Automorphic L-Functions: A Survey

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The purpose of this article is to report on the progress made on analytic properties of automorphic L-functions after Corvallis. The reader who is interested in the work done before that should consult [2], [6], [15], [16], [21], and [26]. For more details and references we refer the reader to the recent book of Gelbart and Shahidi [21]. We finally refer to [17] and [39] for two recent expository articles on the subject. I would like to thank Jean-Pierre Serre for several comments towards the precision of this article. We start with the following conjecture of Langlands.

1. The conjecture. In this section our main reference is Borel's lectures in Corvallis [6].

1.1 Local Langlands L-Functions. Let F be a non-archimedean local field. Denote by O its ring of integers and let P be its maximal ideal. We use q to denote the number of elements in the residue field O/P. If ψ is a non-trivial (additive) character of F, we shall say ψ is *unramified* if O is the largest ideal of F on which ψ is trivial.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. In this section we shall assume that $\mathbf{G}$ is *unramified*. This means that $\mathbf{G}$ is quasi-split to split over an unramified extension L of F. Let ${}^L G$ be the L-group of $\mathbf{G}$ (cf. [6] and [36]) and denote by ${}^L G^0$ its connected component. For our purposes we may assume ${}^L G = {}^L G^0 \rtimes \Gamma_{L/F}$, where $\Gamma_{L/F}$ is the Galois group of L over F. Let τ be the Frobenius conjugacy class of $\Gamma_{L/F}$. Since $\mathbf{G}$ is unramified we can talk of $\mathbf{G}(O)$ and take it as a hyperspecial maximal compact subgroup K of $G = \mathbf{G}(F)$. Let π be an irreducible admissible K-unramified representation of G. This simply means that there exists a vector in the space of π fixed by K. As it is explained in Sections 6 and 7 of [6], to every such π, Satake isomorphism attaches a unique ${}^L G^0$-semisimple conjugacy class $A \rtimes \tau$ in ${}^L G^0 \rtimes \tau$.

By a representation r of ${}^L G$, we shall mean a continuous homomorphism from ${}^L G$ into some $GL_N(\mathbb{C})$ whose restriction to ${}^L G^0$ is a complex analytic map. Let $\tilde{r}$ denote the contragredient of r.

Fix a complex number s and set

$$(1.1) \qquad L(s, \pi, r) = \det(I - r(A \rtimes \tau)q^{-s})^{-1},$$

where $I = I_n$. This is the local Langlands L-function attached to π and r.

1.2. Langlands' conjecture on automorphic L-functions. In this section we let F be an A-field, i.e. either a number field or a function field of one variable over a finite field. Denote by A_F its ring of adeles. We shall always fix a non-trivial character ψ of $F \backslash A_F$.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. Let $\pi = \otimes_v \pi_v$ be an automorphic form on $G = \mathbf{G}(A_F)$. We refer to [7] for its precise definition.

Let $^L G$ and $^L G_v$ denote L-groups of $\mathbf{G}$ and $\mathbf{G} \times_F F_v$ ($\mathbf{G}$ as a group over F_v), respectively. Then there exists a natural homomorphism $\eta_v : {}^L G_v \to {}^L G$. Let r be a representation of $^L G$ as defined in 1.1. Then each $r_v = r \cdot \eta_v$ is one of $^L G_v$.

For almost all the places v of F, $\mathbf{G} \times_F F_v$ is unramified and π_v is unramified with respect to $\mathbf{G}(O_v)$. We always use S to denote a finite set of places of F, including all the archimedean ones, such that for every $v \notin S$, $\mathbf{G} \times_F F_v$, π_v, and ψ_v, $\psi = \otimes_v \psi_v$, are all unramified.

Given a set S as above and a representation r of $^L G$, let

$$(1.2) \qquad L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v),$$

where the factors on the right are defined as in 1.1. As explained in Theorem 13.2 of [6], given π and r, the Euler product (1.2) converges absolutely for $Re(s)$ sufficiently large and therefore defines a non-zero analytic function of s in that region. Langlands' conjecture on automorphic L-functions can then be stated as follows [36]:

CONJECTURE (LANGLANDS). *For every $v \in S$, it is possible to define a local L-function $L(s, \pi_v, r_v)$, inverse of a polynomial in q_v^{-s}, and a local root number $\varepsilon(s, \pi_v, r_v, \psi_v)$, a monomial in q_v^{-s}, in such a way that*

$$(1.3) \qquad L(s, \pi, r) = \prod_v L(s, \pi_v, r_v)$$

extends to a meromorphic function of s on $\mathbb{C}$ with only a finite number of poles if F is number field, and a rational function of q^{-s}, if F is a function field whose field of constants has q elements, satisfying

$$(1.4) \qquad L(s, \pi, r) = \varepsilon(s, \pi, r) L(1 - s, \pi, \widetilde{r}),$$

where

$$(1.5) \qquad \varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r_v, \psi_v),$$

with $\varepsilon(s, \pi_v, r_v, \psi_v) = 1$ if v is unramified, in particular if $v \notin S$.

In what follows we shall explain the progress made on the conjecture since Corvallis.

2. Rankin-Selberg L-functions.
These L-functions generalize those of Rankin [55] and Selberg [57]. They have been studied by Jacquet, Piatetski-Shapiro, and Shalika in a series of papers, but unfortunately their complete results have yet to appear.

