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The Annals of Mathematics, Second Series, Volume 127, Issue 3 (May, 1988), 547-584.

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Annals of Mathematics, 127 (1988), 547–584

On the Ramanujan conjecture and finiteness of poles for certain L-functions

By Freydoon Shahidi*

To My Parents

Introduction

In this paper we prove a number of general results about automorphic L-functions which appear in the constant terms of Eisenstein series for an arbitrary quasi-split connected reductive algebraic group over a number field. These L-functions were first introduced by Langlands for Chevalley groups [20] and through them he was led to make some of his deep conjectures [21]. One significance of these L-functions is that all the automorphic L-functions studied so far are among them.

There are three general results. First, we obtain a uniform line of absolute convergence for all of these L-functions (Theorem 5.1). A uniform estimate for the Hecke eigenvalues of generic cusp forms on many absolutely simple quasi-split connected reductive algebraic groups over number fields (Corollary 5.4) and an improvement on the best available estimate for the Fourier coefficients of Maass wave forms (Corollary 5.5) also follow. Next, we establish a meromorphic continuation and functional equation for each of these L-functions (Theorem 6.1). Finally, in Theorem 6.2, we prove finiteness of poles on the whole complex plane for an important class of these L-functions (Corollaries 6.6 through 6.10).

More precisely, let $\mathbf{G}$ be a quasi-split connected reductive algebraic group over a number field F. Set $G = \mathbf{G}(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ over F and let $\mathbf{U}$ be its unipotent radical. Let $\mathbf{P}$ be a maximal F-parabolic subgroup of $\mathbf{G}$ with $\mathbf{P} \supseteq \mathbf{B}$. In the context of the problems studied here, nothing new will be obtained if one drops the maximality condition on $\mathbf{P}$. Write $\mathbf{P} = \mathbf{MN}$, a Levi decomposition, $\mathbf{N} \subseteq \mathbf{U}$, and let B, U, P, M, and N be the corresponding groups of adelic points. For every place v of F, let $G_v = \mathbf{G}(F_v)$. Similarly we have B_v, U_v, P_v, M_v, and N_v.

Let $\chi = \otimes_v \chi_v$ be a generic character of $\mathbf{U}(F)\backslash U$ (cf. Section 3). Then each χ_v is generic. Let $\pi = \otimes_v \pi_v$ be a cusp form on M. We shall say π is

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*Partially supported by NSF grants MCS-8320317 and DMS-8521179.

χ-generic if equation (3.1) does not vanish for some φ. Finally, we say π is generic, if it is generic with respect to some χ.

Let α be the unique reduced root of the split torus $\mathbf{A}$ of the center of $\mathbf{M}$ in the Lie algebra of $\mathbf{N}$. Denote by $\rho_{\mathbf{P}}$ half the sum of F-roots generating $\mathbf{N}$. Then $\tilde{\alpha} = \langle \rho_{\mathbf{P}}, \alpha \rangle^{-1} \rho_{\mathbf{P}}$ belongs to the complex dual of the real Lie algebra of $\mathbf{A}$. We shall now identify the field of complex numbers $\mathbf{C}$ with a subspace of this complex dual by identifying $s \in \mathbf{C}$ with $s\tilde{\alpha}$. Then s can also be realized as an element in the complex dual of the real Lie algebra of $\mathbf{A}$ as a group over F_v for each v (cf. Section 1).

For each v, there exists a homomorphism H_{P_v} from M_v into the real Lie algebra of $\mathbf{A}$ as a group over F_v (cf. Section 1). Let

$$I(s, \pi_v) = \operatorname*{Ind}_{P_v \uparrow G_v} \pi_v \otimes q_v^{\langle s, H_{P_v}(\,)\rangle} \otimes 1,$$

be the corresponding induced representation, where q_v is the number of elements in the residue class field of F_v (cf. Section 2).

Let $A(s, \pi_v, w)$ be the standard intertwining operator attached to $I(s, \pi_v)$, defined by equation (2.3) of Section 2. If $M(s, \pi)$ is the nontrivial part of the constant term of the Eisenstein series defined by π (cf. [23]), then $M(s, \pi) = \otimes_v A(s, \pi_v, w)$.

We use S to denote a finite set of places of F, including the archimedean ones, such that for $v \notin S$, $\mathbf{G}$, π_v, and χ_v are all unramified (cf. Sections 2 and 3).

Given a finite dimensional complex representation r of ${}^LM$, the L-group of $\mathbf{M}$ (cf. [3]), let r_v be its restriction to ${}^LM_v$, the L-group of $\mathbf{M}$ as a group over F_v. For a finite place v of F with $v \notin S$, let $L(s, \pi_v, r_v)$ be the Langlands L-function attached to π_v, r_v, and $s \in \mathbf{C}$ (cf. equation (2.5) of Section 2). Then the Euler product

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v)$$

always converges absolutely for $\operatorname{Re}(s)$ large enough [3], [21].

There exist m finite dimensional complex representations $r_1, \ldots, r_m$ of ${}^LM$ such that

$$M(s, \pi)f = \bigotimes_{v \in S} A(s, \pi_v, w)f_v \otimes \bigotimes_{v \notin S} \tilde{f}_v \cdot \prod_{i=1}^{m} L_S(is, \pi, \tilde{r}_i)/L_S(1 + is, \pi, \tilde{r}_i),$$

where $f = \otimes_v f_v$, $f_v \in V(s, \pi_v)$, and for every $v \notin S$, f_v and $\tilde{f}_v \in V(-s, \tilde{w}(\pi_v))$ are the $\mathbf{G}(O_v)$ fixed functions, normalized by $f_v(e_v) = \tilde{f}_v(e_v) = 1$ ([19], [20]; cf. equation (2.7) of Section 2). The representations r_i are all irreducible. As we mentioned before, one significance of these L-functions is that all the automorphic L-functions studied so far are among them (for an appropriate choice of $\mathbf{G}$ and $\mathbf{M}$).

Our first result has already been announced in [39]. Its goal is to provide some evidence for the validity of the Ramanujan conjecture ([13], [25], [31]) for generic cusp forms by providing a uniform estimate for their Hecke eigenvalues. More precisely, in Corollary 5.4, we show: *Let* π *be a generic cusp form on* G, *where* G *is the group of adelic points of any of the groups fixed there and for a* $v \notin S$, *let* $A_v \in {}^LT^0$ *represent the corresponding semi-simple conjugacy class in* ${}^LG$. *Finally, let* μ *be a weight for the restriction of the standard representation of* ${}^LG$ *to* ${}^LG^0$. *Then* $|\mu(A_v)| < q_v$.

The strong form of the Ramanujan conjecture to the effect that every π_v is tempered in fact requires $|\mu(A_v)| \leq 1$. For a general group, it is usually very hard to reduce q_v even to $q_v^{1/2}$, and in fact presently only for $\mathbf{G} = \mathrm{GL}_n$ can this be done (cf. Remark 2 after Corollary 6.4 and [15]).

This uniform bound and the fact that none of the non-tempered automorphic representations constructed so far are generic ([13], [25]) suggest that if we restrict ourselves to the generic part of the cuspidal spectrum, then the conjecture holds in the generality of every quasi-split reductive algebraic group over a number field.

The proof is based on the fact that for $v \notin S$ and $\mathrm{Re}(s) \geq 1$, certain local Langlands L-functions are holomorphic (Lemma 5.8). In fact in that lemma we prove this for all the local L-functions $L(s, \pi_v, r_{i,v})$ discussed before (under the assumption that the restriction of π to the center of M is trivial if $m \geq 2$). It then immediately implies our Theorem 5.1 which proves: *Assume* π *is generic. Then for* $\mathrm{Re}(s) > 2$, *the partial L-functions* $L_S(s, \pi, r_i)$, $i = 1, \ldots, m$, *are all absolutely convergent* (with the same assumption for $m \geq 2$).

Another application of Lemma 5.8, Corollary 5.5, gives the estimate $q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$, where $\alpha_v + \alpha_v^{-1}$ is the Hecke eigenvalue of a non-monomial cusp form on $\mathrm{PGL}_2(\mathbb{A}_F)$ at an unramified place v of F. Even when $F = \mathbf{Q}$, this gives an improvement over the estimate $p^{-1/5} \leq |\alpha_p| \leq p^{1/5}$ obtained for the Hecke eigenvalues of Maass wave forms in [27] to strict inequality. (A similar estimate to that of [27], using the same proof, was also known to Serre, as well as to Deligne and Patterson. It has also been observed by M. Ram Murty in [28]. Serre's proof was first explained in a letter to Deshouillers, a copy of which was kindly provided to me by Iwaniec.) Over an arbitrary number field F, the method of [27] and others can only provide us with an exponent of type $\frac{1}{4} - \delta$, where $0 < \delta \leq \frac{1}{20}$ depends on the degree of F over $\mathbf{Q}$ ($\delta = \frac{1}{20}$ only if $F = \mathbf{Q}$). This is due to their use of Landau's lemma. Moreover, unlike their proof, ours does not rely on the unpublished work of Jacquet, Piatetski-Shapiro, and Shalika on the properties of global Rankin-Selberg L-functions (cf. [17], for example). I wish to thank the referee for reminding me of these differences.

Finally in Proposition 5.9, we use Lemma 5.8 to prove a relation between L-functions and intertwining operators in the unramified case. We expect this to be true in general.

Lemma 5.8 is a consequence of our Theorem 5.2 which proves the holomorphy of the intertwining operator $A(s, \pi_v, w)$ for $\mathrm{Re}(s) \geq 1$ if π_v is a component of a generic cusp form (with the same assumption if $m \geq 2$). A local proof (as a generic representation) for this is usually very hard. (Compare Remark 1 after Theorem 5.2 and Proposition 2.2 of [14] for GL_n; no local proof is available for any other group. We must remark that, even though in the case of GL_n our estimate is no better than the local one, our results are global in nature. This is manifest, for example, by our result on Maass wave forms discussed above.)

Theorem 5.2 is proved by means of two important Lemmas 4.2 and 5.7. Lemma 4.2 is our induction lemma which was always desired in this approach to the theory of automorphic L-functions (cf. [3], [20] and Theorem 6.1 here). Its proof is accomplished by reducing the lemma to the absolutely simple cases and then checking all of them. We do this by completing the tables started by Langlands in [20] for all the quasi-split absolutely simple groups of adjoint type. This is the content of Section 4. Clearly these tables are of particular interest even by themselves.

Lemma 5.7 is proved by means of induction provided by Lemma 4.2. It has an interesting implication in the theory of automorphic L-functions. In fact, it proves that outside the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ all the L-functions appearing in $M(s, \pi)$ have possibly only a *finite* number of zeros.

Next, let π be a cusp form on $\overline{\mathbf{M}}(\mathbb{A}_F)$, where $\overline{\mathbf{M}}$ is the adjoint group of $\mathbf{M}$ and let ρ: $\mathbf{M} \to \overline{\mathbf{M}}$ be the natural projection. As another application of our induction Lemma 4.2, in Theorem 6.1, we prove that each of the L-functions $L_S(s, \pi, r_i \cdot {}^L\rho)$ extends to a meromorphic function of s on the whole complex plane, and moreover if π is also generic, satisfies a standard functional equation.

Finally, for the last result of this paper we specialize to those cases where $m = 1$ or $m = 2$ and r_2 is one dimensional. If F_v is archimedean, there exists a homomorphism φ_v: $W_{F_v} \to {}^LM_v$ which determines π_v [24]. We then let our local L-function at v be the Artin L-function $L(s, r_{i,v} \cdot \varphi_v)$ (cf. [22]). Next for $v \in S$, $v < \infty$, we use our local coefficient $C_{\chi_v}(s, \pi_v, w)$ (cf. Section 3 and [34]) to define a local L-function attached to $r_{1,v}$ and π_v at each of these ramified finite places (see the discussion before Theorem 6.2). The corresponding global L-function is then the product of $L_S(s, \pi, r_1)$ with every local L-function mentioned above. Theorem 6.2 then proves that this extension is meromorphic with possibly only a *finite number* of poles in $\mathbf{C}$. The delicate part is to show that multiplying $L_S(s, \pi, r_1)$ with other local factors adds at most only a finite number of new poles. As Corollaries 6.6 through 6.10, we give a large number of interesting and new examples, among them the Rankin triple and Asai 3-fold L-functions. An integral representation for each of them has now been found by Garrett [8], [9] and Piatetski-Shapiro and Rallis [30].

I would like to thank M. F. Vigneras for inviting me to E.N.S.J.F. where Theorem 6.2 was announced. Thanks are also due to S. Gelbart for his keen interest in this paper and several comments towards its improvement.

1. Notation and terminology

Let F be a number field. For every place v of F, let F_v be the corresponding completion. Denote by O_v and P_v, the ring of integers of F_v and its maximal ideal, respectively. Let q_v be the number of elements in the residue field O_v/P_v. Denote by $\mathbb{A}_F$ the ring of adeles of F.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. We shall further assume that $\mathbf{G}$ is quasi-split over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ over F. Write $\mathbf{B} = \mathbf{TU}$ where $\mathbf{T}$ is a maximal torus and $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$, both over F. Finally, let $\mathbf{P}$ be an F-parabolic subgroup of $\mathbf{G}$. Assume $\mathbf{P} \supset \mathbf{B}$. Write $\mathbf{P} = \mathbf{MN}$, a Levi decomposition. Then $\mathbf{N} \subset \mathbf{U}$.

For every place v, let $G_v = \mathbf{G}(F_v)$. Similarly, we use B_v, T_v, U_v, P_v, M_v, and N_v to denote the corresponding groups of F_v-rational points. Set $G = \mathbf{G}(\mathbb{A}_F)$, and let B, T, U, P, M, and N denote the corresponding adelic groups for the subgroups defined before.

When $\mathbf{G}$ is unramified over a place v, we let $K_v = \mathbf{G}(O_v)$. Otherwise we shall fix a special maximal compact subgroup $K_v \subset G_v$. Let $K = \otimes_v K_v$. Then $G = PK$.

