently $V(\nu, \eta)_\infty$ and $V(\nu, \eta)_K$ will denote the smooth and K-finite functions in $V(\nu, \eta)$, respectively. From now on, we shall equip V_∞ with the relative Schwartz topology, With this topology V_∞ is a Fréchet space.

We call a character χ of U non-degenerate if its restriction to every non-trivial subgroup U_w, $U_w = wUw^{-1} \cap U$, $w \in W$, is non-trivial. Clearly $\chi = \prod_{\alpha \in \Delta} \chi_\alpha$, where each χ_α is a non-trivial character of U^α. Here U^α denotes the connected subgroup of U whose Lie algebra is $\mathfrak{g}_\alpha$. Throughout this paper, we shall fix a non-degenerate character χ of U.

Now, let (π, V) be a continuous representation of G as before. We shall call (π, V) a non-degenerate representation if there exists a continuous linear functional λ on V_∞ such that:

$$\lambda(\pi(u)v) = \chi(u)\lambda(v) \qquad u \in U, v \in V_\infty. \tag{1.1}$$

When π is irreducible and unitary, it follows from [15] that λ is unique. More precisely, if V_χ^* denotes the space of continuous linear functionals on V_∞ satisfying (1.1), then $\operatorname{Dim}_{\mathbf{C}} V_\chi^* \leqslant 1$. Such functionals are called Whittaker functionals.

Now, let $P \supset B$ be a parabolic subgroup of G. Put N for its unipotent radical. Fix a Levi component M for P with $M \supset T$. We have $P = MN$. Let π be a continuous representation of M on a Fréchet space V. By restriction χ can be considered as a non-degenerate character of $M \cap B$. Let (I, W) be the continuously induced representation $\operatorname{Ind}_{P\uparrow G} \pi$. We have the following result from [5] (see also section 5.2 in [9]).

THEOREM 1.1. *Let (π, V) and (I, W) be as above; then:*

$$\operatorname{Dim}_{\mathbf{C}} W_\chi^* \leqslant \operatorname{Dim}_{\mathbf{C}} V_\chi^*.$$

COROLLARY. *Let (I, W) be a principal series representation of G; then:*

$$\mathrm{Dim}_{\mathbf{C}} W_{\chi}^{*} \leqslant 1.$$

In fact we have:

PROPOSITION 1.1. *Let $I(\nu, \eta)$ be a principal series representation of G with $\nu \in S(w_l)$. Then $I(\nu, \eta)$ is non-degenerate and a Whittaker functional for $I(\nu, \eta)$ is given by*

$$\lambda(f) = \int_U f(uw_l)\chi(u)\,du \qquad f \in V(\nu, \eta)_\infty. \tag{1.2}$$

Proof. In fact the integral in the right hand side of (1.2) converges absolutely whenever $\mathrm{Re}(\nu(H_\alpha)) > 0$ for all $\alpha \in \psi^+$, and defines a nonzero continuous linear functional on $V(\eta, \nu)_\infty$ satisfying (1.1). More precisely, $V(\eta, \nu)_\infty$ is generated by the functions f_ϕ, $\phi \in C_c^\infty(G)$, defined by

$$f_\phi(g) = \int_{M_0 \times A \times U} \eta(m_0) e^{(\nu+\rho)(\log a)} \phi(gm_0au)\,dm_0\,da\,du. \tag{1.3}$$

Since Bw_lU is open in G, one may choose ϕ with support in Bw_lU such that $\lambda(f_\phi) \neq 0$. Also put $T(\phi) = \lambda(f_\phi)$. Then T is a distribution on G (cf. [12]), and it satisfies the same invariance properties as those in the proof of theorem 1.3 (cf. [13]). It follows from the uniqueness of such distributions that λ is continuous. This completes the proposition.

Now, let G_D be the derived group of G. Denote by $\tilde{G}_D$ the simply connected covering of G_D. There exists a homomorphism α from $\tilde{G}_D$ onto G_D which is an isomorphism on U. For a function f in $V(\nu, \eta)$, we define a function $\tilde{f}$ by:

$$\tilde{f}(g) = f(\alpha(g)) \qquad g \in \tilde{G}_D$$

Then $\tilde{f}$ belongs to a principal series representation of $\tilde{G}_D$. Let $\tilde{\lambda}$ be the Whittaker functional for this principal series representation of $\tilde{G}_D$ defined by (1.2) (we assume that $\nu \in S(w_l)$). Clearly $\lambda(f) = \tilde{\lambda}(\tilde{f})$, $f \in V(\nu, \eta)_\infty$, where λ denotes the Whittaker functional for G again defined by (1.2). Put $\lambda_K = \lambda \mid V(\nu, \eta)_K$. Applying Jacquet's results [3] to $\tilde{G}_D$, one may extend λ_K to an entire function for all values of $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. The purpose of the next several sections is to extend Jacquet's result to λ. More precisely, we shall prove that, as a function of ν, the functional λ defined by (1.2) can be continued to a holomorphic function on $\mathfrak{a}_{\mathbf{C}}^*$.

2. Whittaker functionals for the principal series. In this section we shall assume that G is semi-simple and simply connected. We fix ϕ in $C_c^\infty(G)$ and define:

$$f_\phi(g, \nu, \eta) = \int_{M_0 \times A \times U} \eta(m_0) e^{(\nu+\rho)(\log a)} \phi(gm_0au)\,dm_0\,da\,du. \tag{2.1}$$

Then $f_\phi \in V(\nu,\eta)_\infty$. Now put:

$$W_\phi(g,\nu,\eta) = \int_U f_\phi(guw_l,\nu,\eta)\chi(u)\,du. \tag{2.2}$$

The integral converges absolutely for $\nu \in S(w_l)$. Furthermore:

$$\lambda(f_\phi) = w_\phi(e,\nu,\eta). \tag{2.3}$$

We shall prove

THEOREM 2.1. *Fix $\phi \in C_c^\infty(G)$; then $W_\phi(g,\nu,\eta)$ extends to an entire function of ν on $\mathfrak{a}_{\mathbf{C}}^*$.*

Now put $\lambda(\nu,\eta)$ for the Whittaker functional λ defined by (1.2). As a consequence of theorem 2.1 we shall also prove

THEOREM 2.2. *Suppose G is split and reductive. Then the analytic continuation of $\lambda(\nu,\eta)$ defines a Whittaker functional for each $I(\nu,\eta)$, where $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $\eta \in \hat{M}_0$.*

The proof of theorem 2.1 follows the same line as that of theorem 3.4 of [3] which proves the same result for the K-finite functions. Consequently it is long and we shall do it in several steps. This is done in the next several sections.

2.1. *The case* $G = \mathrm{SL}_2(\mathbf{R})$. Suppose that $G = \mathrm{SL}_2(\mathbf{R})$. Choose ϕ in $C_c^\infty(G/U)$ and put:

$$f_\phi(g,\nu,\eta) = \int_{M_0\times A} \eta(m_0)e^{(\nu+\rho)(\log a)}\phi(gm_0a)\,dm_0\,da.$$

For $\nu \in S(w_l)$ define $W_\phi(g,\nu,\eta)$ using (2.2). Let $V = w_l^{-1}Uw_l$. Then for $\nu \in S(w_l)$ we have

$$W_\phi\left(w_l^{-1},\nu,\eta\right) = \int_{M_0\times A\times V} \eta(m_0)e^{(\nu+\rho)(\log a)}\phi(vm_0a)\chi'(v)\,dm_0\,da\,dv,$$

where $\chi'(v) = \chi(w_lvw_l^{-1})$ is a non-degenerate character of V. As in [12], changing a to a^2, we have

$$W_\phi\left(w_l^{-1},\nu,\eta\right) = 2\int_{M_0\times A\times U} \eta(m_0)e^{2(\nu+\rho)(\log a)}\phi\left(vm_0a^2\right)\chi'(v)\,dm_0\,da\,dv,$$

where we again assume $\nu \in S(w_l)$.

Let us assume that $\eta \equiv 1$, and consequently consider

$$W_\phi\left(w_l^{-1},\nu,1\right) = 2\int_{M_0\times A\times V} e^{2(\nu+\rho)(\log a)}\phi\left(vm_0a^2\right)\chi'(v)\,dm_0\,da\,dv.$$

Now changing v to ava^{-1}, we have

$$W_{\phi}\left(w_l^{-1}, \nu, 1\right) = 2\int_A e^{2\nu(\log a)}\, da \int_V \phi(ava)\chi'(ava^{-1})\, dv$$
$$+2\int_A e^{2\nu(\log a)}\, da \int_V \phi(-ava)\chi'(ava^{-1})\, dv.$$

Further, consider

$$W'_{\phi}\left(w_l^{-1}, \nu, 1\right) = \int_A e^{2\nu(\log a)}\, da \int_V \phi(ava)\chi'(ava^{-1})\, dv. \tag{2.1.1}$$

Now, put

$$a_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \qquad t > 0.$$

Then

$$W'_{\phi}\left(w_l^{-1}, \nu, 1\right) = \int_0^{+\infty} t^{2\nu} f(t)\, d^* t,$$

where

$$f(t) = \int_V \phi(a_t v a_t)\chi'\left(a_t v a_t^{-1}\right) dv. \tag{2.1.2}$$

Again we assume that $\nu \in S(w_l)$ to justify our computation. It is a result of G. Schiffmann [12] that $W'_{\phi}(g, \nu, 1)$, and consequently $W_{\phi}(g, \nu, \eta)$ extend to entire functions of ν on $\mathfrak{a}_{\mathbf{C}}^*$.