2.1. The L-functions. Here $\mathbf{G} = GL(n) \times GL(m)$, where m and n are two positive integers. We may take $^L G = GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$. The representation r is equal to $r = \rho_n \otimes \rho_m$, where ρ_n and ρ_m are standard representations of $GL_n(\mathbb{C})$ and $GL_m(\mathbb{C})$, respectively. Let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be cusp forms on $GL_n(\mathbf{A}_F)$ and $GL_m(\mathbf{A}_F)$, respectively. The homomorphism η_v of 1.2 is the identity. If v is unramified, we set

$$L(s, \pi_v \times \pi'_v) = L\big(s, (\pi_v, \pi'_v), \rho_n \otimes \rho_m\big),$$

where the L-function on the right is as in 1.1. Then

$$L(s, \pi_v \times \pi'_v) = \prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} (1 - \alpha_{i,v} \alpha'_{j,v} q_v^{-s})^{-1},$$

where $A_v = \{\mathrm{diag}(\alpha_{1,v}, \ldots, \alpha_{n,v})\}$ and $A'_v = \{\mathrm{diag}(\alpha'_{1,v}, \ldots, \alpha'_{m,v})\}$ are the semisimple conjugacy classes attached to π_v and π'_v (cf. 1.1), respectively. With notation as in 1.2, we let

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi'_v).$$

This is the partial Rankin-Selberg L-function attached to π, π', and S. For $m = n = 2$, they were studied by Jacquet who generalized results of Rankin [55] and Selberg [57]. On the other hand, if $m = 1$, they are the principal L-functions of Godement and Jacquet (cf. [26]).

2.2. The results. The results can be stated as follows:

a) The partial L-function $L_S(s, \pi \times \pi')$ converges absolutely for $\mathrm{Re}(s) > 1$ ([29]).

b) $L_S(s, \pi \times \pi')$ extends to a meromorphic function of s on $\mathbb{C}$ ([34]).

c) For $m \neq n$, $L_S(s, \pi \times \pi')$ is holomorphic on $\mathrm{Re}(s) \geq 1$ ([29,30]).

d) Assume $m = n$. Let

$$X = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1, \alpha^{s-1} \otimes \pi \cong \widetilde{\pi'}\}.$$

Then $L_S(s, \pi \times \pi')$ has a pole at s_0 with $\mathrm{Re}(s_0) = 1$ if and only if $s_0 \in X$. This pole is simple [30]. Here $\alpha = |\det(\ )|$.

e) For $\mathrm{Re}(s) = 1$, $L_S(s, \pi \times \pi') \neq 0$ ([59], also Theorem 3.2.3 below).

f) If $v < \infty$, the L-function $L(s, \pi_v \times \pi'_v)$ and the root number $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ are defined in [28].

g) If $v = \infty$, let $\varphi_v : W_v \to GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ be the homomorphism attached to $\pi_v \otimes \pi'_v$ by local class field theory [6,37], where W_v is the Weil group $W(\overline{F}_v/F_v)$. Denote by $L(s, r \cdot \varphi_v)$ and $\varepsilon(s, r \cdot \varphi_v, \psi_v)$, the Artin L-function and root number attached to $r \cdot \varphi_v$ [69], where $r = \rho_n \otimes \rho_m$. We then set

$$L(s, \pi_v \times \pi'_v) = L(s, r \cdot \varphi_v)$$

and

$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, r \cdot \varphi_v, \psi_v).$$

h) Let

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v)$$

and

$$\varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v),$$

where the factors are defined as in f) and g). Then

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi'}).$$

This is proved by combining the results in [59], [60], and [62].

i) The L-function $L(s, \pi \times \pi')$ is expected to be entire unless $m = n$ and $\pi \otimes \alpha^s \cong \widetilde{\pi'}$ for some s in which case poles are simple. More precisely $L(s, \pi \times \pi)$ is expected to have simple poles at $s = 0, 1$. A very recent preprint of Waldspurger [71] seems to have answered

this question also positively and therefore the theory must now be complete (cf. [41]).

All the results are proved completely for number fields. Parts a, c, and d are also stated for function fields [29, 30]. Immediate extensions of all other parts to function fields are expected, but have never been stated anywhere.

2.3. Applications.

2.3.1. Classification of automorphic forms for $GL(n)$. Let $G = GL_r(\mathbf{A}_F)$. Fix a cusp form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_u$ of $M = GL_{r_1}(\mathbf{A}_F) \times \cdots \times GL_{r_u}(\mathbf{A}_F)$, $r_1 + \cdots + r_u = r$, where M is considered as the standard Levi subgroup of the standard parabolic subgroup $P = MN$ of G. Let $\xi = \underset{v}{\otimes} \xi_v$ be the representation

$$\xi = \operatorname{Ind}(G, P, \sigma \otimes 1).$$

Similarly assume Q is another standard parabolic subgroup of G and τ a cusp form on its standard Levi subgroup. Set

$$\eta = \operatorname{Ind}(G, Q, \tau \otimes 1).$$

We choose a finite set S of places of F such that for $v \notin S$, σ_v and τ_v are both unramified. Then ξ_v and η_v have the same unramified components if and only if (σ, P) and (τ, Q) are conjugate , i.e. up to a permutation they are equivalent. This is proved in [30] and is a consequence of 2.2.a, 2.2.c, 2.2.d, and 2.2.e. When $M = G$ this is called the *Strong Multiplicity one Theorem*. A stronger version of this case is proved in [42].