Let $\mathbf{A}$ denote the split torus in the center of $\mathbf{M}$. For every group $\mathbf{H}$ defined over F, let $X(\mathbf{H})_F$ be the group of F-rational characters of $\mathbf{H}$. We set

$$\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbf{R}),$$

the real Lie algebra of $\mathbf{A}$. Then

$$\begin{aligned}\mathfrak{a}^* &= X(\mathbf{M})_F \otimes_{\mathbf{Z}} \mathbf{R}\\ &= X(\mathbf{A})_F \otimes_{\mathbf{Z}} \mathbf{R}\end{aligned}$$

and $\mathfrak{a}_{\mathbf{C}}^* = \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C}$ is the complex dual of $\mathfrak{a}$. Let $\mathfrak{z}$ be the real Lie algebra of the split torus in the center $\mathbf{Z}_G$ of $\mathbf{G}$. Then $\mathfrak{z} \subset \mathfrak{a}$. Throughout this paper we shall assume $\mathbf{P}$ is maximal, i.e. $\mathfrak{a}/\mathfrak{z}$ is of dimension 1. No new result is obtained if this assumption is removed.

The imbedding $X(\mathbf{M})_F \hookrightarrow X(\mathbf{M})_{F_v}$ induces an imbedding $\mathfrak{a}_v \hookrightarrow \mathfrak{a}$, where $\mathfrak{a}_v = \mathrm{Hom}(X(\mathbf{M})_{F_v}, \mathbf{R})$. There exists a homomorphism $H_P\colon M \to \mathfrak{a}$, defined by

$$\exp\langle \chi, H_P(m)\rangle = \prod_v |\chi(m_v)|_v,$$

where $\chi \in X(\mathbf{M})_F$ and $m = (m_v)$. We shall extend H_P to G by making it trivial on N and K. If we define H_{P_v}: $M_v \to \mathfrak{a}_v$, by

$$q_v^{\langle \chi, H_{P_v}(m)\rangle} = |\chi(m)|_v,$$

$\chi \in X(\mathbf{M})_{F_v}$, $m \in M_v$, when v is finite, and by

$$\exp\langle \chi, H_{P_v}(m)\rangle = |\chi(m)|_v,$$

otherwise, then

$$\exp\langle \chi, H_P(m)\rangle = \prod_{v=\infty} \exp\langle \chi, H_{P_v}(m_v)\rangle \cdot \prod_{v<\infty} q_v^{\langle \chi, H_{P_v}(m_v)\rangle}.$$

Observe that for almost all v, $m_v \in \mathbf{G}(O_v)$ on which H_{P_v} is trivial. Thus the product is in fact finite.

Let $\mathbf{A}_0$ be the maximal F-split torus in $\mathbf{T}$. Denote by ψ the set of F-roots of $\mathbf{A}_0$. Then $\psi = \psi^+ \cup \psi^-$, where ψ^+ is the set of positive roots, i.e. those generating $\mathbf{U}$. Let $\Delta \subset \psi^+$ be the set of simple roots. We shall identify the roots of $\mathbf{A}$ in $\mathbf{N}$ with a subset of ψ^+. Then the unique reduced root of $\mathbf{A}$ in $\mathbf{N}$ can be identified by an element $\alpha \in \Delta$. Let $\rho_{\mathbf{P}}$ be half the sum of F-roots generating $\mathbf{N}$. Then $\tilde{\alpha} = \langle \rho_{\mathbf{P}}, \alpha\rangle^{-1}\rho_{\mathbf{P}}$ belongs to $\mathfrak{a}^*$. Here, for any pair of roots α and β in ψ^+, the pairing $\langle \alpha, \beta\rangle$ is defined as follows. Let $\tilde{\psi}^+$ be the set of non-restricted roots of $\mathbf{T}$ in $\mathbf{U}$, restricting to ψ^+. Then the set of simple roots $\tilde{\Delta}$ in $\tilde{\psi}^+$ restricts to Δ. Identifying α and β with roots in $\tilde{\psi}^+$, we then set

$$\langle \alpha, \beta\rangle = 2(\alpha, \beta)/(\beta, \beta),$$

where (,) is the standard inner product on $\mathbf{R}^l$, $l = \operatorname{Card} \tilde{\Delta}$. Moreover, for each v, $\tilde{\alpha}$ can be realized as an element in $\mathfrak{a}_v^*$ through $\mathfrak{a}^* \to \mathfrak{a}_v^*$. We shall now identify $\mathbf{C}$ with a subspace of $\mathfrak{a}_{\mathbf{C}}^*$ by identifying $s \in \mathbf{C}$ with $s\tilde{\alpha} \in \mathfrak{a}_{\mathbf{C}}^*$. Then s can also be realized as an element in $(\mathfrak{a}_v)_{\mathbf{C}}^*$ for each v.

2. Eisenstein series and L-functions

Let $\pi = \otimes_v \pi_v$ be a cusp form on M. Given a K-finite function φ in the space of π, we shall extend φ to a function $\tilde{\varphi}$ on G as in Section 2 of [33], and we set

$$\Phi_s(g) = \tilde{\varphi}(g)\exp\langle s + \rho_{\mathbf{P}}, H_P(g)\rangle.$$

The corresponding Eisenstein series is then defined by

$$E(s, \tilde{\varphi}, g, P) = \sum_{\gamma \in \mathbf{P}(F)\backslash \mathbf{G}(F)} \Phi_s(\gamma g) \tag{2.1}$$

(cf. [20], [33]).

Let W be the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Denote the subset of Δ which generates $\mathbf{M}$ by θ. Then $\Delta = \theta \cup \{\alpha\}$. There exists a unique element $\tilde{w} \in W$

such that $\tilde{w}(\theta) \subset \Delta$ while $\tilde{w}(\alpha) \in \psi^-$. Fix a representative $w \in K \cap \mathbf{G}(F)$ for $\tilde{w}$. We shall denote every component of w by w again.

Let $I(s, \pi)$ be the induced representation

$$I(s, \pi) = \operatorname*{Ind}_{MN \uparrow G} \pi \otimes \exp\langle s, H_P(\ )\rangle \otimes 1.$$

Then $I(s, \pi) = \otimes_v I(s, \pi_v)$ with

$$I(s, \pi_v) = \operatorname*{Ind}_{M_v N_v \uparrow G_v} \pi_v \otimes q_v^{\langle s, H_{P_v}()\rangle} \otimes 1,$$

where q_v must be replaced by exp if $v = \infty$.

We let $\mathbf{M}'$ be the subgroup of $\mathbf{G}$ generated by $\tilde{w}(\theta)$. There exists a parabolic subgroup $\mathbf{P}' \supset \mathbf{B}$ which has $\mathbf{M}'$ as its Levi factor. Let $\mathbf{N}'$ be its unipotent radical. Given f in the space of $I(s, \pi)$ and $\operatorname{Re}(s)$ sufficiently large, set

$$(2.2) \qquad M(s, \pi)f(g) = \int_{N'} f(w^{-1}ng)\, dn \qquad (g \in G).$$

Observe that if $f = \otimes_v f_v$, then for almost all v, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. Finally, if at each v we define a local intertwining operator by

$$(2.3) \qquad A(s, \pi_v, w)f_v(g) = \int_{N_v'} f_v(w^{-1}ng)\, dn \qquad (g \in G_v),$$

then

$$(2.4) \qquad M(s, \pi) = \bigotimes_v A(s, \pi_v, w).$$

It is a consequence of the general theory of the Eisenstein series that for $\operatorname{Re}(s) \geq 0$, $M(s, \pi)$ extends to a meromorphic function of $s \in \mathbf{C}$ with only a finite number of simple poles [23].

For every algebraic group $\mathbf{H}$ over F, let ${}^LH$ be its L-group (cf. [3], [21]). It is the semi-direct product of a complex group ${}^LH^\circ$ and the Weil group $W(\bar{F}/F)$. For every place v of F, let ${}^LH_v$ be the L-group of $\mathbf{H}$ as a group defined over F_v. Then there is a natural homomorphism ${}^LH_v \to {}^LH$. Let η_v: ${}^LM_v \to {}^LM$ be this map for $\mathbf{M}$. Given a finite dimensional complex representation r of ${}^LM$, $r_v = r \cdot \eta_v$ becomes one of ${}^LM_v$.

Suppose that $\mathbf{G}$ splits over L, where L is a finite Galois extension of F. For every unramified v, there exists a unique Frobenius conjugacy class in $\operatorname{Gal}(L_w/F_v)$, $w|v$, which we denote by τ_v. Moreover if v is such that π_v and $\mathbf{G}$ are both unramified, then there exists an ${}^LM$ semi-simple conjugacy class in ${}^LM^\circ \times \tau_v$ which determines π_v uniquely ([3], [21]). We may identify this conjugacy class, as we in fact do, with an element $A_v \in {}^LT^\circ$. It may moreover be assumed to be fixed by τ_v (cf. §6.3 and 6.5 of [3]). The local Langlands L-function attached to π_v and r_v is then defined to be (cf. [3], [21]),

$$L(s, \pi_v, r_v) = \det(I - r_v(A_v \rtimes \tau_v)q_v^{-s})^{-1}. \tag{2.5}$$

Let S be a finite set of places of F, including all the archimedean ones, such that for every $v \notin S$, π_v and **G** are both unramified. Set:

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v). \tag{2.6}$$

Next, let ${}^LN$ be the unipotent radical of ${}^LP$ (cf. §3.4 of [3]), and denote by ${}^L\mathfrak{n}$ its Lie algebra. The group ${}^LM$ acts on ${}^L\mathfrak{n}$ by adjoint action. The numbers $\langle \tilde{\alpha}, \beta \rangle$, $\beta \in \psi^+$, where $\beta^\vee$ ranges over those dual roots $\beta^\vee$ for which $X_{\beta^\vee} \in {}^L\mathfrak{n}$, take a string of integers from 1 through m, where m is a positive integer. Given i, $1 \leq i \leq m$, let

$$V_i = \left\{ X_{\beta^\vee} \in {}^L\mathfrak{n} \,|\, \langle \tilde{\alpha}, \beta \rangle = i \right\}.$$

Then for each i, the adjoint action of ${}^LM$ leaves V_i stable. Let r_i be its restriction to V_i. The thrust of the calculations in [19] and [20] is that

$$M(s, \pi)f = \bigotimes_{v \in S} A(s, \pi_v, w)f_v \otimes \bigotimes_{v \notin S} \tilde{f}_v \times \prod_{i=1}^{m} L_S(is, \pi, \tilde{r}_i)/L_S(1 + is, \pi, \tilde{r}_i), \tag{2.7}$$

where $f = \otimes_v f_v$ is such that for each $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ and for each i, $\tilde{r}_i$ denotes the contragredient of r_i, $i = 1, \ldots, m$. Here $\tilde{f}_v$ is the K_v-fixed function in the space of $I(-s, \tilde{w}(\pi_v))$, normalized the same way.

3. Generic representations and non-constant Fourier coefficients

In the first paragraph of this section we shall consider **G** as a group over either a local or a global field, both of characteristic zero. Therefore, we let **G** be a quasi-split connected reductive group over a field F (local or global) of characteristic zero. Fix a finite Galois extension L of F over which **G** splits. Choose **B** = **TU**, a Borel subgroup of **G**. Let $\tilde{\psi}^+$ be the set of non-restricted roots of **T** in **U**, restricting to ψ^+. Then the set of simple roots $\tilde{\Delta}$ in $\tilde{\psi}^+$ restricts to Δ. Given $\alpha \in \tilde{\psi}^+$, we use x_α to denote the corresponding root subgroup map from $\overline{F}$ into U. Then every $\sigma \in \mathrm{Gal}(L/F)$ acts on $\tilde{\psi}^+$ in such a way that $\sigma \cdot x_\alpha(t) = x_{\sigma\alpha}(t^\sigma)$, $t \in L$, where $g \mapsto \sigma \cdot g$ denotes the action of $\mathrm{Gal}(L/F)$ on $\mathbf{G}(L)$. An F-morphism f from **U** to $\overline{F}$ is *non-degenerate or generic*, if for

$u = \prod_{\alpha \in \tilde{\psi}^+} x_\alpha(t_\alpha)$,

$$f(u) = \sum_{\alpha \in \tilde{\Delta}} \kappa_\alpha t_\alpha$$

with $\kappa_\alpha \in L$, all non-zero, satisfying $\kappa_{\sigma\alpha} = \kappa_\alpha^\sigma$ $(\alpha \in \tilde{\Delta})$. If F is global, we shall extend f to a map on $\mathbf{U}(\mathbb{A}_F)$. A character χ of $\mathbf{U}(F)$ ($\mathbf{U}(F) \backslash \mathbf{U}(\mathbb{A}_F)$ if F is global) is then called *non-degenerate* or *generic* if $\chi(u) = \psi(f(u))$, $u \in \mathbf{U}(F)$ ($u \in \mathbf{U}(F) \backslash \mathbf{U}(\mathbb{A}_F)$ if F is global), with ψ a fixed non-trivial additive (unitary) character of F ($F \backslash \mathbb{A}_F$ if F is global) (cf. Appendix of [40]). Suppose F is local. We shall say χ is *unramified* if every $\kappa_\alpha \in O_L^*$ and the largest ideal on which ψ is trivial is O_F.

Now we resume our previous assumption that F is a number field. Let $\chi = \otimes_v \chi_v$ be a generic character of $\mathbf{U}(F) \backslash U$. Then for each v, χ_v is a generic character of U_v and moreover almost all of the χ_v's are unramified.

Let $\mathbf{U}^\circ = \mathbf{U} \cap \mathbf{M}$, and let χ also denote the restriction of χ to U°. Choose a function φ in the space of $\pi = \otimes_v \pi_v$, and $\mathbf{U}^\circ(F) \backslash U^\circ$ being compact, set

$$W_\varphi^\circ(m) = \int_{\mathbf{U}^\circ(F) \backslash U^\circ} \varphi(um)\overline{\chi(u)}\,du \qquad (m \in M). \tag{3.1}$$

We shall say π is χ-generic if $W_\varphi^\circ \neq 0$ for some φ. The representation π is generic if it is χ-generic with respect to some generic χ.