Now, put $\nu = \alpha + i\beta$ and fix a real number $m > 0$. We shall prove the following proposition which is an analogue of lemma 1.11 of [3].

PROPOSITION 2.1.1. *The function $W_{\phi}(g, \nu, \eta)$ vanishes uniformly as $|\beta| \to +\infty$ for $|\alpha| \leqslant m$.*

As we observed we may study $W'_{\phi}(w_1^{-1}, \nu, 1)$ defined by (2.1.1). We need the following result from [12].

PROPOSITION 2.1.2. (G. Schiffmann). *Let f be the function defined by* (2.1.2). *Then f is a continuous function of compact support in* $[0, +\infty)$. *Furthermore, given any positive integer n, there exists a constant $M_n > 0$ such that*

$$|f(t)| \leqslant M_n t^{2n}.$$

Now, put:

$$f_\alpha(t) = t^{2\alpha} f(t) \qquad |\alpha| \leqslant m.$$

Then $\lim_{t\to 0} f_\alpha(t)=0$, and $f_\alpha \in L^1(\mathbf{R})$. Changing t to e^x, we have:

$$W'_\phi(w_l^{-1},\nu,1)=\int_{-\infty}^{+\infty} f_\alpha(x)e^{2i\beta x}\,dx$$
$$=\hat{f}_\alpha(2\beta),$$

where $\hat{f}_\alpha$ denotes formally the Fourier transform of f_α.

We need

LEMMA 2.1.1. *For $|\alpha|\leqslant m$, the family $\{f_\alpha\}_\alpha$ is uniformly equicontinuous.*

Proof. f_α has support inside the support of f. First let $t\to 0$. We may assume $0<|t|<1$. Fix $n_0>m$, an integer; then by proposition 2.1.2 we have

$$|f_\alpha(t)|\leqslant M_{n_0}|t|^{2(n_0-m)},\qquad |\alpha|\leqslant m.$$

Now given $\epsilon>0$, take $\delta=(\epsilon/M_{n_0})^{1/2(n_0-m)}>0$. Hence for $|t|<\delta$, we have $|f_\alpha(t)|<\epsilon$ which implies that f_α is equicontinuous at 0.

Now let $t_0\neq 0$ be arbitrary in the support of f. Fix $\gamma>0$ such that $t_0\in[\gamma,A]$, where $A>0$ is so that support of f is contained in $[0,A]$.

Given $\epsilon'>0$, there exists a step function $s_{\epsilon'}(t)$ such that

$$|f(t)-s_{\epsilon'}(t)|<\epsilon'\qquad t\in[\gamma,A].$$

Now consider the continuous map g

$$g:[\gamma,A]\times[-m,m]\to\mathbf{C}$$

defined by $g(t,\alpha)=t^{2\alpha}$. Put $C=[\gamma,A]\times[-m,m]$, and set $B=\sup_{(t,\alpha)\in C}|g(t,\alpha)|$. Then:

$$|t^{2\alpha}f(t)-t^{2\alpha}s_{\epsilon'}(t)|<\epsilon' B\qquad t\in[\gamma,A],$$

and

$$|t_0^{2\alpha}f(t_0)-t_0^{2\alpha}s_{\epsilon'}(t_0)|<\epsilon' B.$$

From uniform continuity of g we conclude that there exists $\delta>0$ such that

$$|t^{2\alpha}s_{\epsilon'}(t)-t_0^{2\alpha}s_{\epsilon'}(t_0)|<\epsilon'|s_{\epsilon'}(t_0)|$$

for $|t-t_0|<\delta$. Now, set $M=\sup_{t\in[\gamma,A]}|f(t)|$, and $\epsilon=\epsilon'(2B+M)$. Then

$$|f_\alpha(t)-f_\alpha(t_0)|<\epsilon$$

for $|t-t_0|<\delta$. Consequently $f_\alpha(t)$ is equicontinuous everywhere. The uniform equicontinuity follows from the compactness of the support of f. The lemma is now complete.

Proof of proposition 2.1.1. It can be shown that

$$2|\hat{f}_\alpha(\beta)| \leqslant \|f_\alpha - R_{-\pi/\beta} f_\alpha\|_1.$$

where $(R_{-\pi/\beta} f_\alpha)(t) = f_\alpha(t - \pi/\beta)$ and $\| \ \|_1$ denotes the L^1-norm. By lemma 2.1.1., given $\epsilon > 0$, there exists $0 < \delta(\epsilon) < A$ such that

$$|f_\alpha(s) - f_\alpha(t)| < (3A)^{-1} \cdot \epsilon$$

for every s and t in the support of f with $|s - t| < \delta(\epsilon)$. Then

$$\begin{aligned} \|f_\alpha - R_{-\pi/\beta} f_\alpha\|_1 &= \int_{-\infty}^{+\infty} \left| f_\alpha(t) - f_\alpha\left(t - \frac{\pi}{\beta}\right) \right| dt \\ &< (3A)^{-1} \cdot \epsilon \cdot (2A + \delta(\epsilon)) \\ &< \epsilon \end{aligned}$$

for $|\pi/\beta| < \delta(\epsilon)$ and the proposition follows.

Now, put:

$$L(\eta, \nu) = \pi^{-\frac{1}{2}(\epsilon + \nu)} \Gamma\left(\frac{1}{2} (\epsilon + \nu) \right)$$

where

$$\eta(m) = \operatorname{sgn}(m)^\epsilon \qquad \epsilon = 0, 1.$$

Also for a Schwartz function ϕ, put:

$$L_\phi(\eta, \nu) = \int_{\mathbf{R}^*} \phi(t) \eta(t) |t|^\nu \, d^* t.$$

Then from [16] we have:

$$L_\phi(\bar{\eta}, 1 - \nu) = \gamma(\eta, \nu) L_{\hat{\phi}}(\eta, \nu),$$

where

$$\gamma(\eta, \nu) = \epsilon(\eta, \nu) \frac{L(\bar{\eta}, 1 - \nu)}{L(\eta, \nu)}$$

with $\epsilon(\eta, \nu) = i^\epsilon$.

We need the following lemma

LEMMA 2.1.2. *Let a and b be two positive real numbers, $0 < a \leqslant b$. Then $\Gamma(s)$ vanishes uniformly for* $\operatorname{Re}(s) \in [a, b]$ *as* $|\operatorname{Im}(s)|$ *approaches* $+\infty$.

Proof. This follows from the formula

$$\Gamma(s) = \sqrt{2\pi}\, s^{s - 1/2} e^{-s} e^{J(s)}.$$

COROLLARY 2.1.1. *Fix $g \in G$; then*

$$|W_\phi(g,\nu,\eta)| \cdot |L(\bar{\eta}, 1+\nu)|$$

vanishes uniformly for $\mathrm{Re}(\nu) \in [a,b]$, $a > -1$, *as* $|\mathrm{Im}(\nu)|$ *approaches* $+\infty$.

Proof. $|L(\bar{\eta}, 1+\nu)| = \pi^{-1/2(\epsilon+\mathrm{Re}(\nu))}|\Gamma(1/2(\epsilon+\nu+1))|$, and the corollary follows from proposition 2.1.1 and lemma 2.1.2.

2.2. *Intertwining operators.* Suppose now that G is of arbitrary rank. Let $f_\nu \in V(\nu,\eta)_\infty$. Fix $w \in W$ and suppose that $\nu \in S(w)$. As in [12], define:

$$A(\nu,\eta,w)f_\nu(g) = \int_{V_w} f_\nu(gwv)\,dv,$$

where $V_w = V \cap w^{-1}Uw$. The integral converges absolutely since $\nu \in S(w)$ and defines an intertwining operator between $I(\eta,\nu)_\infty$ and $I(w(\nu), w(\eta))_\infty$ (cf. [12]).

Let us first recall certain results from [12]. Put:

$$\Phi_{w(\nu)}(g) = \frac{1}{\Gamma_w(\nu)} \int_{V_w} f_\nu(gwv)\,dv \qquad \nu \in S(w),$$

where

$$\Gamma_w(\nu) = \prod_{\alpha \in \Delta(w)} \Gamma(\nu_\alpha)$$

with $\nu_\alpha = \nu(H_\alpha)$ and

$$\Delta(w) = \{\alpha \in \psi^+ \mid w(\alpha) \in \psi^-\}$$

as before.

Now put $w = w_\alpha$, $\alpha \in \Delta$. Then in the proof of lemma 2.1 of [12], Schiffmann has shown that:

$$\Gamma(\nu_\alpha)\Phi_{w_\alpha(\nu)}(g) = J'_\nu(g) + J''_\nu(g) \tag{2.2.1}$$

with

$$J'_\nu(g) = \int_1^{+\infty} \phi_\nu(g,t)t^{\nu_\alpha - 1}\,dt$$

and

$$J''_\nu(g) = \int_0^1 \phi_\nu(g,t)t^{\nu_\alpha - 1}\,dt.$$

Here ϕ_ν is a function satisfying

LEMMA 2.2.1. (G. Schiffmann). *The function ϕ_ν has the following properties*:
(a) *ϕ_ν is C^∞.*

(b) *For every integer* $n \geqslant 0$, *the function* $(\partial^n/\partial t^n)\phi_\nu(g,t)$ *is a continuous function of* (ν, g, t) *which is analytic with respect to* ν.