2.3.2. Converse theorem. As explained in paragraph 14.6 of [6], it is expected that the analytic properties of these *L*-functions would lead to existence of automorphic forms on $GL_r(\mathbf{A}_F)$. But unfortunately the only published version of this is still [27].

2.3.3. Applications in Base change for $GL(n)$. Almost all the results in 2.2 are used by Arthur and Clozel in [3] to establish base change for forms on $GL(n)$.

3. Langlands' Euler products method.
In a series of lectures in 1967, Langlands expressed constant terms of Eisenstein series on certain split algebraic groups as ratios of products of certain automorphic *L*-functions. From this he deduced the meromorphy of these

L-functions in a number of cases on the whole complex plane. This also gave him the most substantial evidence for his conjecture of Section 1.2. These lectures were later published as a book titled "Euler Products" [34]. Langlands' method was later pursued by Shahidi (cf. [59, 60, 63, 65], for example) who generalized and established further properties of these L-functions for the so called "generic" representations. The recent preprint of Waldspurger [71] must shed new lights on the whole theory, since in the case of $GL(N)$ it combines the results of this method with certain results of Jacquet, Piatetski-Shapiro, and Shalika to prove the holomorphy of Rankin-Selberg L-functions (cf. Section 2) for all $s \neq 0, 1$ (cf. §2.2.1); thus avoiding deeper analysis of the local integral representations of Jacquet, Piatetski-Shapiro, and Shalika for these L-functions.

3.1. The set up. Let **H** be a quasi-split connected reductive algebraic group over a *number field* F. Fix a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of **H** and let $\mathbf{P} = \mathbf{MN}$ be a standard maximal parabolic $(\mathbf{U} \supset \mathbf{N})$ subgroup of **M**. Let ^{L}M be the L-group of **M** and denote by $^{L}\mathfrak{n}$ the Lie algebra of the L-group ^{L}N of **N**. If r denotes the adjoint action of ^{L}M on $^{L}\mathfrak{n}$, we write $r = \overset{m}{\underset{i=1}{\oplus}} r_i$ for its decomposition to irreducible components.

It is the group **M** and the representations r_i for which the conjecture can be addressed. To be in accordance with our general notation from now on we shall use **G** instead of **M**. Let $\mathbf{A}_F$ denote the ring of adeles of F. For every group **L** over F we use L to denote $\mathbf{L}(\mathbf{A}_F)$. Fix a character $\chi = \underset{v}{\otimes}\chi_v$ of $\mathbf{U}(F) \setminus U$. We shall assume χ is *generic*. This simply means that the restriction of χ to every simple root group is non-trivial. Let $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{G}$. We again use χ to denote $\chi|U^0$.

Let $\pi = \underset{v}{\otimes}\pi_v$ be a cuspidal representation of G. We shall say π is *globally χ-generic* if there exists a cusp form φ in the space of π such that

$$\int_{\mathbf{U}^0(F)\setminus U^0} \varphi(ug)\overline{\chi(u)}du \neq 0$$

for some $g \in G$.

Fix a non-trivial character $\psi = \underset{v}{\otimes}\psi_v$ of $F \setminus \mathbf{A}_F$. Then there is a natural generic character χ_0 of $\mathbf{U}(F) \setminus U$ defined by ψ. Changing the splitting on **G** we may assume that π is χ_0-generic. Otherwise said, we can find a cusp form in the L-packet of π, generic with respect

to χ_0. Observe that $\chi = \chi_0 \cdot Ad(a)$, where $a \in \mathbf{A}_0(\overline{F})$ with $Ad(a)$ defined over F. Here $\mathbf{A}_0$ is the maximal split torus of π.

3.2. The results. The theory of local coefficients as developed in [59], [60], [63], and [65] leads to a number of general and deep results both in the theory of automorphic forms and representations of p-adic groups. We shall now state some of these results.

In what follows generic always means globally generic, and changing the splitting we may always assume that our representation is χ_0-generic. The following theorem is Theorem 3.5 of [65].

THEOREM 3.2.1. *([59, 60, 63, 65]) Assume π is χ_0-generic. Then each $L_S(s, \pi, r_i)$ converges absolutely for $\mathrm{Re}(s) > 2$ and extends to a meromorphic function of s on $\mathbb{C}$ (Theorem 5.1 of [63] and Remark 12.4 of [65]). Moreover for each i, $1 \leq i \leq m$, and each $v \in S$, there exists a complex function $\gamma_i(s, \pi_v, \psi_v)$ (which is a rational function of $q_v^{-s}, v < \infty$), satisfying the following properties:*

a) If $v = \infty$ or π_v has a Iwahori fixed vector (in particular if π_v has a vector fixed by a special maximal compact subgroup) and $\varphi'_v : W'_{F_v} \to {}^L G$ is the homomorphism of the Deligne-Weil group attached to π_v, then

$$\gamma_i(s, \pi_v, \psi_v) = \varepsilon(s, r_{i,v} \cdot \varphi'_v, \psi_v) L(1 - s, \widetilde{r}_{i,v} \cdot \varphi'_v) / L(s, r_{i,v} \cdot \varphi'_v),$$

where $L(s, r_{i,v} \cdot \varphi'_v)$ and $\varepsilon(s, r_{i,v} \cdot \varphi'_v, \psi_v)$ are the Artin L-function and root number attached to $r_{i,v} \cdot \varphi'_v$ (cf. [69]).