Suppose π is χ-generic. Then each π_v is χ_v-generic. Consequently each π_v can be realized in a space of smooth functions W_v° satisfying

$$W_v^\circ(um) = \chi_v(u)W_v^\circ(m),$$

where $m \in M_v$ and $u \in U_v^\circ$, the χ_v-Whittaker model $W(\pi_v)$ for π_v (cf. [34], [37]). Moreover, let $\lambda_{\chi_v}(s, \pi_v)$ be the functional $\lambda(s\tilde{\alpha}, \pi_v, \theta_v, \chi_v)$ defined in Proposition 3.1 of [34] (cf. [37] for $v = \infty$) for $I(s, \pi_v)$. Here θ_v is the image of θ under the imbedding $X(\mathbf{M})_F \hookrightarrow X(\mathbf{M})_{F_v}$. Then for each f in the space of $I(s, \pi_v)$, define a function W_f on G_v by

$$W_f(g) = \lambda_{\chi_v}(s, \pi_v)(I(s, \pi_v)(g)f). \tag{3.2}$$

Let

$$W(s, \pi_v, \chi_v) = \{W_f \mid f \in V(s, \pi_v)\}$$

be the χ_v-Whittaker model for $I(s, \pi_v)$.

Choose φ in the space of π and let $E(s, \tilde{\varphi}, g, P)$ be the corresponding Eisenstein series defined by relation (2.1). Assume π is χ-generic. Let

$$\underline{E}_\chi(s, \tilde{\varphi}, g, P) = \int_{\mathbf{U}(F) \backslash U} E(s, \tilde{\varphi}, ug, P)\overline{\chi(u)}\,du, \tag{3.3}$$

be the corresponding non-constant Fourier coefficient. Here g is in G. Let $E_\chi(s, \tilde{\varphi}, g, P)$ be a non-zero K-matrix coefficient of $\underline{\underline{E}}_\chi(s, \tilde{\varphi}, g, P)$ as is explained after Lemma 5.1 of [33]. Choose a finite set S of places of F, including all the archimedean ones, such that for $v \notin S$, π_v, $\mathbf{G}$, and χ_v are all unramified.

Suppose $\tilde{\varphi}$ corresponds to an element $f = \otimes_v f_v$ where each f_v is in the space of $I(s, \pi_v)$. Moreover, assume that for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. Write W_v for $W_{f_v} \in W(s, \pi_v, \chi_v)$. Then by computations in Section 4 of [34],

$$E_\chi(s, \tilde{\varphi}, e, P) = \prod_{v \in S} W_v(e_v) \cdot \prod_{i=1}^{m} L_S(1 + is, \pi, \tilde{r}_i)^{-1}. \tag{3.4}$$

where $e = (e_v)$ is the identity element in G.

We continue with our assumption that $\pi = \otimes_v \pi_v$ is χ-generic. Fix a place v of F and let $A(s, \pi_v, w)$ be the intertwining operator defined by relation (2.3). Observe that whenever $A(s, \pi_v, w)$ is defined (by analytic continuation), it is a map from $I(s, \pi_v)$ into $I(-s, \tilde{w}(\pi_v))$, where in the second representation we have identified s with $-s\tilde{w}(\tilde{\alpha})$. We use $\lambda_{\chi_v}(-s, \tilde{w}(\pi_v))$ to denote the Whittaker functional $\lambda(s\tilde{w}(\tilde{\alpha}), \tilde{w}(\pi_v), \tilde{w}(\theta_v), \chi_v)$ for $I(-s, \tilde{w}(\pi_v))$. Let $C_{\chi_v}(s, \pi_v, w)$ be the local coefficient $C_{\chi_v}(s\tilde{\alpha}, \pi_v, \theta_v, w)$ defined in [34]. More precisely it satisfies

$$\lambda_{\chi_v}(s, \pi_v) = C_{\chi_v}(s, \pi_v, w)\lambda_{\chi_v}(-s, \tilde{w}(\pi_v))A(s, \pi_v, w). \tag{3.5}$$

One of the main results of [34], Theorem 4.1, proves the functional equation

$$\prod_{i=1}^{m} L_s(is, \pi, r_i) = \prod_{v \in S} C_{\bar{\chi}_v}(s, \tilde{\pi}_v, w) \prod_{i=1}^{m} L_S(1 - is, \pi, \tilde{r}_i), \tag{3.6}$$

where $\tilde{\pi}_v$ is the contragredient of π_v which is $\bar{\chi}_v$-generic. In fact we have:

Lemma 3.1. *Let π be an irreducible unitary χ-generic representation of $\mathbf{M}(F)$ where F is a local field. Then $\tilde{\pi}$ is $\bar{\chi}$-generic and*

$$C_{\bar{\chi}}(\bar{s}, \tilde{\pi}, w) = \overline{C_\chi(s, \pi, w)}.$$

Proof. Let λ be a χ-Whittaker functional for $(\pi, H(\pi)_\infty)$, where $H(\pi)_\infty$ is the space of smooth vectors in the Hilbert space $H(\pi)$ of π. For $v \in H(\pi)_\infty$, define $W_v(m) = \lambda(\pi(m)v)$, $m \in \mathbf{M}(F)$, and let $W(\pi, \chi)$ be the set of all W_v's. Define a map j from $H(\pi)$ onto itself which conjugates its complex structure. More precisely $j(cv) = \bar{c}j(v)$ for $c \in \mathbf{C}$ and $v \in H(\pi)$. Then $\tilde{\pi}$ satisfies $\tilde{\pi}(m)j(v) = j(\pi(m)v)$ and moreover $\lambda(j(v)) = \overline{\lambda(v)}$ for $v \in H(\pi)_\infty$. Observe that $\lambda(\tilde{\pi}(u)j(v)) = \bar{\chi}(u)\lambda(j(v))$, $u \in \mathbf{U}^\circ(F)$. Consequently $\lambda \cdot j$ is a $\bar{\chi}$-Whittaker functional for $\tilde{\pi}$ and the set

$$W(\tilde{\pi}, \bar{\chi}) = \{\overline{W} | W \in W(\pi, \chi)\}$$

is a $\bar{\chi}$-Whittaker model for $\tilde{\pi}$. The map j then extends to a map from $W(\pi, \chi)$ onto $W(\tilde{\pi}, \bar{\chi})$ by $j(W) = \overline{W}$.

Next, let $\lambda_{\chi}(s, \pi)$ be the Whittaker functional for $I(s, \pi)$ fixed before. Extend j to $\tilde{j}$ from $I(s, \pi)$ to $I(\bar{s}, \tilde{\pi})$. Then it is easy to check that

$$\lambda_{\bar{\chi}}(\bar{s}, \tilde{\pi})(\tilde{j}(f)) = \overline{\lambda_{\chi}(s, \pi)(f)} \qquad (f \in V(s, \pi)),$$

and therefore

$$\begin{aligned} \lambda_{\bar{\chi}}(\bar{s}, \tilde{w}(\tilde{\pi}))(A(\bar{s}, \tilde{\pi}, w)\tilde{j}(f)) &= \lambda_{\bar{\chi}}(\bar{s}, \tilde{w}(\tilde{\pi}))(\tilde{j}(A(s, \pi, w)f)) \\ &= \bar{\lambda}_{\chi}(s, \tilde{w}(\pi))(A(s, \pi, w)f), \end{aligned}$$

which proves the last statement.

4. The representations r_i

For every reductive algebraic group **H** over F, we use $\overline{\mathbf{H}}$ to denote its adjoint group. The purpose of this section is to determine the representations r_i (or rather $\bar{r}_i$, see below) when **G** is absolutely simple. This means that as a group over $\bar{F}$, $\overline{\mathbf{G}}$ has an irreducible Dynkin diagram. Let $\overline{\mathbf{M}}$ be the adjoint group of **M** and let ρ: $\mathbf{M} \to \overline{\mathbf{M}}$ be the canonical projection. Then for each i, $\bar{r}_i = r_i \cdot {}^L\rho$ is a representation of ${}^L\overline{M}$. We start with the following proposition.

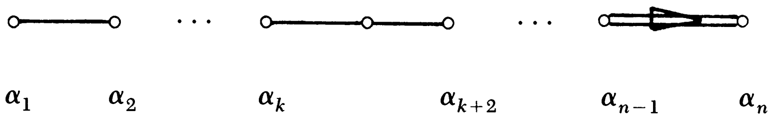
Proposition 4.1. *The representations r_i (as well as $\bar{r}_i$) of ${}^LM$ (of ${}^L\overline{M}$, respectively) are all irreducible.*

Proof. Since **P** is maximal, we may assume **G** is simple. Assume first that **G** is split. Fix i, $1 \leq i \leq m$. The weights of r_i are the roots $\beta^{\vee}$ in ${}^L\mathfrak{n}$ which restrict to $i\alpha^{\vee}$ on ${}^LA^{\circ}$. The adjoint action of $\exp(X_{\gamma^{\vee}}) \in {}^LU^{\circ} \cap {}^LM^{\circ}$ sends $X_{\beta^{\vee}}$ to $X_{\beta^{\vee}+\gamma^{\vee}} \in V$ and therefore to prove that r_i is irreducible by complete reducibility it would be enough to show that V_i contains a longest (or shortest; either is enough) root. When $i = 1$, $\alpha^{\vee}$ is clearly the shortest root since every other root in V_1 is of the form $\alpha^{\vee} + \sum_{\beta^{\vee} \in \theta^{\vee}} c_{\beta^{\vee}} \beta^{\vee}$, where $c_{\beta^{\vee}}$ are non-negative integers. For $i = m$, the longest root in $(\psi^{+})^{\vee}$ will be the highest weight for r_m. This leaves us with a few cases for which $m > 2$. In all such cases **G** is exceptional. To complete the list in [20], in the next paragraph we shall be giving the highest weights of the representation $\bar{r}_i$ for all the remaining cases (we also need this to prove one of the main results of this paper, Theorem 5.1). For this, one must explicitly determine these longest roots which, using the tables in [5], is a fairly easy matter. This, in part, shows the existence of a longest root for each r_i. We shall discuss the case of quasi-split groups separately.

As we just mentioned, we shall now complete the list of the highest weights of the representations $\bar{r}_i$ (when **G** is split and simple) which was started by

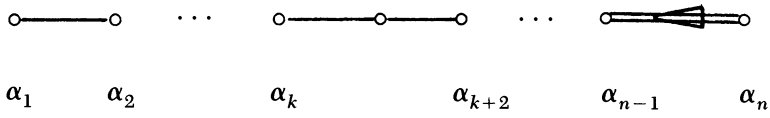
Langlands in [20]. As in [20], in the following table, we shall give the Dynkin diagram of $\overline{\mathbf{G}}$ with the points belonging to the Dynkin diagram of $\overline{\mathbf{M}}$ labeled. More precisely, the simple root which is not labeled is α. Since there are a large number of non-simple Levi factors in the cases considered here, we shall index the simple roots of $\overline{\mathbf{M}}$ with their indices as roots of $\overline{\mathbf{G}}$. The reader must observe that this is in contrast with the convention used in [20], where in all but one case $\overline{\mathbf{M}}$ was simple. Next, for each i, we shall give the highest weight λ_i of $\bar{r}_i$ as a linear combination of the fundamental weights δ_j of ${}^L\overline{M}^\circ$. It must be emphasized that the δ_j's, even though indexed according to the roots of ${}^L\overline{G}^\circ$, are in fact weights of ${}^L\overline{M}^\circ$. For example, in the case (B_n) below, $\bar{r}_1$ is just the tensor product of the first fundamental (standard) representation of $\mathrm{SL}_{k+1}(\mathbf{C})$ and that of $\mathrm{SP}_{2(n-k-1)}(\mathbf{C})$, while $\bar{r}_2$ is the second symmetric power representation of $\mathrm{SL}_{k+1}(\mathbf{C})$. The representations $\bar{r}_i$ are all irreducible (the Weil group always acts trivially since $\mathbf{G}$ is split).

(B_n)



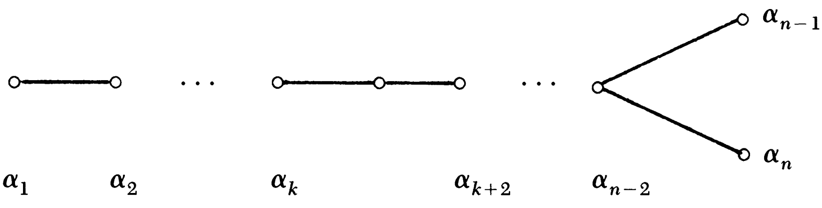
$$\lambda_1 = \delta_1 + \delta_{k+2}$$
$$\lambda_2 = 2\delta_1$$

(C_n)



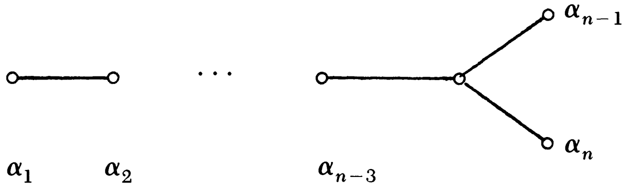
$$\lambda_1 = \delta_1 + \delta_{k+2}$$
$$\lambda_2 = \delta_2$$

$(D_n - 1)$



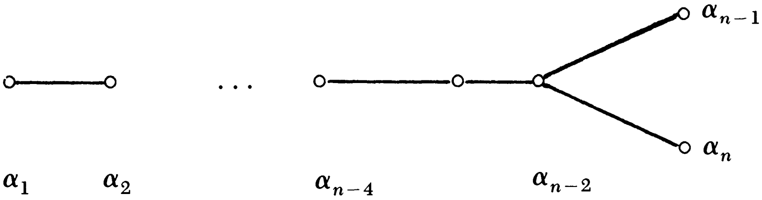
$$\lambda_1 = \delta_1 + \delta_{k+2}$$
$$\lambda_2 = \delta_2$$

$(D_n - 2)$



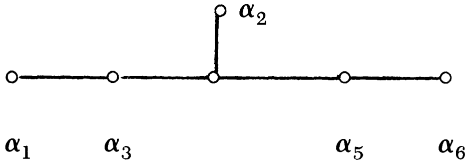
$$\lambda_1 = \delta_1 + \delta_{n-1} + \delta_n$$
$$\lambda_2 = \delta_2$$