(c) *Given* Ω *a compact subset of* G, *and* Ω' *a compact subset of* $\mathfrak{a}^*_{\mathbf{C}}$, *there exists a continuous semi-norm* μ *on* $V(\nu,\eta)_\infty$ *which depends only on* Ω *and* Ω' *such that*:

$$|\phi_\nu(g,t)| \leqslant \mu(f_\nu)|t|^{-\mathrm{Re}(f_\nu)-\rho_\alpha}$$

for $|t| \geqslant 1$, $g \in \Omega$, *and* $\nu \in \Omega'$.

(d) *Given* Ω *and* Ω' *as above and a non-negative integer* n, *there exists a continuous semi-norm* μ_n *on* $V(\nu,\eta)_\infty$ *which depends only on* Ω, Ω' *such that*:

$$\left|\left(\frac{\partial^n}{\partial t^n}\right)\phi_\nu(g,t)\right| \leqslant \mu_n(f_\nu)$$

for $|t| \leqslant 1$, $g \in \Omega$, *and* $\nu \in \Omega'$.

Let $\eta_\alpha = \eta \mid M_\alpha$. Write $\eta_\alpha(m_\alpha) = \mathrm{sgn}(m_\alpha)^{\epsilon_\alpha}$. We now prove

PROPOSITION 2.2.1. (a) *Suppose* $\epsilon_\alpha = 1$; *then for every even non-negative integer* r:

$$\left(\frac{\partial^r}{\partial t^r}\right)\phi_\nu(g,0) = 0. \tag{2.2.2}$$

(b) *Suppose* $\epsilon_\alpha = 0$; *then* (2.2.2) *holds for every odd positive integer* r.

Proof. The possible poles of

$$\Gamma(\nu_\alpha)\Phi_{w_\alpha(\nu)}(g)$$

are provided by $J_\nu''(g)$ which has the following Taylor expansion (cf. [12])

$$\begin{aligned} J_\nu''(g) = {} & \sum_{r=0}^{n-1} \frac{1}{r!\,(r+\nu_\alpha)}\left(\frac{\partial^r}{\partial t^r}\right)\phi_\nu(g,0) \\ & + \frac{1}{(n-1)!}\int_0^1\int_0^1 (1-u)^{n-1}t^{\nu_\alpha+n-1}\left(\frac{\partial^n}{\partial t^n}\right)\phi_\nu(g,ut)\,du\,dt. \end{aligned} \tag{2.2.3}$$

Now, let $\Upsilon(\mathfrak{D},\eta_\alpha,\nu_\alpha)$ be the operator defined by Jacquet in corollary 1.10 of [3], where $\mathfrak{D}$ is a representation of K. Let ${}^0\mathfrak{v}$ and ${}^0\tilde{\mathfrak{v}}$ be two vectors in the space of $\mathfrak{D}$ and its contragredient, respectively, and define the following function in $V(\nu,\eta)_K$

$$f_\nu(g) = \langle \mathfrak{D}(k_g)^0\mathfrak{v}, {}^0\tilde{\mathfrak{v}}\rangle \eta(m_g^{-1})e^{-(\nu+\rho)(\log a_g)}, \tag{2.2.4}$$

where $g = k_g m_g a_g u_g$ and ${}^0\mathfrak{v}$ satisfies:

$$\mathfrak{D}(m)^0\mathfrak{v} = \eta(m^{-1})^0\mathfrak{v}, \qquad m \in M_0$$

We need the following lemma from [13]:

LEMMA 2.2.2. *Suppose f_ν is defined by* (2.2.4). *Then for* $\mathrm{Re}(\nu_\alpha) > 0$:

$$\gamma(\bar{\eta}_\alpha, \nu_\alpha)\int_{V_\alpha} f_\nu(gw_\alpha v)\,dv$$

$$= (w_\alpha\eta)(m_g^{-1})e^{-(w_\alpha\nu+\rho)(\log a_g)}\langle\mathfrak{D}(k_g)\Upsilon(\mathfrak{D}, \eta_\alpha, \nu_\alpha)^0 v, {}^0\tilde{v}\rangle. \tag{2.2.5}$$

where $V_\alpha = V_{w_\alpha}$.

Let us substitute the value of $\gamma(\bar{\eta}_\alpha, \nu_\alpha)$ and use (2.2.1). Then the left hand side of (2.2.5) is equal to

$$\epsilon(\bar{\eta}_\alpha, \nu_\alpha)L(\eta_\alpha, 1-\nu_\alpha)\left[\frac{1}{L(\bar{\eta}_\alpha, \nu_\alpha)}J'_\nu(g) + \frac{1}{L(\bar{\eta}_\alpha, \nu_\alpha)}J''_\nu(g)\right].$$

Now suppose that in the expansion (2.2.3) n is so chosen that $\mathrm{Re}(\nu_\alpha) > -n$; then the poles will appear only at $\nu_\alpha = 0, -1, \ldots, -(n-1)$ (cf. [12]). Suppose $\mathrm{Re}(\nu_\alpha) < 0$. Then $L(\eta_\alpha, 1-\nu_\alpha)$ has no pole and no zero. It has been proved by H. Jacquet [3] that the right hand side of (2.2.5) is analytic for $\mathrm{Re}(\nu_\alpha) < 1$. Consequently the poles of $J''_\nu(g)$ must be among those of $L(\bar{\eta}_\alpha, \nu_\alpha)$, or equally those of $\Gamma(1/2(\nu_\alpha + \epsilon_\alpha))$. Hence when $\epsilon_\alpha = 0$, there should be no pole for odd negative ν_α and consequently (2.2.2) holds for odd positive integers. The case of $\epsilon_\alpha = 1$ follows in the same way. This proves the proposition for the K-finite function. Now, suppose that f_ν is any function in $V(\nu, \eta)_\infty$. Choose a sequence $\{f_\nu^n\}_n$ of K-finite functions in $V(\nu, \eta)_K$ approaching to f_ν in the semi-norms topology. By part (d) of lemma 2.2.1, there exists a semi-norm μ_r such that:

$$\left|\left(\frac{\partial^r}{\partial t^r}\right)\phi_\nu(g, t)\right| \leqslant \mu_r(f_\nu)$$

for g and ν in the compact subsets of G and $\mathfrak{a}_{\mathbf{C}}^*$, respectively, and $|t| \leqslant 1$. Now, if $f_\nu^n \to f_\nu$, then $(\partial^r/\partial t^r)\phi_\nu^n(g, t) \to (\partial^r/\partial t^r)\phi_\nu(g, t)$ for $|t| \leqslant 1$, and the proposition follows for every f_ν in $V(\nu, \eta)_\infty$.

COROLLARY. *Let G be a split, semi-simple, and simply connected real algebraic group. Fix $w \in W$ and write $w = w_{\alpha_m}, \ldots, w_{\alpha_1}$, $\alpha_i \in \Delta$ in the reduced form. Define $A(\nu, \eta, w)$ as before, and put $\nu^1 = \nu$, $\nu^{j+1} = w_{\alpha_j}\nu^j$; $\eta^1 = \eta$, $\eta^{j+1} = w_{\alpha_j}\eta^j$, $j = 1, \ldots, m$. Write $\eta^j_{\alpha_j}(m_{\alpha_j}) = \mathrm{sgn}(m_{\alpha_j})^{\epsilon_{\alpha_j}}$, $m_{\alpha_j} \in M_{\alpha_j}$, $\epsilon_{\alpha_j} = 0, 1$, $j = 1, \ldots, m$; then*

(a) *Suppose $\epsilon_{\alpha_j} = 1$, $j = 1, \ldots, m$, and $\nu^j(H_{\alpha_j})$, $j = 1, \ldots, m$, are non-positive even integers; then $A(\nu, \eta, w)$ has no pole.*

(b) *Suppose $\epsilon_{\alpha_j} = 0$, $j = 1, \ldots, m$, and $\nu^j(H_{\alpha_j})$, $j = 1, \ldots, m$; are odd negative integers; then $A(\nu, \eta, w)$ has no pole.*

2.3 *Analytic continuation of Whittaker functions.* Let ϕ be in $C_c^\infty(G)$ and fix $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $\eta \in \hat{M}_0$. Define f_ϕ by (2.1) and fix $g \in G$. The restriction of $L_{g^{-1}}f_\phi$ to

G_α (the subgroup whose Lie algebra is generated by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{a}$) is a member of $V(\nu_\alpha, \eta_\alpha)_\infty$. Let $\nu \in S(w_\alpha)$ and put

$$W_{\phi,g}(e, \nu_\alpha, \eta_\alpha) = \int_{V_\alpha} f_\phi(gv)\chi'(v)\, dv. \tag{2.3.1}$$

Fix a real number $0 < b < 1$ and let Ω' be a compact subset of $\mathfrak{a}^*_{\mathbf{C}}$ such that $|\mathrm{Re}(\nu_\alpha)| \leqslant b$ for all $\nu \in \Omega'$.