b) For each i, $1 \leq i \leq m$,

$$L_S(s, \pi, r_i) = \prod_{v \in S} \gamma_i(s, \pi_v, \psi_v) L_S(1 - s, \pi, \widetilde{r}_i)$$

c) The factors γ_i are defined locally for every quasi-split local group, a Levi factor of it, and an irreducible admissible χ_0-generic representation σ of this Levi factor. They satisfy

$$\gamma_i(s, \sigma, \psi) \gamma_i(1 - s, \widetilde{\sigma}, \overline{\psi}) = 1,$$

where χ_0 is defined by means of ψ.

d) Together with an inductive property, the conditions a) and b) determine γ_i's uniquely.

The functional equation in part b) is a consequence of Theorem 4.1 of [59] and the inductive results of [63] and [65]. The fact that the local factors $\gamma_i(s, \pi_v, \psi_v)$ at the archimedean places are Artin factors is the main result (Theorem 3.1) of [60].

THEOREM 3.2.2. *([63], [65]) Assume π is χ_0-generic. Then for every $v \notin S$, every local L-function $L(s, \pi_v, r_{i,v})$ is holomorphic if $\mathrm{Re}(s) \geq 1$, $1 \leq i \leq m$.*

THEOREM 3.2.3. *(Theorem 5.1 of [59]) Assume π is χ_0-generic. Then*

$$\prod_{i=1}^{m} L_S(1, \pi, r_i) \neq 0.$$

Lists of all the possible $(\mathbf{H}, \mathbf{G}, r_i)$ are given in [34] and [63]. They include all the cases known by other methods. Examples will be given in the next several sections. We conclude this section with the following:

THEOREM 3.2.4. *(Theorem 6.1 of [63]) Assume $m = 1$ or 2 and moreover if $m = 2$, assume r_2 is one dimensional. Suppose π is χ_0-generic. Then for each $v \in S$, a local L-function $L(s, \pi_v, r_{1,v})$ can be defined in such a way that*

$$L(s, \pi, r_1) = \prod_v L(s, \pi_v, r_{1,v})$$

extends to a meromorphic function of $s \in \mathbb{C}$ with possibly only a finite number of poles, satisfying a functional equation. The factors at the archimedean places are Artin factors (cf. Theorem 3.2.1.a)

COROLLARY. *With assumptions as in Theorem 3.2.4, let S be a finite set of places of F, including all the ramified and archimedean ones, such that if $v \in s$ is ramified, then S contains all other places which lie over the same rational prime as v does. Then the partial L-function $L_S(s, \pi, r_1)$ extends to a meromorphic function of s with possibly only a finite number of poles on all of $\mathbb{C}$.*

3.3. Examples of Theorem 3.2.4.
In all the following examples, besides Theorems 3.2.1–3.2.3, Theorem 3.2.4 applies and consequently the finiteness of poles on all of $\mathbb{C}$ also follows.

3.3.1. Rankin triple products.
(Corollary 6.9 of [63]). Let $\mathbf{H} = \mathrm{Spin}(4, 4)$ and take $\mathbf{G} = GL_2 \times SL_2 \times SL_2$. Fix a cusp form $\pi = \pi_1 \times \pi_2 \times \pi_3$ on $GL_2(\mathbf{A}_F) \times SL_2(\mathbf{A}_F) \times SL_2(\mathbf{A}_F)$. Assume $v \notin S$, $A(\pi_{1,v}) = \mathrm{diag}(\alpha_{1,v}, \alpha_{2,v})$, and $A(\pi_{2,v}) = \mathrm{diag}(\beta_{1,v}, \beta_{2,v})$, $A(\pi_{3,v}) = \mathrm{diag}(\gamma_{1,v}, \gamma_{2,v})$, both modulo the center of $GL_2(\mathbb{C})$. Then

$$(3.3.1.1) \qquad L(s, \pi_v, r_1) = \prod_{i,j,k=1,2} (1 - \alpha_{i,v}\beta_{j,v}\gamma_{k,v}q_v^{-s})^{-1}.$$

This is the first and only example of a triple Rankin product L-function of automorphic forms available at present. An integral representation for this L-function has been obtained by Garrett [13]. Using group representations this has also been treated by Piatetski-Shapiro and Rallis in [49]. We shall discuss these in Section 6 below.