$(D_n - 3)$



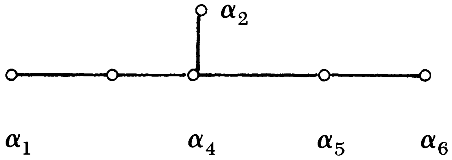
$$\lambda_1 = \delta_1 + \delta_{n-2}$$
$$\lambda_2 = \delta_2$$

$(E_6 - 1)$



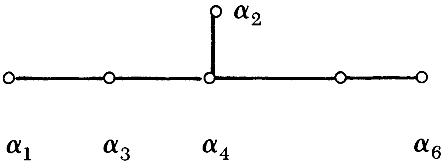
$$\lambda_1 = \delta_1 + \delta_2 + \delta_6$$
$$\lambda_2 = \delta_3 + \delta_5$$
$$\lambda_3 = \delta_2$$

$(E_6 - 2)$



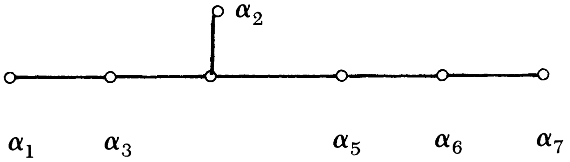
$$\lambda_1 = \delta_1 + \delta_5$$
$$\lambda_2 = \delta_2$$

$(E_6 - 3)$



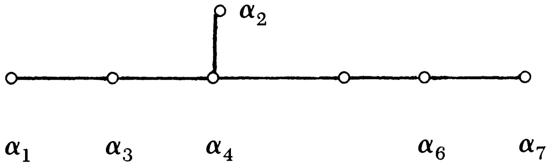
$\lambda_1 = \delta_3 + \delta_6$
$\lambda_2 = \delta_2$

$(E_7 - 1)$



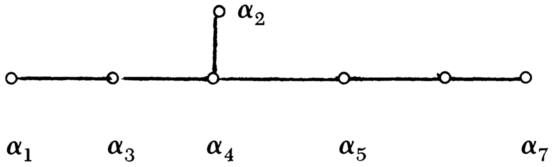
$\lambda_1 = \delta_1 + \delta_2 + \delta_7$
$\lambda_2 = \delta_3 + \delta_6$ $(D_6 - 3$, cf. Lemma 4.2)
$\lambda_3 = \delta_2 + \delta_5$
$\lambda_4 = \delta_1$

$(E_7 - 2)$



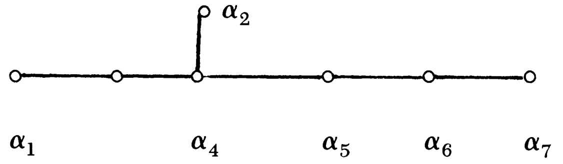
$\lambda_1 = \delta_3 + \delta_7$
$\lambda_2 = \delta_2 + \delta_7$
$\lambda_3 = \delta_1$

$(E_7 - 3)$



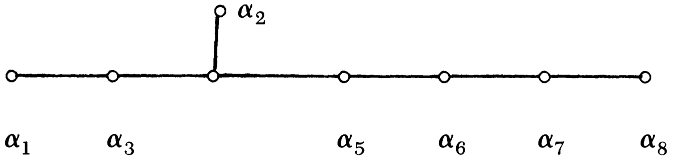
$\lambda_1 = \delta_2 + \delta_7$
$\lambda_2 = \delta_1$

$(E_7 - 4)$



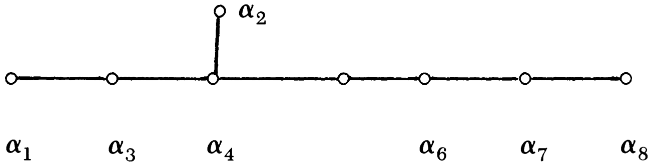
$\lambda_1 = \delta_1 + \delta_6$
$\lambda_2 = \delta_4$
$\lambda_3 = \delta_1$

$(E_8 - 1)$



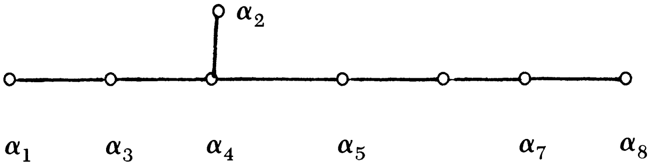
$\lambda_1 = \delta_1 + \delta_2 + \delta_8$
$\lambda_2 = \delta_3 + \delta_7$ $(E_7 - 2)$
$\lambda_3 = \delta_2 + \delta_6$ $(E_6 - 2)$
$\lambda_4 = \delta_1 + \delta_5$
$\lambda_5 = \delta_2 + \delta_3$
$\lambda_6 = \delta_8$

$(E_8 - 2)$



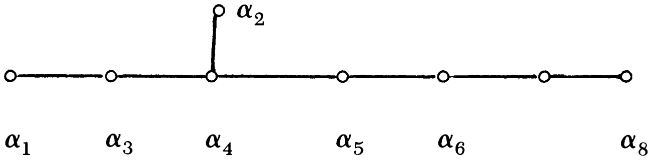
$\lambda_1 = \delta_3 + \delta_8$
$\lambda_2 = \delta_2 + \delta_7$ $(D_8 - 3)$
$\lambda_3 = \delta_1 + \delta_6$
$\lambda_4 = \delta_4$
$\lambda_5 = \delta_8$

$(E_8 - 3)$



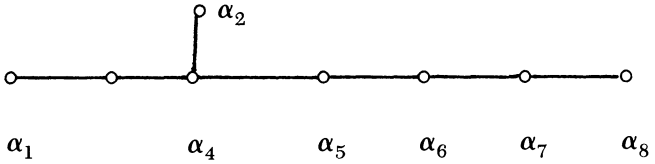
$\lambda_1 = \delta_2 + \delta_8$
$\lambda_2 = \delta_1 + \delta_7$
$\lambda_3 = \delta_5$ ((xxv) of [20])
$\lambda_4 = \delta_8$

$(E_8 - 4)$



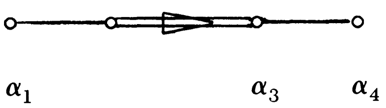
$\lambda_1 = \delta_1 + \delta_8$
$\lambda_2 = \delta_6$ ((xxxi) of [20])
$\lambda_3 = \delta_8$

$(E_8 - 5)$



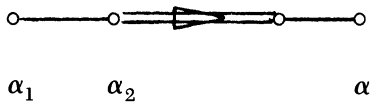
$\lambda_1 = \delta_1 + \delta_7$
$\lambda_2 = \delta_5$ ((xi) of [20])
$\lambda_3 = \delta_1 + \delta_2$
$\lambda_4 = \delta_8$

$(F_4 - 1)$



$\lambda_1 = \delta_1 + \delta_4$
$\lambda_2 = 2\delta_1 + \delta_3$
$\lambda_3 = \delta_1$
$\lambda_4 = \delta_4$

$(F_4 - 2)$



$$\lambda_1 = 2\delta_1 + \delta_4$$
$$\lambda_2 = 2\delta_2$$
$$\lambda_3 = \delta_4$$

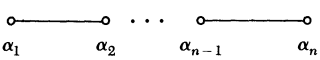
Remark 1. The reader will observe the abundance of the double and triple Rankin products which appear among the r_i's. For example the L-function attached to the case $D_4 - 2$ is that of Rankin triple product attached to three forms on GL_2 (cf. [8], [9], [30]).

Remark 2. The cases B_n, C_n, $D_n - 1$, and $E_6 - 1$ were studied by C. Moreno in [26]. They have also been studied by Gelbart and Piatetski-Shapiro in [11].

Remark 3. The two cases of F_4 were already considered in [34] (Theorem 4.1.2).

Remark 4. In examples (xvii) and (xxi) of [20], the roles of λ_1 and λ_2 must be interchanged, i.e. $\lambda_1 = \delta_1$ while $\lambda_2 = 0$.

Next, we shall give a complete list of representations $\bar{r}_i$ when $\mathbf{G}$ is an absolutely simple non-split quasi-split group over F. In each case, we shall give the Γ-diagram of $\overline{\mathbf{G}}$ (cf. [32]) with the points of $\overline{\mathbf{M}}$ labeled. For each case, we give the sum λ_i of the highest weights of irreducible constituents of $\bar{r}_i^{\circ}$, the restriction of $\bar{r}_i$ to ${}^L\overline{M}^{\circ}$, as a linear combination of the fundamental weights δ_j of ${}^L\overline{M}^{\circ}$. When $\bar{r}_i^{\circ}$ is reducible, we characterize each such highest weight by inserting it inside parentheses. To understand how the diagrams are indexed, we refer the reader to our remarks on the cases of split groups which were made immediately after Proposition 4.1. The Weil group $W(\bar{F}/F)$ always acts through $\mathrm{Gal}(\bar{F}/F)$ and therefore its action can be read off the Γ-diagram. Unless $\overline{\mathbf{G}}$ is of type ${}^2A_{2k}$, it either permutes the root spaces of ${}^L\mathfrak{n}$ or fixes them (as usual we have fixed the standard Chevalley basis for ${}^L\overline{M}^{\circ}$ to accomplish this). Whenever appropriate, we shall give the action of the Galois group on the Γ-diagram by arrows (e.g. orthogonal groups). The representations $\bar{r}_i$ are again always irreducible.

We start with the cases of unitary groups (type ${}^2A_n$). The group $\mathbf{G}$ splits over a quadratic extension L/F and the non-trivial element $\tau \in \mathrm{Gal}(L/F)$ sends α_j to α_{n-j+1} (). The action of $W(\bar{F}/F)$ on ${}^L\mathfrak{n}$

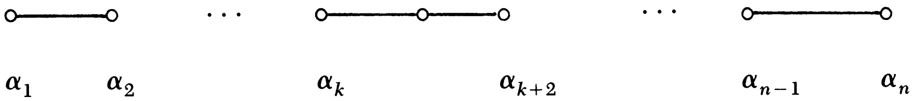
is according to that of τ on roots unless $n = 2k$ in which case τ acts at every X_β by (-1) if β is of the form $\sum_{j=p}^{2k-p+1}\alpha_j$, $1 \le p \le k$ (Lemma 13.6.2 of [6]). The list is as follows:

$({}^2A_n - 1)$

$$\lambda_1 = (\delta_2) + (\delta_{n-1})$$
$$\lambda_2 = 0$$

Remark. λ_1 gives the standard L-function for unitary groups.

$({}^2A_{2k+1} - 2)$

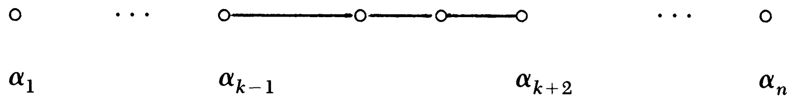


$$\lambda_1 = \delta_1 + \delta_n$$

Remark. This case was studied by Flicker in [7].

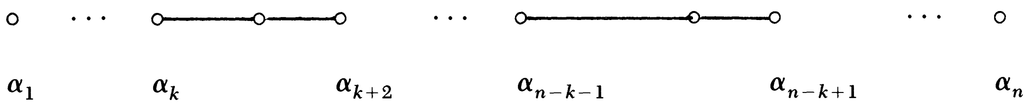
Remark. When $k = 1$ and therefore $n = 3$, this gives the L-function considered by Asai [1]. It is an important part of the Hasse-Weil zeta function of Hilbert-Blumenthal surfaces [12]. Incidently, Theorem 5.1 of [34] when applied to this case gives a direct proof of the non-vanishing of the Asai L-function on the line $\mathrm{Re}(s) = 1$. This fact was used in [12].

$({}^2A_{2k} - 3)$



$$\lambda_1 = (\delta_1) + (\delta_n)$$
$$\lambda_2 = \delta_1 + \delta_n$$

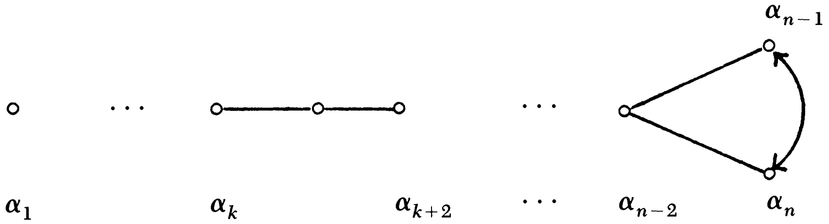
$({}^2A_n - 4)$



$$\lambda_1 = (\delta_1 + \delta_{n-k-1}) + (\delta_{k+2} + \delta_n)$$
$$\lambda_2 = \delta_1 + \delta_n$$

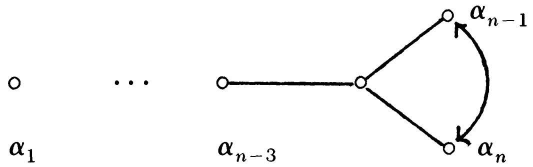
The cases of quasi-split orthogonal groups are:

$({}^2D_n - 1)$



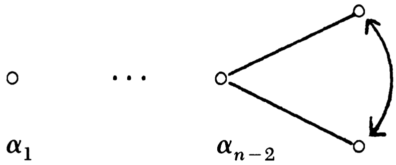
$$\lambda_1 = \delta_1 + \delta_{k+2}$$
$$\lambda_2 = \delta_2$$

$({}^2D_n - 2)$



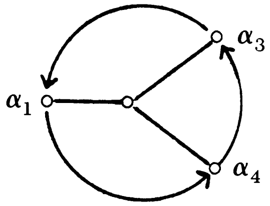
$$\lambda_1 = \delta_1 + \delta_{n-1} + \delta_n$$
$$\lambda_2 = \delta_2$$

$({}^2D_n - 3)$



$$\lambda_1 = (\delta_1) + (\delta_1)$$
$$\lambda_2 = \delta_2$$

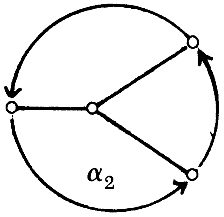
($^3D_4 - 1$)



$$\lambda_1 = \delta_1 + \delta_3 + \delta_4$$
$$\lambda_2 = 0$$

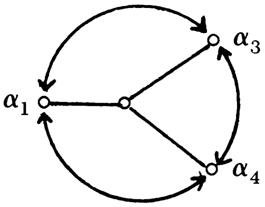
Remark. The corresponding L-function generalizes the Asai L-function (cf. [1], [12]; also see the case $^2A_3 - 1$) from the case of quadratic extensions to that of cubic ones (cf. [9], [30]).