We need the following lemma which is the analogue of part (iii) of lemma 3.2 of [3].

LEMMA 2.3.1. *Fix $0 < b < 1$ and Ω' as before; then there exists a constant $B_\phi(\Omega') > 0$ such that*

$$|W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)| \leqslant B_\alpha(\Omega')c(\nu_\alpha)\int_{V_\alpha} f_{\nu_b}(gv)\, dv \tag{2.3.2}$$

for all $\nu \in \Omega'$, where $c(\nu_\alpha)$ is a continuous function of ν_α for $|\mathrm{Re}(\nu_\alpha)| \leqslant b$ and

$$f_{\nu_b}(km_0au) = e^{-(\rho + \nu_b)(\log a)}$$

with

$$\nu_b = \mathrm{Re}(\nu) - \frac{1}{2}(\mathrm{Re}(\nu_\alpha) - b).$$

Proof. The function

$$W_{\phi,g}(e, \nu_\alpha, \eta_\alpha) \cdot L(\bar{\eta}_\alpha, 1 + \nu_\alpha)$$

is analytic on the strip $|\mathrm{Re}(\nu_\alpha)| \leqslant \mathrm{b} < 1$, and furthermore by corollary 2.1.1 vanishes uniformly as $|\mathrm{Im}(\nu_\alpha)| \to +\infty$. Therefore we may apply the maximal principle to conclude that for $|\mathrm{Re}(\nu_\alpha)| \leqslant b$

$$\begin{aligned} &|W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)| \\ &\quad\leqslant |L(\bar{\eta}_\alpha, 1 + \nu_\alpha)|^{-1} \cdot \sup_{|\mathrm{Re}(\nu_\alpha)| = b} |W_{\phi,g}(e, \nu_\alpha, \eta_\alpha) \cdot L(\bar{\eta}_\alpha, 1 + \nu_\alpha)| \end{aligned} \tag{2.3.3}$$

Here ν is a fixed element of Ω' and the sup is being taken over the set:

$$\{\nu' \in \mathfrak{a}^*_{\mathbf{C}} \mid \nu'_\beta = \nu_\beta \quad \text{for} \quad \beta \in \Delta,\ \beta \neq \alpha, |\mathrm{Re}(\nu'_\alpha)| = b\}.$$

Now we shall compute the right hand side of (2.3.3).

Suppose first that $\mathrm{Re}(\nu_\alpha) = b$. Then from (2.3.1), it follows that:

$$\begin{aligned} |W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)| &\leqslant \int_{V_\alpha} |f_\phi(gv)|\, dv \\ &\leqslant \nu_{K,1}(f_\phi)\int_{V_\alpha} f_{\nu_b}(gv)\, dv \end{aligned} \tag{2.3.4}$$

where

$$\nu_{K,1}(f_\phi) = \sup_{g \in K} |f_\phi(g)| \tag{2.3.5}$$

and

$$f_{\nu_b}(km_0au) = e^{-(\nu_b+\rho)(\log a)}$$

Let us now introduce some notation from [12]. Given a compact subset Ω of G, a compact subset Ω' of $\mathfrak{a}_{\mathbf{C}}^*$, and a distribution X on G with support at the origin, we define

$$\nu_{\Omega,\Omega',X}(f_\phi) = \sup_{\substack{g \in \Omega \\ \nu \in \Omega'}} |X^* f_\phi(g)|.$$

Here we have identified the universal enveloping algebra with the algebra of distribution with support at the origin so that the topology of these semi-norms is the same as the Schwartz topology of $V(\nu, \eta)_\infty$ explained in section 1. This justifies the indices in (2.3.5) where we have eliminated $\{\nu\}$.

Now, set

$$A'(\nu, \eta, w_\alpha) = \gamma(\bar{\eta}_\alpha, \nu_\alpha) A(\nu, \eta, w_\alpha),$$

and suppose $\mathrm{Re}(\nu_\alpha) < 0$. Take $f \in V(\nu, \eta)_\infty$; then:

$$\int_{V_\alpha} A'(\nu, \eta, w_\alpha) f(gv) \chi'(v)\, dv$$

is absolutely convergent and defines a holomorphic function of ν. More precisely, this follows from proposition 2.2.1 which states that $A'(\nu, \eta, w_\alpha)f$ is holomorphic whenever $\mathrm{Re}(\nu_\alpha) < 0$. We need the following lemma.

LEMMA 2.3.2. *Suppose that* $\mathrm{Re}(\nu_\alpha) < 0$. *Then*:

$$W_{\phi,g}(e, \nu_\alpha, \eta_\alpha) = \int_{V_\alpha} A'(\nu, \eta, w_\alpha) f_\phi(gv) \chi'(v)\, dv. \tag{2.3.6}$$

Proof. Put

$$W_f'(g) = \int_{V_\alpha} A'(\nu, \eta, w_\alpha) f(gv) \chi'(v)\, dv.$$

Then the map $f \longmapsto W_f'$ is continuous. Choose a sequence $\{f_n\}_n$ with $f_n \in V(\nu, \eta)_K$ so that $f = \lim_n f_n$. Consequently $\lim_n W_{f_n}' = W_f'$. Using Jacquet's functional equation (cf. [3]) together with lemma 2.2.2, we conclude that:

$$W_{f_n}(g) = W_{f_n,\, g}(e, \nu_\alpha, \eta_\alpha) = W_{f_n}'(g).$$

Hence the map $f \longmapsto W_f$ is also continuous for $f \in V(\nu, \eta)_K$. Consequently:

$$W_f = \lim_n W_{f_n} = \lim_n W'_{f_n} = W'_f$$

for all $f \in V(\nu, \eta)_\infty$. This completes the lemma.

Remark (1). The left hand side of (2.3.6) is defined by analytic continuation.

Remark (2). We have used $W_{f,g}(e, \nu_\alpha, \eta_\alpha)$ to denote $W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)$.

Now suppose $\mathrm{Re}(\nu_\alpha) = -b$. Using (2.3.6) we have:

$$|W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)| \leqslant |L(\eta_\alpha, 1 - \nu_\alpha)| \cdot \int_{V_\alpha} |\Phi_\phi(gv)|\, dv, \tag{2.3.7}$$

where Φ_ϕ is defined by

$$\Phi_\phi(g) = L(\bar{\eta}_\alpha, \nu_\alpha)^{-1} A(\nu, \eta, w_\alpha) f_\phi(g),$$

which itself is holomorphic for $\mathrm{Re}(\nu_\alpha) < 0$ by proposition 2.2.1 and its corollary.

From (2.3.7) we conclude:

$$|W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)| \leqslant |L(\eta_\alpha, 1 - \nu_\alpha)| \cdot \nu_{K,1}(\Phi_\phi) \cdot \int f_{\nu_b}(gv)\, dv \tag{2.3.8}$$

Now, set:

$$B_\phi(g, \nu, \eta) = \Phi_\phi(g) \cdot L(\bar{\eta}_\alpha, 1 + \nu_\alpha) \cdot L(\eta_\alpha, 1 - \nu_\alpha); \tag{2.3.9}$$

then we have

LEMMA 2.3.3. *Suppose* $\mathrm{Re}(\nu_\alpha) = -b$ *and let* ϕ *and* Ω' *be as before. Then there exists a continuous semi-norm* μ *depending only upon* Ω' *and* b *such that*:

$$\sup_{\substack{g \in K \\ \mathrm{Re}(\nu_\alpha) = -b}} |B_\phi(g, \nu, \eta)| \leqslant \pi^{-1/2(1+b+\epsilon_\alpha)} \Gamma\left(\frac{1}{2}(1 + b + \epsilon_\alpha)\right) \mu(c_{\nu_\alpha} f_\phi). \tag{2.3.10}$$

for all $\nu \in \Omega'$, *where*

$$c_{\nu_\alpha} = \frac{L(\bar{\eta}_\alpha, 1 + \nu_\alpha)}{L(\bar{\eta}_\alpha, \nu_\alpha)} \tag{2.3.11}$$

is the corresponding normalizing factor for $A(\nu, \eta, w_\alpha)$.

Proof. Using (2.2.1) and (2.2.3), (2.3.9) can be written as

$$B_\phi(g, \nu, \eta) = L(\eta_\alpha, 1 - \nu_\alpha) \cdot c_{\nu_\alpha} \left\{ \int_1^\infty \phi_\nu(g, t) t^{\nu_\alpha - 1}\, dt \right.$$

$$\left. + \frac{1}{\nu_\alpha} \phi_\nu(g, 0) + \int_0^1 \int_0^1 t^{\nu_\alpha} \left(\frac{\partial}{\partial t}\right) \phi_\nu(g, ut)\, du\, dt \right\}. \tag{2.3.12}$$

Now by lemma 2.2.1, we have the following inequalities.

$$|\phi_\nu(g,t)| \leqslant C \cdot \nu_{K,\Omega,1}(f_\phi)|t|^{-\mathrm{Re}(\nu_\alpha)-\rho_\alpha}, \tag{2.3.13}$$

with $g \in K$, $\nu \in \Omega'$, and $|t| \geqslant 1$, where C is a constant depending only upon $\mathrm{Re}(\nu_\alpha) = -b$ and Ω' (cf. the proof of lemma 2.2.1). Also there are constants $a_i \geqslant 0$, depending only upon $\mathrm{Re}(\nu_\alpha)$, and distributions X_i with support at the origin such that

$$\left|\left(\frac{\partial}{\partial t}\right)\phi_\nu(g,t)\right| \leqslant \sum_i a_i \nu_{K,\Omega',X_i}(f_\phi) \tag{2.3.14}$$

for $g \in K$, $\nu \in \Omega'$, and $|t| \leqslant 1$.