3.3.2. Twisted triple products. (Cases $^3D_4 - 1$ and $^6D_4 - 1$ of [63]). Here **H** is the quasi-split orthogonal group of type D_4 defined by a separable extension E of degree 3 over F. We can take **G** such that there exists a surjection $\mathbf{G} \xrightarrow{\rho} \mathrm{Res}_{E/F} PGL_2 \longrightarrow 0$. The representation $r_1 \cdot {}^L\rho$ is an irreducible 8-dimensional representation of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rtimes \Gamma_{E/F}$ and $L(s, \pi_v, r_{1,v})$ generalizes the Asai's L-function (cf. [4,23]). If $v \notin S$ is inert, and $A(\pi_v) = (\mathrm{diag}(\alpha_v, \alpha_v^{-1}), I_2, I_2)$, then

$$(3.3.2.1) \quad L(s, \pi_v, r_{1,v})$$
$$= (1 - \alpha_v q_v^{-s})^{-1}(1 - \alpha_v^{-1} q_v^{-s})^{-1}(1 - \alpha_v q_v^{-3s})^{-1}(1 - \alpha_v^{-1} q_v^{-3s})^{-1}.$$

According as E/F is normal or not, this is the case $^3D_4 - 1$ or $^6D_4 - 1$ of [63]. Again we refer to [13] and [49] for an integral representation.

3.3.3. Second symmetric or exterior power L-functions for GL_n.
Using the cases (viii) and (iv) of [34], one can show that the results of the previous section all hold for $L_S(s, \pi, r)$, where π is a cusp form on $GL_n(\mathbf{A}_F)$ and r is either the symmetric or exterior square of the standard representation of $GL_n(\mathbb{C})$. They are also subject of a work in progress of Jacquet and Shalika, and Bump and Friedberg [9]. Finally when $n = 3$, we refer to [46] (cf. Section 7.1 for an application).

3.3.4. Exterior cube L-function for GL_6. Let **H** be the simply connected split group of type E_6. There is a parabolic subgroup whose Levi factor **G** is isomorphic to $(GL_1 \times SL_6)/\{\pm 1\}$. Let π_0 be a cusp form on $GL_6(\mathbf{A}_F)$ with central character ω. Then we use π to denote any irreducible component of $\omega^3 \otimes (\pi | SL_6(\mathbf{A}_F))$. It is a cusp form on G. If $\wedge^3 \rho_6$ denotes the exterior cube of the standard representation of $GL_6(\mathbb{C})$ (this is a 20-dimensional irreducible representation), then for $v \notin S$

$$L(s, \pi_v, r_1) = L(s, \pi_{0,v}, \wedge^3 \rho_6).$$

In fact if $A(\pi_{0,v}) = \mathrm{diag}(\alpha_{1,v}, \ldots, \alpha_{6,v})$, then

$$L(s, \pi_{0,v}, \wedge^3 \rho_6) = \prod_{\substack{i \neq j,k \\ j \neq k \\ i,j,k=1}}^{6} (1 - \alpha_{i,v}\alpha_{j,v}\alpha_{k,v}q_v^{-s})^{-1}.$$

Now all the results of the previous section apply to $L_S(s, \pi_0, \wedge^3 \rho_6)$. This is example (x) of [34] (cf. Corollary 6.8 of [63] for PGL_6).

We should remark that in all these cases there is no restriction on the form since all the cusp forms on the group GL_n are globally generic [66] (with respect to any character).

3.4. Applications. The results of Section 3.2 can be used to compute Plancherel measures for quasi-split group [59, 65]. In particular, it leads to a proof of a conjecture of Langlands [35] on Plancherel measures [62], [64], [65]. Such results can be used to obtain deep results on non-supercuspidal tempered spectrum of many quasi-split groups (cf. [64] and [65], for example).

4. Symmetric power L-functions for GL_2. Let $\pi = \otimes_v \pi_v$ be a cusp form on $GL_2(\mathbf{A}_F)$. Then for $v \notin S$, $A(\pi_v) = \mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbf{C})$. Given a positive integer m, let r_m denote the m-th symmetric power of the standard representation of $GL_2(\mathbf{C})$. This is an $(m+1)$-dimensional irreducible representation. Then, for $v \notin S$

$$L(s, \pi_v, r_m) = \prod_{0 \leq j \leq m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1}.$$

The L-functions $L_S(s, \pi, r_m)$ are quite important. They, basically, comprise all the automorphic L-functions for GL_2. Besides:

a) Assume that for every $m \in \mathbf{Z}^+$, $L_S(s, \pi, r_m)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Then in [36], Langlands showed that for every $v \notin S$, $|\alpha_v| = |\beta_v| = 1$. This is the Ramanujan-Petersson's conjecture for π. One of the deepest and most difficult conjectures in number theory, whose proof in the case of holomorphic forms is due to Deligne [10]. For non-holomorphic forms the problem is still open (cf. 4.1.3 below), except for those forms which correspond to Galois representations (cf. the remark after Theorem 10.1 of [72]).

b) Assume a) and in addition that for every $m \in \mathbf{Z}^+$, the L-function $L_S(s, \pi, r_m)$ is non-zero and holomorphic for $\mathrm{Re}(s) = 1$. Then Sato-Tate's conjecture is valid [58]. It is a result of K. Murty [44] that if one

knows the holomorphy for all m, then one has the non-vanishing for all m, and therefore the conjecture follows only from the holomorphy of all of these L-functions for $\mathrm{Re}(s) \geq 1$.

4.1. Results. Except for Murty's result mentioned above, there are no general results known about these L-functions. All that we know is for $m \leq 5$ which we shall now explain. We shall leave out the classical case $m = 1$ for which the conjecture is known following Hecke, Jacquet-Langlands, and Weil.