($^3D_4 - 2$)



$$\lambda_1 = (\delta_2) + (\delta_2) + (\delta_2)$$
$$\lambda_2 = (0) + (0) + (0)$$
$$\lambda_3 = \delta_2$$

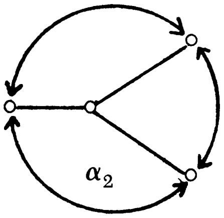
($^6D_4 - 1$)



$$\lambda_1 = \delta_1 + \delta_3 + \delta_4$$
$$\lambda_2 = 0$$

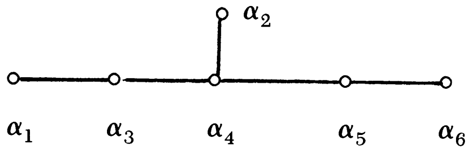
Remark. This is the non-Galois version of the case ${}^3D_4 - 1$.

$({}^6D_4 - 2)$



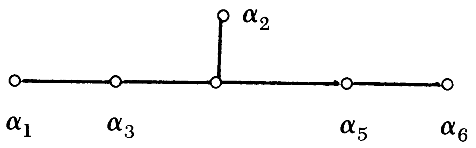
$$\lambda_1 = (\delta_2) + (\delta_2) + (\delta_2)$$
$$\lambda_2 = (0) + (0) + (0)$$
$$\lambda_3 = \delta_2$$

The quasi-split group ${}^2E_6$ whose Γ-diagram is of type E_6 splits over a quadratic extension L/F and the non-trivial element $\tau \in \mathrm{Gal}(L/F)$ while sending α_1 and α_3 to α_6 and α_5, respectively, fixes α_2 and α_4,



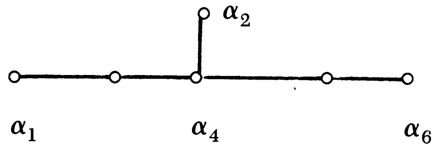
The cases are as follows:

$({}^2E_6 - 1)$



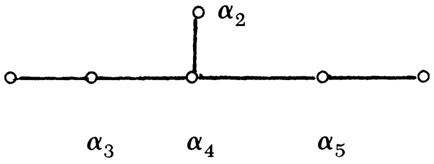
$$\lambda_1 = \delta_1 + \delta_2 + \delta_6$$
$$\lambda_2 = \delta_3 + \delta_5$$
$$\lambda_3 = \delta_2$$

$({}^2E_6 - 2)$



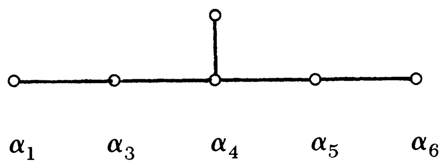
$$\lambda_1 = (\delta_1 + \delta_2) + (\delta_2 + \delta_6)$$
$$\lambda_2 = \delta_1 + \delta_4 + \delta_6$$
$$\lambda_3 = (\delta_1) + (\delta_6)$$
$$\lambda_4 = \delta_2$$

$({}^2E_6 - 3)$



$$\lambda_1 = (\delta_3) + (\delta_5)$$
$$\lambda_2 = \delta_2$$

$({}^2E_6 - 4)$



$$\lambda_1 = \delta_4$$
$$\lambda_2 = 0$$

To conclude this section we shall now state and prove a crucial induction lemma which we shall later use to prove one of our main results (Theorem 5.1). It is also of interest by itself.

LEMMA 4.2. *Suppose* **G** *has no factor which is obtained by restriction of scalars from either a group of type* ${}^2A_{2k}$ *or a group of type* F_4 *if the part of* **M** *in this factor is obtained by restricting from the group generated by* $\{\alpha_1, \alpha_3, \alpha_4\}$ *(Case* $F_4 - 1$*). Fix* i, $2 \leq i \leq m$. *Then, there exists a quasi-split connected reductive F-group* $\mathbf{G}_i$, *unramified outside* S, *a maximal F-parabolic subgroup with a Levi factor* $\mathbf{M}_i \subset \mathbf{G}_i$ *whose adjoint group* $\overline{\mathbf{M}}_i$ *embeds into* $\overline{\mathbf{M}}$ *by an F-rational map* $j\colon \overline{\mathbf{M}}_i \to \overline{\mathbf{M}}$ *such that if* $r_1', \ldots, r_{m_i}'$ *are the corresponding representations of* ${}^LM_i$, *then*

$$r_i \cdot {}^L\rho = r_1' \cdot {}^L\rho_i \cdot {}^Lj,$$

where $\rho_i\colon \mathbf{M}_i \to \overline{\mathbf{M}}_i$ *is the natural projection. Moreover* $m_i < m$.

Proof. We first observe that every semi-simple adjoint F-group is a direct product of its simple factors (cf. [4]). When the simple factors are also absolutely simple the proof is done case by case. We do this by checking the simple factors of **G**, using the tables for $\bar{r}_i$'s. In fact, in our table, we have given a choice for the pair $(\mathbf{G}_i, \mathbf{M}_i)$, whenever we have felt it may not be clear, by giving the case number for $(\mathbf{G}_i, \mathbf{M}_i)$ in parentheses to the right of λ_i. We remark that $\mathbf{G}_i$ need not necessarily be absolutely simple. Otherwise we may assume $\mathbf{G} = \mathrm{Res}_{E/F}\tilde{\mathbf{G}}$ (cf. [4]), $[E : F] = n$, and $\tilde{\mathbf{G}}$ is absolutely simple over E. Choose $\tilde{\mathbf{M}} \subset \tilde{\mathbf{G}}$ such that $\mathbf{M} = \mathrm{Res}_{E/F}\tilde{\mathbf{M}}$. Then $\overline{\mathbf{M}} = \mathrm{Res}_{E/F}\overline{\tilde{\mathbf{M}}}$. Fix $2 \le i \le m$ for $\mathbf{G}$ and $\mathbf{M}$. The L-group ^LM is isomorphic to $^L\tilde{M}^\circ \times \cdots \times {}^L\tilde{M}^\circ \rtimes W(\bar{F}/F)$, where the number of copies of $^L\tilde{M}^\circ$ is equal to n. Restriction of r_i to every copy of $^L\tilde{M}^\circ$ is the same. The action of $W(\bar{F}/F)$ on each $^L\tilde{M}^\circ$ is through $\mathrm{Gal}(\bar{F}/E)$, while different copies of $^L\tilde{M}^\circ$ are permuted by monomorphism of E into $\bar{F}$ (each $^L\tilde{M}^\circ$ is attached to a fixed coset representative of $\mathrm{Gal}(\bar{F}/E)$ in $\mathrm{Gal}(\bar{F}/F)$). For each k, $1 \le k \le n$, embed $^L\tilde{M} = {}^L\tilde{M}^\circ \rtimes W(\bar{F}/E)$ into ^LM by embedding $^L\tilde{M}^\circ$ into the k-th factor and $W(\bar{F}/E)$ into $W(\bar{F}/F)$. Restriction of r_i to the image of this embedding is independent of k. Let $\tilde{r}_i$ be this restriction. Now, choose E-groups $\tilde{\mathbf{G}}_i$ and $\tilde{\mathbf{M}}_i$ by the absolutely simple case such that $\tilde{j}\colon \tilde{\mathbf{M}}_i \to \tilde{\mathbf{M}}$ is an E-embedding, and let $\tilde{r}_1'$ be the corresponding representation of $^L\tilde{M}_i$. Then it satisfies

$$\tilde{r}_i \cdot {}^L\tilde{\rho} = \tilde{r}_i' \cdot {}^L\tilde{\rho}_i \cdot {}^L\tilde{j}, \tag{4.2.1}$$

where $\tilde{\rho}\colon \tilde{\mathbf{M}} \to \overline{\tilde{\mathbf{M}}}$ and $\tilde{\rho}_i\colon \tilde{\mathbf{M}}_i \to \overline{\tilde{\mathbf{M}}}_i$ are natural projections, and $\tilde{m}_i < \tilde{m} = \underline{m}$. Let $\mathbf{M}_i = \mathrm{Res}_{E/F}\tilde{\mathbf{M}}_i$ and $\mathbf{G}_i = \mathrm{Res}_{E/F}\tilde{\mathbf{G}}_i$. Observe that $\overline{\mathbf{M}}_i = \mathrm{Res}_{E/F}\overline{\tilde{\mathbf{M}}}_i$ and $\underline{m}_i = \tilde{m}_i$. The representation $\bar{r}_1'$ of $^L\overline{M}_i$, when restricted to each embedding of $^L\tilde{M}_i^\circ \rtimes W(\bar{F}/E)$ is equal to $\tilde{r}_1' \cdot {}^L\tilde{\rho}_i$. The map $\tilde{j}$ defines an F-embedding $j\colon \overline{\mathbf{M}}_i \to \overline{\mathbf{M}}$ commuting with $\mathrm{Gal}(\bar{F}/F)$. Consequently by (4.2.1), $r_i \cdot {}^L\rho$ and $r_1' \cdot {}^L\rho_i \cdot {}^Lj$ both have the same restriction to each embedding of $^L\tilde{M}^\circ \ltimes W(\bar{F}/E)$ into $^L\overline{M}$. But now the lemma is a consequence of the fact that the action of $\mathrm{Gal}(\bar{F}/F)$ on $^L\overline{M}^\circ$ and $^L\overline{M}_i^\circ$ commutes with ^Lj, and therefore $r_i \cdot {}^L\rho$ and $r_1' \cdot {}^L\rho_i \cdot {}^Lj$, when restricted to $W(\bar{F}/E)$, permute the factors of $^L\mathfrak{n}$ the same way.

5. A weak Ramanujan conjecture

In this section we shall state and prove one of the main results of this paper, namely:

THEOREM 5.1. *Suppose π is cuspidal and generic.*

a) *Assume $m = 1$. Then for* $\mathrm{Re}(s) > 2$, *$L_S(s, \pi, r_1)$ is absolutely convergent.*

b) *Suppose $m > 1$. Assume further that the restriction of π to the center of M is trivial. Then for* $\mathrm{Re}(s) > 2$, *the partial L-functions $L_S(s, \pi, r_i)$ are all absolutely convergent, $i = 1, \ldots, m$.*

This is a consequence of the following result.

THEOREM 5.2. *Suppose* π *is cuspidal and generic.*

a) *Assume* $m = 1$. *Then for each finite* $v \in S$, $A(s, \pi_v, w)$ *is holomorphic for* $\mathrm{Re}(s) \geq 1$.

b) *Suppose* $m > 1$. *Assume further that the restriction of* π *to the center of* M *is trivial. Then for each finite* $v \in S$, $A(s, \pi_v, w)$ *is holomorphic for* $\mathrm{Re}(s) \geq 1$.

Remark 1. When $\mathbf{G} = \mathrm{GL}_{n+r}$, $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_r$, $m = 1$, and Theorem 5.2 becomes Proposition 2.2 of [14], whose proof is based on a classification theorem for generic representations of GL_n which is a fairly deep result of Bernstein and Zelevinski [2], [41] (cf. [15]), and certain properties of local Rankin product L-functions [15].

Remark 2. The assumption that σ be trivial on the center of M (when $m > 1$) is made so that the necessary induction can be established in general (Lemma 4.2). However there are many cases with $m > 1$ for which the assumption is not necessary. This is due to our better understanding of $L_S(s, \pi, r_i)$, $i = 2, \ldots, m$, in these special cases (cf. Corollary 5.4).

COROLLARY 5.3. *Suppose* π *is cuspidal and generic.*

a) *Assume* $m = 1$. *Then for* $\mathrm{Re}(s) \geq 1$, *the poles of the corresponding Eisenstein series are exactly the poles of*

$$\bigotimes_{v=\infty} A(s, \pi_v, w) \cdot L_S(s, \pi, \tilde{r}_1)/L_S(1 + s, \pi, \tilde{r}_1).$$

b) *Suppose* $m > 1$. *Assume further that the restriction of* π *to the center of* M *is trivial. Then for* $\mathrm{Re}(s) \geq 1$, *the poles of the corresponding Eisenstein series are exactly the poles of*

$$\bigotimes_{v=\infty} A(s, \pi_v, w) \cdot \prod_{i=1}^{2} L_S(is, \pi, \tilde{r}_i)/L_S(1 + is, \pi, \tilde{r}_i).$$

Moreover, for $\mathrm{Re}(s) > 1$, *in both cases the poles of the corresponding Eisenstein series are exactly those of*

$$\bigotimes_{v=\infty} A(s, \pi_v, w) \cdot L_S(s, \pi, \tilde{r}_1).$$

COROLLARY 5.4. *Let* $\mathbf{G}$ *be either* $\mathrm{GL}(n)$, $\mathrm{U}(n, n)$, $\mathrm{U}(n + 1, n)$, $\mathrm{SP}(2n)$, *or* $\mathrm{SO}(m, n)$, $m = n, n + 1, n + 2$, *their groups of similitudes, their adjoint groups and those of split groups of types* E_6 *and* E_7 *over a number field. Let* $\pi = \otimes_v \pi_v$ *be a* χ*-generic cusp form on* $G = \mathbf{G}(\mathbb{A}_F)$. *At each place* v *of* F *where* π_v, χ_v *and* $\mathbf{G}$ *are all unramified, let* A_v *be an element in* ${}^LT^\circ$, *representing the corresponding* ${}^LG^\circ$*-semi-simple conjugacy class in* ${}^LG^\circ \times \tau_v$. *Finally, let* μ *be a weight for the restriction of the standard representation (first fundamental representation) of* ${}^LG$ *to* ${}^LG^\circ$. *Then* $|\mu(A_v)| < q_v$, *where* q_v *is the number of elements in* O_v/P_v.

Remark 1. The strong Ramanujan conjecture is equivalent to $|\mu(A_v)| \leq 1$.

Remark 2. When $\mathbf{G} = \mathrm{GL}_n$, Theorem 5.2 applied to GL_{2n} with the Levi factor $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_n$ and the representation $\pi \otimes \pi$ of M will immediately imply that $|\mu(A_v)| < q_v^{1/2}$. This is Corollary 2.5 of [15].