Let us now compute $|B_\phi(g,\nu,\eta)|$ by means of (2.3.12), (2.3.13), and (2.3.14); then

$$|B_\phi(g,\nu,\eta)| \leqslant |L(\eta_\alpha, 1-\nu_\alpha)\cdot c_{\nu_\alpha}| \cdot \left\{ C\nu_{K,\Omega',1}(f_\phi)\int_1^\infty t^{-\rho_\alpha-1}\,dt \right.$$

$$\left. + \frac{1}{|1+\nu_\alpha|} \sum_i a_i \nu_{K,\Omega',X_i}(f_\phi) \right\}, \tag{2.3.15}$$

where we are assuming that $\mathrm{Re}(\nu_\alpha) = -b > -1$, $g \in K$, and $\nu \in \Omega'$. Observe that we are considering the case $\epsilon_\alpha = 1$ which implies $\phi_\nu(g,0) = 0$ by proposition 2.2.1. The other case can be treated the same way.

Now for each distribution X with support at the origin, choose a function $\phi_X \in C_c^\infty(G)$ such that

$$|X^*\phi| \leqslant \phi_X.$$

Then clearly:

$$|X^* f_\phi(g)| \leqslant |f_{\phi_X}(g)|,$$

where $f_{\phi_X} \in V(\mathrm{Re}(\nu), 1)_\infty$ is defined by

$$f_{\phi_X}(g) = \int_{M_0\times A\times U} \phi_X(gm_0au)e^{(\mathrm{Re}(\nu)+\rho)(\log a)}\,dm_0\,da\,du.$$

Consequently

$$\sup_{\mathrm{Re}(\nu_\alpha)=-b} \nu_{K,\Omega',X}(f_\phi) \leqslant \nu_{K,\Omega',1}(f_{\phi_X}) \tag{2.3.16}$$

Now as $|\mathrm{Im}(\nu_\alpha)| \to +\infty$, $\mathrm{Re}(\nu_\alpha) = -b$ and ν is fixed in Ω', the coefficients:

$$|L(\eta_\alpha, 1-\nu_\alpha)\cdot c_{\nu_\alpha}|$$

and

$$\left| \frac{L(\eta_\alpha, 1-\nu_\alpha)}{(1+\nu_\alpha)} \cdot c_{\nu_\alpha} \right|$$

will approach to zero, while the terms $a_i \nu_{K,\Omega',X_i}(f_\phi)$ and $C \cdot \nu_{K,\Omega',1}(f_\phi)$ remain bounded by (2.3.16). Consequently there are constants $b_j > 0$ and another compact set Ω'' of $\mathfrak{a}^*_{\mathbf{C}}$, depending upon Ω' and b, such that

$$\sup_{\substack{g \in K \\ \mathrm{Re}(\nu_\alpha) = -b}} |B_\phi(g, \nu, \eta)| \leqslant \pi^{-1/2(1+b+\epsilon_\alpha)} \Gamma(1/2(1+b+\epsilon_\alpha)) \sum_j b_j \nu_{K,\Omega'',X_j}(c_{\nu_\alpha} \cdot f_\phi).$$

Now put:

$$\mu = \sum_j b_j \nu_{K,\Omega'',X_j}$$

to complete lemma 2.3.3.

Proof of lemma 2.3.1. For $\mathrm{Re}(\nu_\alpha) = b$, it follows from (2.3.4) and (2.3.16) that for $\nu \in \Omega'$

$$\sup_{\mathrm{Re}(\nu_\alpha)=b} |L(\bar{\eta}_\alpha, 1+\nu_\alpha) \cdot W_{\phi,\,g}(e, \nu_\alpha, \eta_\alpha)|$$

$$\leqslant \pi^{-1/2(1+b+\epsilon_\alpha)} \Gamma(1/2(1+b+\epsilon_\alpha)) \cdot \nu_{K,\Omega',1}(f_{\phi_1}) \int_{V_\alpha} f_{\nu_b}(gv)\, dv,$$

where $\phi_1 \in C_c^\infty(G)$ and satisfies $|\phi| \leqslant \phi_1$.

Suppose now that $\mathrm{Re}(\nu_\alpha) = -b$; then using (2.3.8) and lemma 2.3.3, we conclude that for $\nu \in \Omega'$

$$\sup_{\mathrm{Re}(\nu_\alpha)=-b} |L(\bar{\eta}_\alpha, 1+\nu_\alpha) W_{\phi,g}(e, \nu_\alpha, \eta_\alpha)|$$

$$\leqslant \sup_{\substack{g \in K \\ \mathrm{Re}(\nu_\alpha) = -b}} |B_\phi(g, \nu, \eta)| \cdot \int_{V_\alpha} f_{\nu_b}(gv)\, dv$$

$$\leqslant \pi^{-1/2(1+b+\epsilon_\alpha)} \Gamma(1/2(1+b+\epsilon_\alpha))\, \mu(c_{\nu_\alpha} \cdot f_\phi) \cdot \int_{V_\alpha} f_{\nu_b}(gv)\, dv.$$

Now set

$$B_\phi(\Omega') = \pi^{-1/2(1+b+\epsilon_\alpha)} \Gamma(1/2(1+b+\epsilon_\alpha)) \sup\{ \nu_{K,\Omega',1}(f_{\phi_1}), \mu(c_{\nu_\alpha} \cdot f_{\phi_1}) \}$$

and

$$c(\nu_\alpha) = L(\bar{\eta}_\alpha, 1+\nu_\alpha)^{-1}$$

to conclude lemma 2.3.1.

Now choose $\phi \in C_c^\infty(G)$ and define $f_\phi \in V(\nu, \eta)_\infty$. Suppose $\nu \in S(w_l)$. Then the integral:

$$\int_V f_\phi(gv)\chi'(v)\,dv$$

is absolutely convergent and defines a function $W_{f_\phi}(g, \nu, \eta)$, analytic in $\nu \in S(w_l)$. Here V is $w_l^{-1}Uw_l$ and $\chi'(v) = \chi(w_l v w_l^{-1})$.

As before put:

$$A'(\nu, \eta, w_\alpha) = \gamma(\bar{\eta}_\alpha, \nu_\alpha)A(\nu, \eta, w_\alpha)$$

for $\alpha \in \Delta$. We have:

Proposition 2.3.1. *Fix $\phi \in C_c^\infty(G)$ and define $f_\phi \in V(\nu, \eta)_\infty$ as before; then $W_{f_\phi}(g, \nu, \eta)$ extends to a holomorphic function of ν on the convex hull of $S(w_l) \cup w_\alpha S(w_l)$. Furthermore it satisfies*

$$W_{f_\phi}(g, \nu, \eta) = W_{A'(\nu, \eta, w_\alpha)f_\phi}(g, w_\alpha \nu, w_\alpha \eta). \tag{2.3.17}$$

Lemma 2.3.4. *Suppose $\nu \in w_\alpha S(w_l)$. Then*

$$W_{f_\phi}(g, \nu, \eta) = \int_V A'(\nu, \eta, w_\alpha)f_\phi(gv)\chi'(v)\,dv$$

Proof. This may be proved the same way as lemma 3.2, but this time we have to use proposition 3.3 of [3].

Proof of proposition 2.3.1. Suppose first that $\nu \in w_\alpha S(w_l)$. Then $A'(\nu, \eta, w_\alpha)$ is holomorphic, and therefore by lemma 2.3.4 $W_{f_\phi}(g, \nu, \eta)$ is also analytic. Now suppose $\nu \in S(w_l)$. Then

$$W_{f_\phi}(g, \nu, \eta) = \int_V f(gv)\chi'(v)\,dv. \tag{2.3.18}$$

Let V_θ be the subgroup of B generated by $\theta = \psi^- - \{-\alpha\}$. Then (2.3.18) can be written as

$$W_{f_\phi}(g, \nu, \eta) = \int_{V_\theta}\int_{V_\alpha} f_\phi(gv_\theta v_\alpha)\chi'(v_\alpha)\chi'(v_\theta)\,dv_\alpha\,dv_\theta$$

$$= \int_{V_\theta} W_{\phi, gv_\theta}(e, \eta_\alpha, \eta_\alpha)\chi'(v_\theta)\,dv_\theta. \tag{2.3.19}$$

All the computations are justified since $\nu \in S(w_l)$. Now fix a real number b, $0 < b < 1$. Let ν^0 be in the convex hull of $S(w_l) \cup w_\alpha S(w_l)$ with $\mathrm{Re}(\nu_\alpha^0) = 0$. Choose a compact neighborhood Ω' of ν^0 in the convex hull of $S(w_l) \cup w_\alpha S(w_l)$ with $|\mathrm{Re}(\nu_\alpha)| \leqslant b$ for all $\nu \in \Omega'$. Then by lemma 2.3.1 there exists $B_\phi(\Omega') > 0$ such that:

$$|W_{\phi,gv_\theta}(e,\nu_\alpha,\eta_\alpha)| \leqslant B_\phi(\Omega')c(\nu_\alpha)\int_{V_\alpha} f_{\nu_b}(gv_\theta v_\alpha)\,dv_\alpha \tag{2.3.20}$$

for all $\nu \in \Omega'$.