4.1.1. The case $m = 2$. This is the only non-classical case which we know the conjecture for $L(s, \pi, r_m)$ (cf. (1.3)). In fact it was proved by Shimura [68] that if π comes from a classical modular form, then $L(s, \pi, r_2)$ is entire unless there exists a non-trivial quadratic character η of $Q^* \setminus \mathbf{A}_Q^*$ such that $\pi \otimes \eta \cong \pi$, i.e. π is *monomial*. This was later extended to cusp forms on any $GL_2(\mathbf{A}_F)$ by Gelbart and Jacquet [18], where F is a $\mathbf{A}$-field.

4.1.2. Gelbart-Jacquet lift. Using the results of [18], it is now a simple application of the converse theorem for GL_3 (cf. [27]) that given a cusp form π on $GL_2(\mathbf{A}_F)$, there exists an automorphic representation Π on $GL_3(\mathbf{A}_F)$ such that

$$L(s, \Pi \otimes \omega) = L(s, \pi, r_2),$$

where the L-function on the left is the standard L-function of $\Pi \otimes \omega$ (cf. [26]) and ω is the central character of π. The representation Π is cuspidal unless $\pi \otimes \eta \cong \pi$ with η as in 4.1.1. The representation Π is what we call the *Gelbart-Jacquet lift* of π. We refer to [11] for a different approach using the trace formula.

4.1.3. Best estimates for Fourier coefficients. Now assume $\pi = \underset{v}{\otimes}\pi_v$ is a non-monomial (cf. 4.1.1) cusp form on $GL_2(\mathbf{A}_F)$. (For the monomial cusp forms the Ramanujan-Petersson's conjecture is automatically valid.) For $v \notin S$, let $A(\pi_v) = \mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbb{C})$. Then using Gelbart-Jacquet lift Π of π, it can be shown that

$$q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$$

and

$$q_v^{-1/5} < |\beta_v| < q_v^{1/5},$$

(cf. 4.a above). When $F = \mathbf{Q}$ and in the form $p^{-1/5} \leq |\alpha_p| \leq p^{1/5}$, this was first proved by Serre in a letter to Deshouillers, but was

never published. Published versions of the proof can be found in [43] and [45]. Unfortunately their proofs make use of certain unpublished results of Jacquet, Piatetski-Shapiro, and Shalika on Rankin-Selberg L-functions (cf. Section 2, here). It was precisely for this reason that Serre never published his proof.

In general (i.e. for $GL_2(\mathbf{A}_F)$ with F any number field) and with strict inequality, this is basically Corollary 5.5 of [63]. Its proof is complete and requires no unpublished results.

At the archimedean places, the best results are due to Gelbart-Jacquet [18] and Iwaniec [25]. We refer to [63] and [40] for estimates for other groups.

4.1.4. The case $m = 3$. This is one of the cases of Theorem 3.2.4 (observed by Langlands in [34]) which when mixed with the results of 4.1.2 and 2.2 leads to a functional equation with Artin L-functions at every place of F. Moreover, it can be shown that under a non-vanishing hypothesis for Jacquet-Langlands L-function $L(s, \pi)$ on an interval parallel to $[1/2, 1)$ (parallel to $(1/2, 1)$ if π is on $PGL_2(\mathbf{A}_F)$), $L(s, \pi, r_3)$ is entire [64]. The fact that $L(s, \pi, r_3) \neq 0$ for $\text{Re}(s) \geq 1$ is basically proved in [59] (cf. Theorem 3.2.3).

4.1.5. The cases $m = 4$ and 5. Both L-functions extend to meromorphic functions of s on $\mathbf{C}$, each satisfying a functional equation (Theorem 3.2.1). Moreover, it is proved in [59], that for $\text{Re}(s) = 1$, the L-function $L_S(s, \pi, r_4)$ is non-zero. When $m = 5$, it is proved in [64], that $L_S(s, \pi, r_5) \neq 0$ for $\text{Re}(s) = 1$, except possibly for a simple zero at $s = 1$. As it is explained in [64], even this leads to non-trivial results in the direction of Sato-Tate's conjecture for holomorphic forms (cf. 4.b). This is due to Serre.

5. The work of Piatetski-Shapiro and Rallis. In around 1980, Waldspuger [70], using an ingenious method, described the Shimura correspondence [67] between automorphic forms on $\overline{SL_2}(\mathbf{A}_F)$, the two fold metaplectic covering of $SL_2(\mathbf{A}_F)$, and $PGL_2(\mathbf{A}_F)$ (a dual reductive pair; cf. [24]), by means of integration against the restriction of a theta function on a bigger group to $\overline{SL_2} \times PGL_2$. This idea was later generalized by Rallis [54], who, using a result of Kudla [33] and the Siegel-Weil formula expressed the norm of the corresponding lift F_f as an integral of an Eisenstein series on a bigger group against the product of f by itself (cf. [21], Section III.1.1 for more detail). These are the type of integrals which appear in the work of Piatetski-Shapiro and Rallis which we shall now explain.