Finally, we shall prove the following result. We refer the reader to the introduction for its significance and relation with the Ramanujan-Petersson conjecture for non-holomorphic Maass wave forms on GL_2.

COROLLARY 5.5. *Let π be a non-monomial cuspidal representation of* $\mathrm{PGL}_2(\mathbb{A}_F)$. *At each unramified v, let* $A_v = \mathrm{diag}(\alpha_v, \alpha_v^{-1})$ *be the corresponding semi-simple conjugacy class in* $\mathrm{SL}_2(\mathbf{C})$. *Then* $q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$.

Proof of Theorem 5.2. We shall start with the following lemma.

LEMMA 5.6. *Suppose π is cuspidal and generic. Then for* $\mathrm{Re}(s) \geq 0$, *the product*

$$\prod_{i=1}^{m} L_S(1 + is, \pi, \tilde{r}_i)$$

has only a finite number of simple zeros which are all on the real axis if π comes from $\overline{M}$.

Proof. Suppose π is χ-generic, $\chi = \otimes_v \chi_v$. Given $v \in S$, choose $W_v \in W(s, \pi_v, \chi_v)$ such that $W_v(e_v) \neq 0$ (cf. [34], [37]). Then by relation (3.4), the poles of $\prod_{i=1}^{m} L_S(1 + is, \pi, \tilde{r}_i)^{-1}$ are the same as those of $E_\chi(s, \tilde{\varphi}, e, P)$. But, if $\mathrm{Re}(s) \geq 0$, this last function has only a finite number of simple poles, all on the real axis (cf. [23]). This completes the proof of the lemma.

Next, suppose $m \geq 2$ and π is trivial on the center $Z = \mathbf{Z}(\mathbb{A}_F)$ of M. Regard π as a representation of M/Z. The quotient of $\overline{M} = \overline{\mathbf{M}}(\mathbb{A}_F)$ by M/Z is compact (locally finite). Choose an automorphic cuspidal representation $\bar{\pi}$ of $\overline{M}$ whose restriction to M/Z contains π. Then at each place v, we may choose $\overline{A}_v$ such that ${}^L\rho(\overline{A}_v \rtimes \tau_v) = A_v \rtimes \tau_v$, and consequently if r is a complex representation of ${}^LM$, then

$$L_S(s, \pi, r) = L_S(s, \bar{\pi}, r \cdot {}^L\rho).$$

LEMMA 5.7. *For* $\mathrm{Re}(s) \geq 1$, $L_S(s, \pi, \tilde{r}_i)$ *and the quotient*

$$L_S(s, \pi, \tilde{r}_i)/L_S(1 + s, \pi, \tilde{r}_i)$$

both have only a finite number of poles and zeros, $1 \leq i \leq m$.

Proof. We first assume that we are not in any of the cases excluded in Lemma 4.2. More precisely, we assume that $\mathbf{G}$ has no factor which is obtained by restriction of scalars from either a group of type ${}^2A_{2k}$ or a group of type F_4 if the part of $\mathbf{M}$ in this factor is obtained by restricting from the group generated by $\{\alpha_1, \alpha_3, \alpha_4\}$ (case $F_4 - 1$). We shall treat these two cases later. We now prove the lemma by induction on m. Assume that for $\mathrm{Re}(s) \geq 1$ and each i, $2 \leq i \leq m$, $L_S(s, \pi, \tilde{r}_i)/L_S(1 + s, \pi, \tilde{r}_i)$ has a finite number of zeros and poles. Then by relation (2.7), we conclude that $L_S(s, \pi, \tilde{r}_1)/L_S(1 + s, \pi, \tilde{r}_1)$ has only a finite number of poles when $\mathrm{Re}(s) \geq 1$. In fact, if $v \in S$, and $v < \infty$, $A(s, \pi_v, w)f_v$ can be made non-zero, and if $v = \infty$, then for each s, f_v can be chosen so that $A(s, \pi_v, w)f_v \not\equiv 0$ and our assertion follows. Now, using the line of absolute convergence, one concludes that every $L_S(s, \pi, \tilde{r}_i)$, $1 \leq i \leq m$, has only a finite number of poles, again when $\mathrm{Re}(s) \geq 1$. Moreover, applying this for $2 \leq i \leq m$ to Lemma 5.6, one concludes that $L_S(s, \pi, \tilde{r}_1)$ has only a finite number of zeros for $\mathrm{Re}(s) \geq 1$. We now deduce that for $\mathrm{Re}(s) \geq 1$, the quotient $L_S(s, \pi, \tilde{r}_1)/L_S(1 + s, \pi, \tilde{r}_1)$ has only a finite number of zeros as well. To use induction we now only have to observe (after the discussion before the lemma) that by Lemma 4.2, for every i, $2 \leq i \leq m$, there exists a pair $(\mathbf{G}_i, \mathbf{M}_i)$ with j: $\overline{\mathbf{M}}_i \to \overline{\mathbf{M}}$ such that if $r_1', \ldots, r_{m_i}'$ are the corresponding representations of ${}^L M_i$, then

$$\begin{aligned} L_S(s, \pi, \tilde{r}_i) &= L_S\big(s, \bar{\pi}, \tilde{r}_i \cdot {}^L\rho\big) \\ &= L_S\big(s, \bar{\pi}, \tilde{r}_1' \cdot {}^L\rho_i \cdot {}^L j\big) \\ &= L_S\big(s, \bar{\pi} \cdot j, \tilde{r}_1' \cdot {}^L\rho_i\big) \end{aligned}$$

and $m_i < m$. This completes the lemma except when $\mathbf{G}$ and $\mathbf{M}$ are among the cases excluded before, which we shall now consider. Since F is arbitrary, we may in fact assume that $(\mathbf{G}, \mathbf{M})$ is either ${}^2A_{2k} - 1$, ${}^2A_{2k} - 3$, ${}^2A_{2k} - 4$, or $F_4 - 1$.

We shall start with ${}^2A_{2k}$ cases and we shall treat the case ${}^2A_{2k} - 3$, since the other two cases are similar. In this case $\lambda_2 = \delta_1 + \delta_n$, $\overline{\mathbf{M}} = \mathrm{Res}_{L/F}\mathrm{PGL}_k$. Let π be a cuspidal representation of $\overline{M} = \overline{\mathbf{M}}(\mathbb{A}_F)$. It is automatically generic. Denote by η the character of $\mathbb{A}_F^*$ attached by class field theory to L/F. Extend η to a character $\tilde{\eta}$ of $L^* \setminus \mathbb{A}_L^*$. The group $U(k, k) = \mathbf{U}(k, k)(\mathbb{A}_F)$ has a Levi factor isomorphic to $\mathrm{GL}_k(\mathbb{A}_L)$. We shall consider $\pi \otimes \tilde{\eta}$ as a representation of this Levi factor. By the case ${}^2A_{2k-1} - 2$, we have

$$L_S(s, \pi, r_2) = L_S(s, \pi \otimes \tilde{\eta}, r_1),$$

where the L-function on the right hand side is defined by the case ${}^2A_{2k-1} - 2$. The finiteness of poles and zeros for $L_S(s, \pi, r_2)$ whenever $\mathrm{Re}(s) \geq 1$ is now immediate.

It remains to consider the case $F_4 - 1$. The problem is that there is no pair to represent r_2 for which $\lambda_2 = 2\delta_1 + \delta_3$. Here $\overline{\mathbf{M}} = \mathrm{PGL}_2 \times \mathrm{PGL}_3$. Let $\pi \otimes \sigma$

be a cusp form on $\mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{GL}_3(\mathbb{A}_F)$, trivial on the center. First assume π is not monomial (cf. [10]). Let Π be the Gelbart-Jacquet lift [10] of π to $\mathrm{PGL}_3(\mathbb{A}_F)$. Then

$$L_S(s, \pi \otimes \sigma, r_2) = L_S(s, \Pi \times \sigma),$$

where the L-function on the right-hand side is the Rankin product L-function of π and σ as forms on $\mathrm{GL}_3(\mathbb{A}_F)$ (cf. [15]). But now the lemma is a consequence of its validity for Rankin L-functions (at most a simple pole at $s = 1$ and no zeros or poles otherwise for $\mathrm{Re}(s) \geq 1$). Next assume π is monomial. Then the lift Π is still automorphic but no longer cuspidal. In fact, $\pi \cong \pi \otimes \eta$, with $\eta \neq 1$, $\eta^2 = 1$, a grossencharacter of $\mathbb{A}_F^*$. Let E be the quadratic extension of F defined by η through class field theory. By Proposition 6.5 of [18] (also see Section 3.7 of [10]), there exists a character Ω of $\mathbb{A}_E^*/E^*$ such that $\pi = \pi(\Omega)$, where $\pi(\Omega)$ is the automorphic representation whose local factor at v is the one attached to the representation of the local Weil group induced from Ω_v. Denote by Ω' the conjugate of Ω. Let $\mathbf{P} \subset \mathrm{GL}_3$ be the parabolic subgroup of GL_3, containing the subgroup $\mathbf{B}$ of upper triangulars, whose Levi factor is defined by the partition (2, 1) of 3. Then using Section (3.7) of [10], we have

$$\Pi = \mathrm{Ind}\big(\mathrm{GL}_3(\mathbb{A}_F), \mathbf{P}(\mathbb{A}_F), \pi(\Omega\Omega'^{-1}), \eta\big).$$

Assume first that $\Omega\Omega'^{-1}$ does not factor through the norm. Then $\pi(\Omega\Omega'^{-1})$ is cuspidal and

$$L_S(s, \pi \otimes \sigma, r_2) = L_S\big(s, \pi(\Omega\Omega'^{-1}) \times \sigma\big) L_S(s, \sigma \otimes \eta),$$

and again the lemma follows since both L-functions on the right have no zeros or poles for $\mathrm{Re}(s) \geq 1$ (cf. [16], [34]). Otherwise $\Omega\Omega'^{-1} = \rho \cdot N_{E/F}$ for some unitary character $\rho = \otimes_v \rho_v$ of $\mathbb{A}_F^*/F^*$, and

$$\Pi = \bigotimes_v \mathrm{Ind}\big(\mathrm{GL}_3(F_v), \mathbf{B}(F_v), \rho_v, \rho_v\eta_v, \eta_v\big).$$

Thus

$$L_S(s, \pi \otimes \sigma, r_2) = L_S(s, \sigma \otimes \rho) L_S(s, \sigma \otimes \rho\eta) L_S(s, \sigma \otimes \eta)$$

and the lemma is again clear. This completes the proof of the lemma.

Remark. The reader must observe the significance of the lemma in implying the finiteness of zeros outside the critical strip in a fairly general situation.

We shall now complete the proof of Theorem 5.2.

By Lemma 5.7 and equation (2.7) we conclude that except for a finite number of poles, $M(s, \pi)f$ and $\otimes_{v \in S} A(s, \pi_v, w) f_v$ have the same poles at least when $\mathrm{Re}(s) \geq 1$. Fix $v \in S$, $v < \infty$. For every $u \in S$, $u \neq v$, $u < \infty$, we can choose f_u such that $A(s, \pi_u, w) f_u(w)$ becomes a non-zero constant, independent of s. Now, suppose for some s, with $\mathrm{Re}(s) \geq 1$, $A(s, \pi_v, w)$ has a pole. For each such pole and each $u = \infty$, choose f_u such that $A(s, \pi_u, w)f_u(w) \neq 0$. The operator $A(s, \pi_v, w)$ having values which are rational functions of q_v^{-s}, will then have infinitely many poles parallel to the imaginary axis. Consequently, $M(s, \pi)f$ must also have infinitely many such poles. This is a contradiction to the finiteness of poles for $M(s, \pi)$ when $\mathrm{Re}(s) \geq 0$ which completes the proof of Theorem 5.2.

Proof of Theorem 5.1. We only need the following lemma.

LEMMA 5.8. *Suppose v is unramified. Then for* $\mathrm{Re}(s) \geq 1$, *each* $L(s, \pi_v, \tilde{r}_{i,v})$ *is holomorphic,* $1 \leq i \leq m$.

Proof. We shall use induction to prove that for each i,

$$L(s, \pi_v, \tilde{r}_{i,v})/L(1 + s, \pi_v, \tilde{r}_{i,v})$$

is holomorphic and non-zero when $\mathrm{Re}(s) \geq 1$. In fact, by Lemma 4.2 we shall assume this for $2 \leq i \leq m$ (for the exceptional cases in Lemma 4.2 the validity of the hypothesis can be verified directly). Then including v in S and using Theorem 5.2, one concludes that for $i = 1$, the corresponding quotient is holomorphic when $\mathrm{Re}(s) \geq 1$. Starting with s and $\mathrm{Re}(s)$ large, one can conclude, inductively, that $L(s, \pi_v, \tilde{r}_{1,v})$ is holomorphic, when $\mathrm{Re}(s) \geq 1$. To complete the induction, one only has to notice that the zeros of the quotient are in fact among the poles of $L(1 + s, \pi_v, \tilde{r}_{1,v})$ which is holomorphic for $\mathrm{Re}(s) \geq 0$.

Proof of Corollary 5.4. From the tables in Section 4 and those in [20], in every case there exists an absolutely simple, quasi-split F-group $\tilde{\mathbf{G}}$ with a Levi subgroup $\tilde{\mathbf{M}}$ and a homomorphism φ: $\tilde{\mathbf{M}} \to \mathbf{G}$, with a kernel of at most dimension one, such that the number m of representations r_i for the adjoint action of ${}^L\tilde{M}$ is either 1 or 2, and moreover $r_1 \cdot {}^L\varphi$ is always the standard representation of ${}^LG$ (the representation whose restriction to ${}^L\overline{G}$ has highest weight equal to δ_1) while in those cases for which $m = 2$, r_2 is one-dimensional. Since the assertion of Lemma 5.7 always holds for every Hecke L-function, it is true for $L_S(s, \pi, r_1 \cdot {}^L\varphi)$. Consequently Theorem 5.2 holds for every such $(\tilde{\mathbf{G}}, \tilde{\mathbf{M}})$ pair. Now the corollary is a consequence of Lemma 5.8.