Using (2.3.20), (2.3.19) implies that for $\nu \in \Omega'$

$$|W_{f_\phi}(g,\nu,\eta)| \leqslant B_\Phi(\Omega')c(\nu_\alpha)\int_V f_{\nu_b}(gv)\,dv.$$

As ν varies over Ω', ν_b ranges over a compact subset of $S(w_l)$ and the last integral converges uniformly. Consequently (2.3.19) converges uniformly for $\nu \in \Omega$ and defines $W_{f_\phi}(g,\nu,\eta)$ as a holomorphic function of ν in a neighborhood of ν^0. This completes the proposition.

Remark. This is the substitute of proposition 3.3 of [3].

Proof of theorem 2.1. The proof follows exactly the same method as that of theorem 3.4 of [3]. Clearly:

$$W_\phi(g,\nu,\eta) = W_{f_\phi}(gw_l,\nu,\eta).$$

For $q > 0$, let B_q be the union of all $wS(w_l)$, $w \in W$, with $l(w) \leqslant q$, and the common walls between them of the type

$$S_\alpha = \{\nu \in \mathfrak{a}_{\mathbf{C}}^* \mid \mathrm{Re}(\nu(H_\alpha)) = 0\}, \qquad \alpha \in \Delta.$$

For every $\alpha \in \Delta$, W_{f_ϕ} has an analytic continuation to $S(w_l)$, $w_\alpha S(w_l)$, and S_α by proposition 2.3.1. Put

$$A_\alpha = S(w_l) \cup w_\alpha S(w_l) \cup S_\alpha.$$

Then $\bigcap_\alpha A_\alpha = S(w_l)$ which is connected. Consequently W_{f_ϕ} extends to a holomorphic function of ν on B_1. The proof is now an induction on q. As we just observed it is true for $q = 1$. Now suppose W_{f_ϕ} has an analytic extension to B_q. Following [3], we shall show that it extends to an analytic function on $B_q \cup w_\alpha B_q$ for every $\alpha \in \Delta$.

Suppose $\nu \in w_\alpha B_q$, but $\nu \notin B_q$. By (3.4.8) of [3], and proposition 2.2.1, $A'(\nu,\eta,w)$ is holomorphic. Define:

$$\psi(\nu) = W_{A'(\nu,\eta,w_\alpha)f_\phi}(g,w_\alpha\nu,w_\alpha\eta).$$

Now by induction hypothesis $\psi(\nu)$ is holomorphic for $\nu \in w_\alpha B_q$, but $\nu \notin B_q$. Furthermore by proposition 2.3.1.,

$$\psi(\nu) = W_{f_\phi}(g,\nu,\eta)$$

$\nu \in A_\alpha$. Consequently $\psi(\nu)$ defines an analytic continuation of W_{f_Φ} to a holomorphic function of ν on $B_q \cup w_\alpha B_q$.

From (3.4.7.1) of [3], it follows that $B_q \cup w_\alpha B_q$ and $B_q \cup w_\beta B_q$, α, $\beta \in \Delta$, have a connected intersection. Thus W_{f_Φ} extends to an analytic function on the union of all $B_q \cup w_\alpha B_q$, $\alpha \in \Delta$.

Now, an application of Hartog's theorem shows that W_{f_Φ} extends to an analytic function on the convex hull of this union which contains B_{q+1} by (3.4.6) of [3]. This completes the induction.

Let m be the maximum length of the elements of W, i.e. $m = l(w_l)$. Then B_m contains $\bigcup_{w \in W} wS(w_l)$. Again an application of Hartog's theorem extends W_{f_Φ} to the convex hull of $\bigcup_{w \in W} wS(w_l)$ which is $\mathfrak{a}_{\mathbf{C}}^*$. This completes the theorem.

Proof of theorem 2.2. We only have to show that the extension of $\lambda(\nu, \eta)$ by means of theorem 2.1. is continuous for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. For $\nu \in \overline{S(w_l)}$, the closure of $S(w_l)$, this follows from proposition 1.1 and the fact that the pointwise limit of distributions is again a distribution. For other values of ν, the theorem is a consequence of the functional equation (2.3.17) which now holds for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, lemma 2.2.1, lemma 3.4.2 of [3], and the same statement for $\nu \in \overline{S(w_l)}$.

COROLLARY. *Let (I, W) be a principal series representation of G; then*:

$$\mathrm{Dim}_{\mathbf{C}} W_\chi^* = 1.$$

Remark. This has also been proved by a completely different approach by B. Kostant [7].

3. Whittaker functionals for non-degenerate representations. In this section, we shall make certain observations concerning the Whittaker functionals of certain non-degenerate representations, and prove a result on non-degeneracy of the representations induced from such representations (cf. [11] for p-adic groups). Throughout this section, we shall make the following assumption on G (except at the end of this section).

Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of $\mathfrak{g}$. Denote by $G_{\mathbf{C}}$ a complex connected group containing G whose Lie algebra is $\mathfrak{g}_{\mathbf{C}}$. Let $\mathrm{Ad}(G_{\mathbf{C}})$ be the corresponding adjoint group, and set $\mathfrak{g} = \mathrm{ad}(\mathfrak{g})$. Put

$$G_{\mathrm{Max}} = \{ g \in \mathrm{Ad}(G_{\mathbf{C}}) \mid g \cdot \mathfrak{g} \subset \mathfrak{g} \}$$

Then as a Lie group, $G_0 = \exp(\mathfrak{g}_0)$ is the identity component of G_{Max}, and $G_{\mathrm{Max}} = FG_0$, where F is the set of all $a \in \exp(\mathrm{ad}\, \mathfrak{a}_{\mathbf{C}})$ such that $a^2 = \mathrm{Id}$ (cf. [7]).

Now let $Z(G)$ be the center of G. *We shall assume* $G \cong Z(G) \times G_{\mathrm{Max}}$. This includes $\mathrm{GL}_n(R)$ and $\mathrm{PGL}_n(R)$, but, for example, not $\mathrm{SL}_2(R)$.

Let $\mathfrak{u}$ denote the Lie algebra of U. We fix a non-singular homomorphism μ of Lie algebras from $\mathfrak{u}$ into $\mathbf{C}$ (cf. [7]). As before, we put $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_{\mathbf{C}})$.

Definition. *A $\mathfrak{U}(\mathfrak{g})$-module V is said to be (algebraically) non-degenerate, if there exists a linear function $\lambda \in V'$ (algebraic dual of V) such that*

$$X \cdot \lambda = \mu(X)\lambda \qquad \forall X \in \mathfrak{u}.$$

Now, from the results of B. Kostant [7] and D. Vogan [17], it follows that for every $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and every $\eta \in \hat{M}_0$, $V(\nu, \eta)_K$ has a unique non-degenerate subquotient which is a subrepresentation if $\nu \in \overline{S(w_l)}$ (the closure of the positive Weyl Chamber). We denote this subrepresentation by V_K^0. Now, let $\lambda(\nu, \eta)$ be the Whittaker functional defined by theorem 2.2. Define a Lie algebra homomorphism μ from $\mathfrak{u}$ to $\mathbf{C}$ by $\exp(\mu(X)) = \chi(\exp(X))$, $X \in \mathfrak{u}$.

Let $B(\nu, \eta)$ be the intertwining map $v \to w_v$ between $V(\nu, \eta)_\infty$ and its Whittaker model (cf. Introduction). Then using μ, $B(\nu, \eta)$ and consequently $\lambda(\nu, \eta)$ do not vanish on V_K^0. Let V_∞^0 be the closure of V_K^0 in $V(\nu, \eta)_\infty$. Then $\lambda(\nu, \eta) \mid V_\infty^0$ is a Whittaker functional for V_∞^0. If we denote by δ^0 the map

$$V(w_l\nu, w_l\eta)_K \xrightarrow{\delta^0} V_K^0 \longrightarrow 0 \qquad \nu \in \overline{S(w_l)},$$

then by a result of W. Casselman and N. Wallach, δ^0 is continuous with respect to the Schwartz topologies on $V(\nu, \eta)_\infty$ and $V(w_l\nu, w_l\eta)_\infty$, and consequently $\mathrm{Dim}_{\mathbf{C}}(V_\infty^0)_\chi^* = 1$.