5.1. The set up. Let $\mathbf{G}$ be a connected reductive algebraic group over F whose center $\mathbf{C}$ is anisotropic, i.e. $\mathbf{C}(F) \setminus \mathbf{C}(\mathbf{A}_F)$ is compact. Assume there exists another reductive group $\mathbf{H}$ over F in which $\mathbf{G} \times \mathbf{G}$ can be embedded. Let $\mathbf{G}^d$ be the image of $\mathbf{G}$ under the diagonal embedding of $\mathbf{G}$ in $\mathbf{H}$. Fix a parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$. Then $\mathbf{G} \times \mathbf{G}$ acts on $\mathbf{P} \setminus \mathbf{H} = X$. An orbit $X' \subset X$ of $\mathbf{G} \times \mathbf{G}$ is called *negligible* if the stabilizer $R' \subset \mathbf{G} \times \mathbf{G}$ of a point $x' \in X'$ contains the unipotent radical of a proper parabolic subgroup of $\mathbf{G} \times \mathbf{G}$. Let $x_0 \in X$ correspond to the coset $\mathbf{P} \cdot e$ and denote by X_0 its $\mathbf{G} \times \mathbf{G}$-orbit. This is called the main orbit. Then the stabilizer R_0 of x_0 in $\mathbf{G} \times \mathbf{G}$ is $\mathbf{P} \cap (\mathbf{G} \times \mathbf{G})$. We shall now assume that the following two conditions are satisfied:

$\quad$ a) $\quad R_0 = \mathbf{G}^d$, and

$\quad$ b) $\quad$ every $X' \neq X_0$ is negligible.

If s is a complex number, there is a natural character ω_s of $\mathbf{P}(F) \setminus \mathbf{P}(\mathbf{A}_F)$ which is trivial on $\mathbf{G}^d(\mathbf{A}_F)$. Fix a function f in the space of $\mathop{\mathrm{Ind}}\limits_{\mathbf{P}(\mathbf{A}_F) \uparrow \mathbf{H}(\mathbf{A}_F)} \omega_s$. We then let:

$$E(\omega_s, f, h) = \sum_{\gamma \in \mathbf{P}(F) \setminus \mathbf{H}(F)} f(\gamma h),$$

where $h \in \mathbf{H}(\mathbf{A}_F)$.

Let π be an irreducible cuspidal representation of $G = \mathbf{G}(\mathbf{A}_F)$. Choose a pair of cusp forms φ_1 and φ_2 in the spaces of π and its contragredient, respectively. Consider

(5.1.1) $\quad Z(\omega_s, \varphi_1, \varphi_2, f)$

$$= \int_{\mathbf{G}(F) \times \mathbf{G}(F) \setminus G \times G} \varphi_1(g_1) \varphi_2(g_2) E(\omega_s, f, (g_1, g_2)) dg_1 dg_2.$$

It is easy to see that under assumptions a) and b) above,

(5.1.2) $\quad Z(\omega_s, \varphi_1, \varphi_2, f) = \int_G f(g, 1) < \pi(g)\varphi_1, \varphi_2 > dg,$

where $(g, 1)$ is considered as an element of H by the embedding of $G \times G$ into H. Choosing φ_1 and φ_2 appropriately, we may assume (5.1.2) is Eulerian. Replacing E by the normalized $E^*($ cf. [47])

which has only a finite number of poles (this is accomplished if one multiplies E by a product of abelian L-functions which eliminates the infinitely many unwanted poles of E), we shall see that (5.1.1) provides us with an integral representation for certain automorphic L-functions. This would then lead to a proof of the finiteness of poles for these L-functions (since E^* has only a finite number of poles). We should remark that this may be considered as a generalization of the work of Godement and Jacquet on principal L-functions [26].

5.2. The results. [47, 48, 50] Choosing **H** appropriately, (5.1.1) will provide us with an integral representation for $L(s, \pi, r)$, where π is a cusp form on $\mathbf{A}_F$-points of either of the groups $\mathbf{G} = Sp_{2n}$, O_n, or U_n, and r is the standard representation of the corresponding L-group $^L G$. We should remark that the cusp forms no longer have to be generic. As mentioned above, this proves the finiteness of poles for each $L_S(s, \pi, r)$ on $\mathbb{C}$. The local factors at the ramified primes have not yet been all defined and therefore the conjecture of 1.2 has not yet been completely verified in these cases. We should remark that, using classical methods for holomorphic forms, some of these results have also been obtained by Andrianov [1], Gritsenko [22], as well as Böcherer and Schulze-Pillot (cf. [5]). We finally refer the reader to [8] for an integral representation for the L-function $L(s, \pi, r)$ where π is a globally generic cusp form on $GSp_6(\mathbf{A}_F)$, trivial on the center, and r is the irreducible eight dimensional representation of $^L GSp_6$ (which is isogenous to $\mathrm{Spin}(7, \mathbb{C})$). This L-function can also be found by the method of Chapter 3.

6. Rankin triple products. One of the striking developments in the theory of automorphic L-functions in the past few years has been the work of Paul Garrett [13] who has obtained an integral representation for the Rankin triple product L-functions (cf. §3.3.1 here). Even though many properties of these L-functions could already be concluded from the results of Chapter 3, this was the first time that an integral representation for these L-functions could be found, forty eight years after Rankin's work [55] on double L-functions. This was later on generalized in [14] to include the twisted cases as well. After his results were explained, it became clear that this is one of the cases that can be obtained from the Piatetski-Shapiro-Rallis' theory. This was done in [49], generalizing the work of Garrett to non-holomorphic forms. We shall now explain both works.