Proof of Corollary 5.5. We only need to apply Lemma 5.8 to a split group of type F_4 (case $F_4 - 2$) with $\mathbf{M}$ generated by $\{\alpha_1, \alpha_2, \alpha_4\}$, and the representation $\Pi \otimes \pi$ of M, where Π is the Gelbart-Jacquet [10] lift of π to $\mathrm{PGL}_3(\mathbb{A}_F)$ discussed before. It can also be proved using the case $E_6 - 1$ and the representation $\Pi \otimes \pi \otimes \Pi$ of M.

We conclude this section by proving a result which connects the poles of L-functions to those of intertwining operators, at least in the unramified case. It is again a consequence of Lemma 5.8. We expect this to be true in general.

Fix a place $v \notin S$. Write

$$L(s, \pi_v, r_{i,v}) = \prod_j \left(1 - \alpha_{ij} q_v^{-s}\right)^{-m_j}$$

with $\alpha_{i,j} \in \mathbf{C}$, m_j a positive integer and $\alpha_{ij} \neq a_{ik}$ if $j \neq k$. Let

$$L^{\circ}(s, \pi_v, r_{i,v}) = \prod_j \left(1 - \alpha_{ij} q_v^{-s}\right)^{-1}.$$

We then have:

PROPOSITION 5.9. *Let π_v be an unramified component of a generic cusp form $\pi = \otimes_v \pi_v$ on M whose restriction to the center of M is trivial, if $m \geq 2$. Then the poles of $L^{\circ}(s, \pi_v, \tilde{r}_{1,v})$ are among those of $A(s, \pi_v, w)$.*

Proof. Let f_v be the unique K_v-fixed function in the space of $I(s, \pi_v)$ satisfying $f_v(e_v) = 1$. Then

$$A(s, \pi_v, w) f_v = \prod_{i=1}^{m} \frac{L(is, \pi_v, \tilde{r}_{i,v})}{L(1 + is, \pi_v, \tilde{r}_{i,v})} \tilde{f}_v, \tag{5.9.1}$$

where $\tilde{f}_v$ is the K_v-fixed function in the space of $I(-s, \tilde{w}(\pi))$, normalized the same way.

Suppose now that for some j the factor $(1 - \alpha_{1j}^{-1} q_v^{-s})^{-1}$ cancels off with a factor of $L(1 + is, \pi_v, \tilde{r}_{i,v})$ in (5.9.1), $1 \leq i \leq m$. Then

$$|\alpha_{1j}|^i = |\alpha_{ik} q_v|$$

for some k. The local coefficient $C_{\chi_v}(s, \pi_v, w)$ is given by

$$C_{\chi_v}(s, \pi_v, w) = \prod_{i=1}^{m} L(1 - is, \pi_v, r_{i,v}) / L(is, \pi_v, \tilde{r}_{i,v}).$$

Now assume that the same factor $(1 - \alpha_{1j}^{-1} q_v^{-s})^{-1}$ cancels off with a factor of L$(1 - ls, \pi_v, r_{l,v})$ for some l. Then

$$|\alpha_{1j}|^l = |\alpha_{lt} q_v^{-1}|$$

for some t and therefore $|\alpha_{ik} q_v|^l = |\alpha_{lt} q_v^{-1}|^i$. But now by Lemma 5.8

$$|\alpha_{ik} q_v|^l > 1$$

and

$$|\alpha_{lt} q_v^{-1}|^i < 1,$$

which is a contradiction. The proposition is now a consequence of Proposition 3.3.1 of [34] which states that the zeros of $C_{\chi_v}(s, \pi_v, w)$ are also among the poles of $A(s, \pi_v, w)$.

6. Finiteness of poles

We start by extending our partial L-functions to include local factors at the archimedean places. More precisely, we let S be the set of all finite ramified places. For every archimedean place v of F, let φ_v: $W_{F_v} \to {}^L M_v$ be the corresponding homomorphism (cf. [24]) attached to π_v, $\pi = \otimes_v \pi_v$. Then each $r_{i,v} \cdot \varphi_v$ becomes a representation of W_{F_v}, where $r_{i,v} = r_i \cdot \eta_v$, η_v: ${}^L M_v \to {}^L M$. Let $L(s, r_{i,v} \cdot \varphi_v)$ be the corresponding Artin L-function (cf. [22]). Now set

$$L_S(s, \pi, r_i) = \prod_{v=\infty} L(s, r_{i,v} \cdot \varphi_v) \prod_{\substack{v \notin S \\ v < \infty}} L(s, \pi_v, r_{i,v}).$$

As before let ρ: $\mathbf{M} \to \overline{\mathbf{M}}$ be the projection of $\mathbf{M}$ onto its adjoint group. One of the consequences of Lemma 4.2 is the following result (for the exceptional cases the result can be proved directly).

THEOREM 6.1. *Let π be a cuspidal automorphic representation of $\overline{\mathbf{M}}$. Then every L-function $L_S(s, \pi, r_i \cdot {}^L\rho)$, $1 \le i \le m$, extends to a meromorphic function of s on* $\mathbf{C}$. *Moreover, if π is generic, then each $L_S(s, \pi, r_i \cdot {}^L\rho)$ also satisfies a standard functional equation.*

Our next goal is to specialize to a particular class of L-functions and define the local L-functions at the ramified places in such a way that the corresponding L-functions, while satisfying a standard functional equation, only have a finite number of poles in the whole complex plane.

We shall assume the pair $(\mathbf{G}, \mathbf{M})$ is such that either $m = 1$ or 2, and if $m = 2$, r_2 is of dimension 1. Moreover, we shall assume $\mathbf{G}$ is absolutely simple over F. This includes many interesting cases which we shall discuss after the proof of our result.

When $\dim r_2 = 1$, there exists a character ω of $\mathbb{A}_F^*/F^*$, unramified for every finite $v \notin S$, such that

$$L_S(s, \pi, r_2) = L_S(s, \omega),$$

where the partial L-function on the right is the Hecke L-function attached to ω. More precisely, if $L(s, \omega_v)$ is the corresponding local Hecke L-function, then

$$L_S(s, \omega) = \prod_{v \notin S} L(s, \omega_v).$$

Next, choose a Galois extension $E \supset F$ over which $\mathbf{G}$ splits. To every separable finite field extension L/F of local fields and every non-trivial character ψ of F, in [22], Langlands has attached a complex number $\lambda(L/F, \psi)$.

Now for each reduced root $\alpha \in \psi^{+}$ for which $\exp(X_\alpha)$ belongs to $\mathbf{N}$, let $\mathbf{G}_\alpha$ be the corresponding rank one subgroup of $\mathbf{G}$. If $\tilde{\mathbf{G}}_\alpha$ is the simply connected covering of the derived group of $\mathbf{G}_\alpha$, then either $\tilde{\mathbf{G}}_\alpha \cong \mathrm{SU}(2,1)$, defined by a quadratic extension E^α/F, $F \subset E^\alpha \subset E$, or $\tilde{\mathbf{G}}_\alpha \cong \mathrm{Res}_{E^\alpha/F}\mathrm{SL}_2$, where $F \subset E^\alpha \subset E$. At each place v of F, we let

$$\lambda(\mathbf{G}, \psi_v) = \prod_\alpha \lambda(E_v^\alpha/F_v, \psi_v)^2 \prod_\alpha \lambda(E_v^\alpha/F_v, \psi_v),$$

where the first product is over all those α for which $\tilde{\mathbf{G}}_\alpha \cong \mathrm{SU}(2,1)$ defined by E^α/F, while the second one runs over all those α for which $\tilde{\mathbf{G}}_\alpha \cong \mathrm{Res}_{E^\alpha/F}\mathrm{SL}_2$. Observe that $\lambda(\mathbf{G}, \psi_v) = 1$, whenever v remains unramified in E and ψ_v is unramified. Moreover $\prod_v \lambda(\mathbf{G}, \psi_v) = 1$.

Since the choice of w does not matter globally, we shall fix $w \in K \cap \mathbf{G}(F)$ so that its local components are as in Theorem 3.1 of [37]. Now, given $v \in S \cup \{\infty\}$, Theorem 3.1 of [37] and the calculations of [38] lead us to define

$$\gamma(s, \pi_v, r_1, \bar{\chi}_v) = \lambda(\mathbf{G}, \bar{\psi}_v)\gamma(2s, \omega_v, \bar{\psi}_v)^{-1} C_{\bar{\chi}_v}(s, \tilde{\pi}_v, \omega), \tag{6.1}$$

where

$$\gamma(s, \omega_v, \psi_v) = \varepsilon(s, \omega_v, \psi_v)\frac{L(1-s, \omega_v^{-1})}{L(s, \omega_v)}.$$

Here $\varepsilon(s, \omega_v, \psi_v)$ is the root number attached to ω_v and ψ_v. It satisfies

$$L(s, \omega) = \prod_{v \in S \cup \{\infty\}} \varepsilon(s, \omega_v, \psi_v) L(1-s, \omega^{-1}), \tag{6.2}$$

where $L(s, \omega) = L_\varnothing(s, \omega)$ with $\varnothing$ denoting the empty set.

Now, using the functional equation (3.6) and (6.2) we have

$$\prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v, r_{1,v}, \bar{\chi}_v)\gamma(1-s, \pi_v, \tilde{r}_{1,v}, \chi_v) = 1.$$

Again Theorem 3.1 of [37] implies that

$$\prod_{v \in S} \gamma(s, \pi_v, r_{1,v}, \bar{\chi}_v)\gamma(1-s, \pi_v, \tilde{r}_{1,v}, \chi_v) = 1. \tag{6.3}$$

We shall now extend S as follows. For every $v \in S$, add to S all those places of F which lie over the same rational prime (of $\mathbf{Q}$) as v does. We still use S to denote this larger set.

For every $v \in S$, $\gamma(s, \pi_v, r_{1,v}, \bar{\chi}_v)$ is a rational function of q_v^{-s}. Therefore for each $v \in S$, choose a polynomial P_v in the variable q_v^{-s}, satisfying $P_v(0) = 1$, such that $\prod_{v \in S} P_v(q_v^{-s})$ is the numerator of

$$\gamma_S(s, \pi, r_1, \bar{\chi}) = \prod_{v \in S} \gamma(s, \pi_v, r_{1,v}, \bar{\chi}_v).$$

Here, we are assuming that $\gamma_S(s, \pi, r_1, \bar{\chi})$ is reduced with relatively prime numerator and denominator. Let p be a rational prime. Then $\prod_{v|p,\, v\in S} P_v(q_v^{-s})$ is unique. Next, choose polynomials $\tilde{P}_v$, $\tilde{P}_v(0) = 1$, such that $\prod_{v\in S}\tilde{P}_v(q_v^{-s})$ is the numerator of $\gamma_S(s, \pi, \tilde{r}_1, \chi)$. Now by (6.3),

$$\gamma_S(s, \pi, r_1, \bar{\chi}) = \varepsilon_S(s, \pi, r_1, \bar{\chi}) \prod_{v\in S} \frac{\tilde{P}_v\left(q_v^{-(1-s)}\right)^{-1}}{P_v\left(q_v^{-s}\right)^{-1}}, \tag{6.4}$$

where $\varepsilon_S(s, \pi, r_1, \bar{\chi})$ is a product of monomials in q_v^{-s}, $v \in S$.

Suppose π is also χ'-generic, $\chi' = \otimes_v \chi'_v$, where χ'_v is unramified for $v \notin S$. It then follows from the functional equation that for every rational prime p,

$$\prod_{\substack{v|p \\ v\in S}} \gamma(s, \pi_v, r_{1,v}, \bar{\chi}_v)/\gamma(s, \pi_v, r_{1,v}, \bar{\chi}'_v)$$

is a monomial in p^{-s} and therefore $\prod_{v|p,\, v\in S} P_v(q_v^{-s})$ and $\prod_{v|p,\, v\in S}\tilde{P}_v(q_v^{-s})$ are both independent of χ, as long as its local components outside S are unramified.

Thus let S be a finite set of finite places such that for every finite place $v \notin S$, everything is unramified. Set

$$L(s, \pi, r_1, S) = \prod_{v\in S} P_v(q_v^{-s})^{-1} \cdot L_S(s, \pi, r_1) \tag{6.5}$$

and

$$L(s, \pi, \tilde{r}_1, S) = \prod_{v\in S} \tilde{P}_v(q_v^{-s})^{-1} L_S(s, \pi, \tilde{r}_1). \tag{6.6}$$

Observe that since at present we cannot show that for an unramified v, $L(s, \pi, r_{1,v})^{-1}$ and $L(1 - s, \pi, \tilde{r}_{1,v})^{-1}$ are relatively prime as polynomials in q_v^{-s}, we cannot rule out the possible dependence of $L(s, \pi, r_1, S)$ and $L(s, \pi, \tilde{r}_1, S)$ on S. At any rate

$$L(s, \pi, r_1, S) = \varepsilon_S(s, \pi, r_1) L(1 - s, \pi, \tilde{r}_1, S), \tag{6.7}$$

where $\varepsilon_S(s, \pi, r_1) = \varepsilon_S(s, \pi, r_1, \bar{\chi})$ is just a product of monomials in q_v^{-s}, $v \in S$, and therefore an entire and non-zero function of s.

Remark. Suppose v is unramified. Write

$$L(s, \pi_v, r_{1,v}) = \prod_i \left(1 - \alpha_{i,v} q_v^{-s}\right)^{-1}$$

and

$$L(s, \pi_v, \tilde{r}_{1,v}) = \prod_j \left(1 - \tilde{\alpha}_{j,v} q_v^{-s}\right)^{-1}.$$

Assume that for all i and j, $q_v^{-1/2} < |\alpha_{i,v}| < q_v^{1/2}$ and $q_v^{-1/2} < |\tilde{\alpha}_{j,v}| < q_v^{1/2}$. Then it is easy to show that as polynomials in q_v^{-s}, $L(s, \pi_v, r_{1,v})^{-1}$ and $L(1-s, \pi_v, \tilde{r}_{1,v})^{-1}$ are relatively prime, and therefore the factors in the functional equation are independent of S. In view of Remark 2 after Corollary 5.4, this is the case for the Jacquet-Godement L-functions for GL(n).