Fix $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, $\eta \in \hat{M}_0$, and $w \in W$. Write $w = w_{\alpha_m} \dots w_{\alpha_1}$, $\alpha_i \in \Delta$, as in the corollary of proposition 2.1.1. Define

$$\gamma_w(\eta, \nu) = \prod_{j=1}^{m} \gamma\left(\eta_{\alpha_j}^j, \nu_{\alpha_j}^j\right).$$

where $\nu_{\alpha_j}^j = \nu^j(H_{\alpha_j})$. Now set

$$A'(\nu, \eta, w) = \gamma_w(\bar{\eta}, \nu) \cdot A(\nu, \eta, w).$$

When $w = w_l$, it follows from lemma 2.2.1 of this paper and lemma 3.4.2 of [3] that $A'(w_l\nu, w_l\eta, w_l)$ is holomorphic whenever $\nu \in \overline{S(w_l)}$. Now write

$$V(w_l\nu, w_l\eta)_K = W_1 \oplus W_2$$

with W_1 indecomposable and

$$W_1 \xrightarrow{\delta^0} V_K^0 \longrightarrow 0,$$

where we still use δ^0 for its restriction to W_1. Let A' denote the restriction of $A'(w_l\nu, w_l\eta, w_l)$ to W_1. Then, by Jacquet's functional equation $A' \not\equiv 0$. The following proposition is a simple consequence of the uniqueness of the non-degenerate quotient.

PROPOSITION 3.1. *Let $\nu \in \overline{S(w_l)}$. Then the image of A' in $V(\nu,\eta)_K$ is V_K^0, and furthermore every intertwining operator whose image is V_K^0, is in fact only a multiple of A'. In particular we may assume $\delta^0 = A'$.*

COROLLARY 3.1. *The image of the operator A' in $V(\nu,\eta)_K$, $\nu \in \overline{S(W_l)}$, is irreducible.*

Now, let π be a non-degenerate irreducible admissible representation of G on a Fréchet space V. Then V_K is algebraically non-degenerate. By Casselman's subrepresentation theorem (cf. [1, 9, 18]) and [8], choose $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $\eta \in \hat{M}_0$ such that

$$0 \longrightarrow V_K \xrightarrow{\delta'} V(\nu,\eta)_K$$

and

$$V(w\nu, w\eta)_K \xrightarrow{\delta} V_K \longrightarrow 0$$

for some $w \in W$. Observe that we may actually assume $\nu \in \overline{S(w_l)}$.

Let us call π a *fine* representation if every infinitesimal imbedding δ' of π into a principal series extends to a bicontinuous isomorphism $\bar{\delta}' : V_\infty \cong V_\infty^0$.

Remark. When the group $G = \mathrm{GL}_n(\mathrm{R})$ or $\mathrm{GL}_n(C)$, then it is a result of W. Casselman and N. Wallach that every irreducible unitary representation of G is fine.

Suppose now that π is fine. Then, the imbedding δ' of V_K inside $V(\nu,\eta)_K$ induces an isomorphism $\bar{\delta}' : V_\infty \cong V_\infty^0$, and consequently $\mathrm{Dim}_{\mathbf{C}} V_\chi^* = 1$.

Furthermore, we may normalize a Whittaker functional λ on π such that $\lambda_K \cdot \delta = \lambda(w\nu, w\eta)_K$; or by Jacquet's functional equation and proposition 3.1

$$\lambda_K = \lambda(\nu,\eta)_K \cdot \delta'. \tag{3.1}$$

Finally, suppose G is any *split reductive group*. Let P be a parabolic subgroup of G containing B. We fix a Levi factor M with $M \supset T$. Write $P = MN$. *We assume $M \cong Z(M) \times M_{\mathrm{Max}}$.* Let π be a non-degenerate irreducible admissible fine representation of M on a Fréchet space. Then

PROPOSITION 3.2. *The continuously induced representation* $\mathrm{Ind}_{P\uparrow G}\,\pi$ *is non-degenerate.*

Proof. Let $I_M(\nu,\eta)$ be a principal series representation of M such that

$$V_M(w_l^0\nu, w_l^0\eta)_K \xrightarrow{\delta_M} V_K \longrightarrow 0,$$

where w_l^0 denotes the longest element of the Weyl group of (M, T), $\nu \in \overline{S(w_l^0)}$, and $\eta \in \hat{\mathrm{M}}_0$. By theorem 1.1, we may assume $V_M(w_l^0\nu, w_l^0\eta)_K$ is indecomposable.

Then, the map $\bar{\delta}_M$ is in fact $A_M(w_l^0\nu, w_l^0\eta, w_l^0)$. Now, by inducing in stages,

$$\operatorname*{Ind}_{P\uparrow G} I_M(\nu,\eta)_\infty \overset{\alpha'}{\cong} I(\nu,\eta)_\infty,$$

and

$$\operatorname*{Ind}_{P\uparrow G} I_M\big(w_l^0\eta, w_l^0\eta\big)_\infty \overset{\alpha}{\cong} I\big(w_l^0\eta, w_l^0\eta\big)_\infty,$$

where

$$\alpha(f)(gm) = f(g)(m).$$

We need

LEMMA 3.1. $\alpha' \cdot (\bar{\delta}_M) \cdot \alpha^{-1} = A'(w_l^0\nu, w_l^0\eta, w_l^0)$.

Proof. Let $\alpha : f \to \tilde{f}$; then $\tilde{f}(g) = f(g)(e)$. Put $\beta = \alpha^{-1}$. Then $\operatorname{Ind}(\bar{\delta}_M)(f)(g) = \bar{\delta}_M(f(g))$ for $f \in \operatorname{Ind}_{P\uparrow G} I_M(w_l^0\nu, w_l^0\eta)_\infty$. Now, take $f = \beta(\tilde{f})$; then

$$\big(\operatorname{Ind}\big(\bar{\delta}_M\big) \cdot \beta\big)(\tilde{f})(g) = \bar{\delta}_M(\beta(\tilde{f})(g)). \tag{3.2}$$

But for $\nu \in S(w_l^0)$ and $m \in M$

$$\bar{\delta}_M(\beta(\tilde{f})(g))(m) = \gamma_{w_l^0}\big(w_l^0\bar{\eta}, w_l^0\nu\big)\int_{U_{w_l^0}} \tilde{f}\big(gmuw_l^0\big)\,du.$$

Set $\gamma = \operatorname{Ind}(\bar{\delta}_M) \cdot \beta$; then by (3.2) we have

$$(\alpha' \cdot \gamma)(\tilde{f})(g) = \gamma(\tilde{f})(g)(e) = A'\big(w_l^0\nu, w_l^0\eta, w_l^0\big)\tilde{f}(g)$$

and the lemma follows by analytic continuation.

Proof of proposition 3.2. Let δ'_M be the original imbedding of V_K in $V_M(\nu,\eta)_K$, and put $\theta = \operatorname{Ind}(\bar{\delta}'_M)$. Suppose $\operatorname{Ind}_{P\uparrow G}\pi_\infty$ is degenerate. Then $\lambda(\nu,\eta)\cdot\theta$ vanishes on $\operatorname{Ind}_{P\uparrow G}\pi_\infty$. Finally, using Jacquet's functional equation

$$\lambda\big(w_l^0\nu, w_l^0\eta\big) = \lambda(\nu,\eta)A'\big(w_l^0\nu, w_l^0\eta, w_l^0\big) = \lambda(\nu,\eta)\cdot\theta = 0.$$

which is a contradiction.

COROLLARY 3.2. *A Whittaker functional* λ *for* $\operatorname{Ind}_{P\uparrow G}\pi_\infty$ *is given by*

$$\lambda = \lambda(\nu,\eta)\cdot\operatorname{Ind}\big(\bar{\delta}'_M\big).$$

with $\nu \in \overline{S(w_l^0)}$.

Appendix. Quasi-split groups. In what follows, we shall explain how to extend the results of section 2, to certain quasi-split reductive real algebraic groups. Since the other results explained in section 3 are already true for such groups, this will extend the results of this paper to these groups. Observe that the results of section 3 only hold for certain quasi-split groups, namely when $G = Z(G) \times G_{\mathrm{Max}}$. Therefore let us still resume this assumption.

This extension follows the same lines as in section 2, but we have to limit ourselves only to the results of [12]. In fact, we shall prove all the results that we need from [3] for this larger class of groups. This is done by means of the results of [12].

We make the assumption that for any root α, 2α is not a root. Using the previous notation, the torus M_0A is no longer necessarily split. For the sake of simplicity we also assume $\eta = 1$, $\eta \in \hat{M}_0$. First observe that all the results proved in 2.1 hold for any group of rank one.