We now prove:

THEOREM 6.2. *Let $\pi = \otimes_v \pi_v$ be a χ-generic cusp form on M (as well as $\overline{M}$). Let S be a finite set of finite places such that for every $v \notin$ S, every datum (including χ_v) is unramified. Define $L(s, \pi, r_1, \mathrm{S})$, $L(s, \pi, \tilde{r}_1, \mathrm{S})$, and $\varepsilon_{\mathrm{S}}(s, \pi, r_1)$ as before. Then $L(s, \pi, r_1, \mathrm{S})$ extends to a meromorphic function of s on* **C** *with possibly only a finite number of poles. It satisfies*

$$L(s, \pi, r_1, \mathrm{S}) = \varepsilon_{\mathrm{S}}(s, \pi, r_1) L(1-s, \pi, \tilde{r}_1, \mathrm{S}).$$

Moreover, if for every $v = \infty$, π_v is either tempered or spherical, then these poles are all on the real axis, provided that π comes from $\overline{M}$.

Proof. Suppose $m = 2$. Given an archimedean place v, let

$$\mathscr{A}(s, \pi_v, w) = \frac{L(1+2s, \omega_v^{-1}) L(1+s, \tilde{r}_{i,v} \cdot \varphi_v)}{L(2s, \omega_v^{-1}) L(s, \tilde{r}_{i,v} \cdot \varphi_v)} A(s, \pi_v, w). \tag{6.2.1}$$

Then equation (2.7) can be written as

$$\begin{aligned} &L_{\mathrm{S}}(s, \pi, \tilde{r}_1) \underset{v=\infty}{\otimes} \mathscr{A}(s, \pi_v, w) f_v \otimes \bigotimes_{v \in \mathrm{S}} \frac{L(1+2s, \omega_v^{-1})}{L(2s, \omega_v^{-1})} \\ &\qquad \times A(s, \pi_v, w) f_v \otimes \bigotimes_{v \notin \mathrm{S} \cup \{\infty\}} \tilde{f}_v \\ &\quad = \frac{L(1+2s, \omega^{-1})}{L(2s, \omega^{-1})} L_{\mathrm{S}}(1+s, \pi, \tilde{r}_1) M(s, \pi) f. \end{aligned} \tag{6.2.2}$$

For every $v = \infty$, with appropriate choice of f_v, the normalized operator defined by (6.2.1) will only have a finite number of zeros for Re(s) > 0 (no zeros if π_v is tempered). Now inductive application of (6.2.2), together with the properties of Hecke L-functions and the finiteness of poles of $M(s, \pi)$ for Re(s) ≥ 0, will show that for Re(s) $\geq \frac{1}{2}$, $L_{\mathrm{S}}(s, \pi, \tilde{r}_1)$ has only a finite number of poles. The case $m = 1$ is similar, but this time the result is true for Re(s) ≥ 0. The delicate part is to show that multiplication by $\Pi_{v \in \mathrm{S}} P_v(q_v^{-s})^{-1}$ will add, possibly, only finitely many new poles. This we will answer next. Observe that each non-constant P_v has infinitely many zeros in s.

For every $v \in S$, there exists a polynomial Q_v in q_v^{-s}, $Q_v(0) = 1$, such that

$$Q_v(q_v^{-s})A(s, \pi_v, w) \tag{6.2.3}$$

is a holomorphic and non-zero operator (Theorem 2.2.2 of [34], for example). We have:

Lemma 6.3. *Suppose* $\operatorname{Re}(s) \geq \frac{1}{2}$. *Then for* $\operatorname{Im}(s)$ *large enough, the zeros of* $\prod_{v \in S} Q_v(q_v^{-s})$ *are among those of* $L_S(s, \pi, \tilde{r}_1)$.

Proof. This is a consequence of the definition of Q_v and relation (6.2.2).

Lemma 6.4. *As a function of s the zeros of* $\prod_{v \in S} P_v(q_v^{-s})$ *are among those of* $\prod_{v \in S} Q_v(q_v^{-s})$.

Proof. By part (b) of Proposition 3.3.1 of [34] and Lemma 3.1 of the present paper the zeros of $\prod_{v \in S} C_{\chi_v}(s, \pi_v, w)$ are among those of $\prod_{v \in S} Q_v(q_v^{-s})$. If $m = 1$, the lemma is an immediate consequence of (6.1) and (6.4). Next suppose $m = 2$. Write

$$L(s, \omega_v) = (1 - \alpha_v q_v^{-s})^{-1},$$

where $\alpha_v \in \mathbf{C}$. Then $\alpha_v^{-1} = \bar{\alpha}_v$ and

$$L(s, \omega_v^{-1}) = (1 - \alpha_v^{-1} q_v^{-s})^{-1}.$$

Now, using (6.1), assume that the factor $1 - \alpha_v^{1/2} q_v^{1/2} q_v^{-s}$ coming from $L(1 - s, \omega_v^{-1})^{-1}$ does not cancel with any factor in the denominator of $\prod_{v|p,\, v \in S} C_{\chi_v}(s, \pi_v, w)$. Then by Lemma 3.1, the factor $1 - \bar{\alpha}_v^{1/2} q_v^{1/2} q_v^{-s}$ will also appear in the numerator of $\prod_{v|p,\, v \in S} C_{\bar{\chi}_v}(s, \tilde{\pi}_v, w)$. Consequently $1 - \alpha_v^{1/2} q_v^{1/2} q_v^{-s}$ and $1 - \bar{\alpha}_v^{1/2} q_v^{1/2} q_v^{-s}$ will divide $\prod_{v|p,\, v \in S} P_v(q_v^{-s})$ and $\prod_{v|p,\, v \in S} \tilde{P}_v(q_v^{-s})$, respectively, and therefore $1 - \bar{\alpha}_v^{1/2} q_v^{1/2} q_v^{-(1-s)} = 1 - \alpha_v^{-1/2} q_v^{-1/2} q_v^{s}$ will also divide $\prod_{v|p,\, v \in S} \tilde{P}_v(q_v^{-(1-s)})$. But then so does $1 - \alpha_v^{1/2} q_v^{1/2} q_v^{-s}$ which is a contradiction (unless $\alpha_v = 0$). Similarly, $1 + \alpha_v^{1/2} q_v^{1/2} q_v^{-s}$ must also cancel with a factor in the denominator of $\prod_{v|p,\, v \in S} C_{\chi_v}(s, \pi_v, w)$. Therefore again $\prod_{v \in S} P_v(q_v^{-s})$ divides the numerator of $\prod_{v \in S} C_{\chi_v}(s, \pi_v, w)$ and therefore its zeros are among those of $\prod_{v \in S} Q_v(q_v^{-s})$, completing the lemma.

The theorem is now a consequence of Lemmas 6.3 and 6.4, and the functional equation (6.7).

Remark. We would like to make a correction in an earlier article [35] on the same subject. Throughout the paper (Lemma 2.1, Proposition 2.1, Theorems 2.1 and 4.1, and also Corollary 6.1.3 of [36]), the polynomial $1 - \omega_1 \omega_2^{-1}(\bar{\omega}_v) q_v^{-ms}$ must be replaced by $Q_v(q_v^{-s})$, where $Q_v(q_v^{-s})$ which is a divisor (as polynomials in q_v^{-s}) of $1 - \omega_1 \omega_2^{-1}(\bar{\omega}_v) q_v^{-ms}$, is defined as in (6.2.3) of the present paper.

COROLLARY 6.5. *Assume* $\omega = 1$. *Furthermore suppose that for* $v = \infty$, π_v *is either spherical or tempered. Then, if* $L(s, \pi, r_1, s)$ *is holomorphic at* $s = \frac{3}{2}$, *then so is it at* $s = \frac{1}{2}$.

Proof. One only has to exploit equation (6.2.2), together with the simplicity of the poles of $M(s, \pi)$ and the existence of a simple pole for $L(s, 1)$ at $s = 1$.

Examples. We shall now give some interesting examples of L-functions satisfying Theorem 6.2.

COROLLARY 6.6. *Let* $\mathbf{G}$ *be any of the groups in Corollary* 5.4, *and let* r *be the standard representation of* ${}^LG$. *Then for every generic cusp form* π *on* G, *the* L*-function* $L(s, \pi, r, S)$ *extends to a meromorphic function of* s *on* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies a functional equation.*

Remark. For classical groups, this result must also follow from the work of Piatetski–Shapiro and Rallis [29]. The generic assumption is no longer necessary in their work.

COROLLARY 6.7. *Let* π *be a cusp form on* $G = \mathrm{GL}_n(\mathbb{A}_F)$ *and let* r *be either the second exterior power* (*i.e. the representation whose restriction to* $\mathrm{SL}_n(\mathbf{C}) \subset \mathrm{GL}_n(\mathbf{C}) = {}^LG$ *has highest weight equal to* δ_2) *or the second symmetric power representation* (*i.e. the representation whose restriction to* $\mathrm{SL}_n(\mathbf{C})$ *has highest weight equal to* $2\delta_1$) *of* $\mathrm{GL}_n(\mathbf{C}) = {}^LG$. *Then the* L*-function* $L(s, \pi, r, S)$ *extends to a meromorphic function of* s *on* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies a standard functional equation.*

Proof. These are cases (viii) and (iv) of [20], respectively.

Remark. For r equal to the second exterior power representation, the recent work of Jacquet, Piatetski-Shapiro, and Shalika on a lifting from GSP(4) to GL(4) (which is still in preparation) determines the exact location of the poles for the corresponding L-function, at least when $n = 4$. Finiteness of poles for both of these L-functions seems to have an application in the recent work of Gelbart and Piatetski-Shapiro on L-functions for $\mathrm{G} \times \mathrm{GL}_n$, where G is a classical group [11].

COROLLARY 6.8. *Let* π *be a cusp form on* $G = \mathrm{PGL}_6(\mathbb{A}_F)$ *and let* r *be the third exterior power representation of* ${}^LG = \mathrm{SL}_6(\mathbf{C})$ (r *is of dimension* 20; *its highest weight is* δ_3). *Then the* L*-function* $L(s, \pi, r, S)$ *extends to a meromorphic function of* s *on* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies a functional equation.*

Proof. This is the case (x) of [20].

Next, consider the group $\mathbf{G} = \mathrm{SO}(4,4)$. It has a Levi subgroup $\mathbf{M}$ isomorphic to $\mathrm{GL}_2 \times \mathrm{SO}(2,2)$. Their Dynkin diagrams are as in the case $D_4 - 2$. For every field K with $\operatorname{char} K \neq 2$, let $\mathbf{M}_2(K)$ be the algebra of 2 by 2 matrices with entries in K. Then $\mathbf{M}_2(K) \cong K^4$. For every $x \in \mathbf{M}_2(K)$, $x \mapsto \det x$ is a $(2,2)$ form on $\mathbf{M}_2(K)$. Given g_1 and g_2 in $\mathrm{SL}_2(K)$, the map $\lambda(g_1, g_2)$: $x \mapsto g_1 x g_2^{-1}$ is a linear isomorphism of K^4. It fixes $\det x$ and thus belongs to $\mathrm{SO}(2,2)(K)$. In fact this defines a homomorphism from $\mathrm{SL}_2 \times \mathrm{SL}_2$ onto $\mathrm{SO}(2,2)$ whose kernel is isomorphic to $\mathbf{Z}_2$. This implies that $\mathrm{PSO}(2,2) \cong \mathrm{PGL}_2 \times \mathrm{PGL}_2$ and consequently there exists an F-rational homomorphism φ: $\mathbf{M} \to \mathrm{GL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Similarly for the pair $(\mathbf{G}, \mathbf{M})$ in the case ${}^2D_4 - 2$ with $\mathbf{G}$ the quasi-split group $\mathrm{SO}(5,3)$ which splits over a quadratic extension E of F, there exists an F-rational map φ: $\mathbf{M} \to \mathrm{GL}_2 \times \mathrm{Res}_{E/F}\mathrm{PGL}_2$. Let ρ_2 denote either the standard representation of $\mathrm{GL}_2(\mathbf{C})$ or its restriction to $\mathrm{SL}_2(\mathbf{C})$, whichever applies. Moreover, denote by r the representation of $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C}) \rtimes W(\bar{F}/F)$, the L-group of $\mathrm{Res}_{E/F}\mathrm{PGL}_2$, given in the case ${}^2A_3 - 2$ (see the remark after this case in relation to the Asai L-function). Finally, if π is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$, considered as a form on the first factor of M, then ω of Theorem 6.2 is equal to ω_π, the central character of π. We then have:

COROLLARY 6.9. a). *Let $\pi \otimes \pi' \otimes \pi''$ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{PGL}_2(\mathbb{A}_F) \times \mathrm{PGL}(\mathbb{A}_F)$. Then the Rankin triple L-function $L(s, \pi \otimes \pi' \otimes \pi'', \rho_2 \otimes \rho_2 \otimes \rho_2, S)$ extends to a meromorphic function of s on* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies* (6.7). *If π' and π'' are holomorphic, then the triple L-function is independent of S and moreover if $\omega_\pi = 1$, it is holomorphic at $s = \frac{1}{2}$.*

b). *Let $\pi \otimes \pi'$ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{PGL}_2(\mathbb{A}_E)$, $[E : F] = 2$. Then the L-function $L(s, \pi \otimes \pi', \rho_2 \otimes r, S)$ extends to a meromorphic function of s in* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies* (6.7).

Finally, using the cases ${}^3D_4 - 1$ and ${}^6D_4 - 1$, we have:

COROLLARY 6.10. *Let π be a cusp form on $\overline{\mathbf{M}}(\mathbb{A}_F) = \mathrm{PGL}_2(\mathbb{A}_E)$, where E is either a Galois or a non-Galois cubic extension of F. Let r be the representation of ${}^L\overline{M}$ as in the cases ${}^3D_4 - 1$ or ${}^6D_4 - 1$, according as E/F is Galois or non-Galois, respectively. Then the function $L(s, \pi, r, S)$ is a meromorphic function of s on* $\mathbf{C}$ *with possibly only a finite number of poles. It also satisfies* (6.7).

Remark. As we mentioned before, these L-functions have also been studied by Garrett [8], [9], and Piatetski-Shapiro and Rallis [30] by means of certain (similar) integral representations.

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(Received September 9, 1986)