Now, let $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and fix an irreducible unitary representation σ of K on the Hilbert space H_σ. Denote by $P(\sigma)$, the projection operator on the subspace of the vectors $v \in H_\sigma$ satisfying

$$\sigma(m)v = v \qquad m \in M_0.$$

Consider the function $f_{\nu,\sigma}$ on G defined by

$$f_{\nu,\delta}(kamu) = \sigma(k)e^{-(\nu+\rho)(\log a)}p(\sigma).$$

For $w \in W$ and $\nu \in S(w)$, put

$$T(\nu, \sigma, w) = \int_{V_W} \sigma(wk_v)e^{-(\nu+\rho)(\log a_v)}P(\sigma)\,dv,$$

where $v = k_v a_v u_v$. Then proposition 3.2 of [12] implies that for $\nu \in S(w)$

$$f_{w\nu,\sigma}T(\nu, \sigma, w) = f^*_{\nu,\sigma}\Phi_{\nu,\sigma,w},$$

where $\Phi_{\nu,\sigma,w}$ is the measure defined in paragraph 1.8 of [12]. Now, we define a Whittaker function of type σ by

$$W_\sigma(g, \nu, 1) = \int_V f_{\nu,\sigma}(gv)\chi'(v)\,dv.$$

Suppose first that G has real rank 1. Then for ν not a nonpositive integer, theorem 3.3 of [12] implies

$$W_\sigma(g, w\nu, 1)T(\nu, \sigma, w) = W_\sigma(g, \nu, 1)\chi'(\Phi_{\nu,\sigma,w}). \tag{A.1}$$

Here w denotes the non-trivial element of W and $\chi'(\Phi_{\nu,\sigma,w})$ is the Fourier transform of $\Phi_{\nu,\sigma,w}$ with respect to χ'. Let $\mathfrak{v}$ denote the Lie algebra of $V = V_w$. Put $v = \exp(Y)$ with $Y \in \mathfrak{v}$. Observe that we are assuming that V is abelian. Suppose that

$$\chi'(\exp(Y)) = e^{-2\pi i\theta(Y)}$$

with $\theta \in \mathfrak{v}^*$, the real dual of $\mathfrak{v}$. Now, let

$$\gamma_\rho(1,\nu) = \pi^{-1/2(1-2\nu)} \frac{\Gamma(1/2(\rho-\nu))}{\Gamma(1/2\nu)},$$

where $\rho = p = \dim V$. Observe that when the dimension of V is one, $\gamma_1(1,\nu) = \gamma(1,\nu)$ defined in 2.1. Let $\| \ \|$ denote the Euclidean norm in $\mathfrak{v}^*$; we prove

LEMMA A.1. *Assume* $\|\theta\| = 1$; *then* $\chi'(\Phi_{\nu,\sigma,w}) = \gamma_\rho(1,\nu)^{-1}$

Proof. By (3.4.2) of [12]

$$\Phi_{\nu,\sigma,w} = N(v)^{\nu-\rho} = \|Y\|^{\nu-\rho}.$$

where $\| \ \|$ also denotes the Euclidean norm in $\mathfrak{v}$. Then

$$\chi'(\Phi_{\nu,\sigma,w}) = \int_{\mathfrak{v}} \|Y\|^{\nu-\rho}\chi'(\exp(Y))\,dY$$

$$= \int_V N(v)^{\nu-\rho}\chi'(v)\,dv.$$

But

$$N(v)^{\nu-\rho} = \frac{1}{\Gamma(1/2(\rho-\nu))}\int_0^{+\infty} t^{1/2(\rho-\nu)}e^{-tN(v)^2}\,d^*t$$

$$= \frac{4}{\Gamma(1/2(\rho-\nu))}\int_A e^{(2\nu-2\rho)(\log a)}e^{-N(ava^{-1})^2}\,da.$$

where $t = e^{-4\log a}$. Thus

$$\chi'(\Phi_{\nu,\sigma,w}) = \frac{4}{\Gamma(1/2(\rho-\nu))}\int_A e^{2\nu\log a}\int_V e^{-N(v)^2}\chi'(a^{-1}va)\,dv\,da$$

$$= \frac{4}{\Gamma(1/2(\rho-\nu))}\int_0^{+\infty} t^{2\nu}\left(\int_{-\infty}^{+\infty} e^{-\|Y\|^2}e^{-2\pi i t^2\theta(Y)}\,dY\right)d^*t$$

$$= \frac{4\sqrt{\pi}}{\Gamma(1/2(\rho-\nu))}\int_0^{+\infty} t^{2\nu}e^{-\pi^2t^4\|\theta\|^2}\,d^*t,$$

where all the computations are justified for $\mathrm{Re}(\nu) > 0$. By assumption $\|\theta\| = 1$, and consequently

$$\chi'(\Phi_{\nu,\sigma,w}) = \frac{2\sqrt{\pi}}{\Gamma(1/2(\rho-\nu))}\int_0^{+\infty} t^{\nu}e^{-\pi^2t^2}\,d^*t$$

$$= \gamma_\rho(1,\nu)^{-1},$$

and the lemma follows.

Remark. In the case of split groups this is just lemma 2.2.2.

Let e and e' be two vectors in H_σ. Define a function in $V(\nu, 1)_K$ by

$$f_{e,e',\nu}(kau) = e^{-(\nu+\rho)\log a}(e \mid \sigma(k)P(\nu)e'),$$

where $(\mid)$ denotes the inner product in H. Then

$$A(\nu, \eta, w) f_{e,e',\nu} = f_{e, T(\nu,\sigma,w)e', w\nu},$$

which is an element of $V(w\nu, 1)_K$. Consequently (A.1) implies

$$\chi'(\Phi_{\nu,\sigma,w}) W_f(g, \nu, 1) = W_{A(\nu,1,w)f}(g, w\nu, 1) \tag{A.2}$$

for every $f \in V(\nu, 1)_K$. Now set

$$A'(\nu, 1, w) = \gamma_\rho(1, \nu) A(\nu, 1, w),$$

same notation as in the split case. Then lemma A.1 and relation (A.2) imply

$$W_f(g, \nu, 1) = W_{A'(\nu,1,w)f}(g, w\nu, 1) \tag{A.3}$$

first for $f \in V(\nu, 1)_K$ and then by continuity for all $f \in V(\nu, 1)_\infty$.

LEMMA A.2. *Suppose* $\mathrm{Re}(\nu) < 0$; *then* $A'(\nu, 1, w)$ *is holomorphic.*

Proof. Suppose $\mathrm{Re}(\nu) < 0$. By a new form of proposition 3.1, the image of $A'(\nu, 1, w)$ is non-degenerate and furthermore the Whittaker integrals on $I(w\nu, 1)_\infty$ do not vanish on this image. Now the lemma follows from the holomorphicity of $W_f(g, \nu, 1)$ for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and the relation (A.3).

Now suppose G is of arbitrary rank. Lemma 2.3.2 and consequently lemma 2.3.1 are now immediate, using the same methods as in section 2. We also observe that the corollary to proposition 2.2.1 holds for such groups.

The proof of lemma 2.3.4 is not of any more difficulty. In fact, for $\nu \in w_\alpha S(w)$

$$\int_V A'(\nu, \eta, w_\alpha) f_\phi(gv) \chi'(v)\, dv \qquad (\phi \in C_c^\infty(G)) \tag{A.4}$$

is holomorphic (after lemma A.2). Also when ν is close to the wall between $S(w_l)$ and $w_\alpha S(w_l)$, the proof of the second part of proposition 2.3.1, shows that $W_{f_\phi}(g, \nu, \eta)$ is holomorphic in a small neighborhood of ν. The equality of $W_{f_\phi}(g, \nu, \eta)$ and (A.4) in this neighborhood follows from lemma 2.3.2. It clearly extends to the whole $w_\alpha S(w_l)$ by analyticity of (A.4). Now proposition 2.3.1, and consequently theorems 2.1 and 2.2 follow immediately.

REFERENCES

1. W. CASSELMAN, *The differential equation satisfied by matrix coefficients*, manuscript, 1975.
2. W. CASSELMAN AND J. A. SHALIKA, *The unramified principal series of P-adic groups* II; *The Whittaker functions*, 1977, to appear.
3. H. JACQUET, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France **95** (1967), 243–309.

4. H. Jacquet and R. P. Langlands, *Automorpic forms on* GL(2) I, Lecture Notes in Math. No. 114, Springer-Verlag, 1970.
5. H. Jacquet and J. A. Shalika, *Hecke theory for* GL_3, to appear.
6. ———, *A non-vanishing theorem for zeta functions of* GL_n, Invent. Math. **38** (1976), 1–16.
7. B. Kostant, *On Whittaker vectors and representation theory*, preprint.
8. R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Notes, Institute for Advanced Study.
9. D. Milicić, *Asymptotic behaviour of the matrix coefficients of the discrete series*, to appear.
10. M. E. Novodvorsky, *Automorphic L-function for symplectic group* GSP(4), based on a talk given at the Amer. Math. Soc. Summer Institute Corvallis, 1977.
11. F. Rodier, *Whittaker models for admissible representations*, Lecture Conference on Harmonic Analysis, Williamstown, 1974.
12. G. Schiffmann, *Intégrales d'enterlacement et fonctions de Whittaker*, Bull. Soc. Math. France **99** (1971), 3–72.
13. F. Shahidi, *Functional equation satisfied by certain L-functions*, Compositio Mathematica, **37** (1978), 171–208.
14. ———, *On certain L-functions*, in preparation.
15. J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. **100** (1974), 171–193.
16. J. Tate, *Fourier analysis in number fields and Hecke's zeta function*, thesis, Princeton, 1950.
17. D. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Institute for Advanced Study, preprint.
18. N. R. Wallach, *Representations of reductive Lie groups*; Proceedings of the Amer. Math. Soc. Summer Institute at Corvallis, 1977.
19. G. Warner, *Harmonic Analysis on Semi-simple Lie Groups* I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

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