6.1. The work of Garrett [13]. Let φ_1, φ_2, and φ_3 be three

holomorphic cusp forms on the upper half plane H. Denote by H_3 the Siegel upper half space of degree 3. Then $H \times H \times H$ can be embedded in H_3. Let $E(z, s)$ be the abelian Eisenstein series on H_3. Consider

$$(6.1.1) \qquad \int_{H \times H \times H} \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)E((z_1, z_2, z_3), s)dz_1 dz_2 dz_3.$$

Then in [13], Garrett shows that (6.1.1) is in fact Eulerian and if $E(z, s)$ is normalized (cf. Section 5.1) properly, the factors (for $v \notin S$) are equal to the Rankin triple product $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$, where π_1, π_2 and $\pi_3(\pi_i = \otimes_v \pi_{i,v})$ are automorphic representations attached to φ_1, φ_2, and φ_3, respectively. The L-function $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ is the L-function defined by the right hand side of (3.3.1.1). Even though not proved in [13], this must at least lead to the finiteness of poles on $\mathbb{C}$ for $L_S(s, \pi_1 \times \pi_2 \times \pi_3)$ (cf. Section 5.1).

6.2. The work of Piatetski-Shapiro and Rallis [49]. Let K be a semi-simple abelian algebra of degree 3 over F. Then either

$$(6.2.1) \qquad K = F \oplus F \oplus F,$$
$$(6.2.2) \qquad K = E \oplus F, \qquad [E : F] = 2, \text{ or}$$
$$(6.2.3) \qquad K = K \qquad\qquad [K : F] = 3.$$

Let $V = K \oplus K$ and define an alternating form A on V by

$$A[(x, y), (x', y')] = xy' - x'y,$$

where (x, y) and (x', y') are in V. Set $A' = tr_{K/F}A$. Then A' is a F-valued skew symmetric form on V and $GS_p(A') = GSp_6(F)$. The group $GL_2(K)$ acts on V. Let $GL_2(K)^0$ be the points of $GL_2(K)$ which under this action belong to $GS_p(A')$. Then, for example

$$GL_2(K)^0 = \{(g_1, g_2, g_3)|g_i \in GL_2(F), \det g_1 = \det g_2 = \det g_3\}$$

if we are in case (6.2.1), while

$$GL_2(K)^0 = \{g \in GL_2(K)| \det g \in F^*\}$$

in case (6.2.3). Next, let $\mathbf{P} = \mathbf{MN}$ be the parabolic subgroup of GSp_6 with $\mathbf{M} = GL_3 \times GL_1$. With notation as in Section 5.1, we choose f in

the space of $\underset{P(\mathbf{A}_F)\uparrow GSp_6(\mathbf{A}_F)}{\text{Ind}}\ \omega_s$ and let $E(\omega_s, f, x), x \in GSp_6(\mathbf{A}_F)$, be
the corresponding Eisenstein series. Finally let Π be a cusp form on
the adelized version of $GL_2(K)^0$ which we denote by $GL_2(K)^0(\mathbf{A}_F)$.
Then $\Pi = \pi_1 \times \pi_2 \times \pi_3$ with each π_i a cusp form on $GL_2(\mathbf{A}_F)$ if we
are in case (6.2.1), while Π is a cusp form on $GL_2(\mathbf{A}_K)$ in the other
extreme. If $\mathbf{Z}'$ is the center of GSp_6 and φ is in the space of Π, we
set

$$Z(s, \varphi, f) = \int_{\mathbf{Z}'(\mathbf{A}_F)GL_2(K)^0 \backslash GL_2(K)^0(\mathbf{A}_F)} \varphi(x)E(\omega_s, \varphi, x)dx.$$

We are now in the situation of Section 5.1 and we must study the
right orbits of $GL_2(K)^0$ in $\mathbf{P}(F) \backslash GSp_6(F)$. Conditions a) and b)
of 5.1 are satisfied and $Z(s, \varphi, f)$ becomes Eulerian. Normalizing
$E(\omega_s, f, x)$ appropriately then shows that for $v \notin S$ the local fac-
tors are equal to $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ in case (6.2.1) (cf. §3.3.1) and
are defined by

$$(1 - \alpha_v q_v^{-s})^{-1}(1 - \beta_v q_v^{-s})^{-1}(1 - \alpha_v \beta_v^2 q_v^{-3s})^{-1}(1 - \alpha_v^2 \beta_v q_v^{-3s})^{-1},$$

if we are in case (6.2.3) and v is inert; as in §3.3.2. We refer the reader
to case (ii) in page 96 of [21] for the case (6.2.2) (cf. Corollary 6.9.b
of [63]). We finally remark that case (6.2.3) extends Asai's result [4]
from quadratic to cubic extensions. The results are formulated as the
following theorem in [49]. Here we use $L_S(s, \Pi)$ to denote the product
of the local factors discussed above.

THEOREM 6.2. [49]. *Under the assumption that F is totally real
(and an assumption on the central character of Π), the partial L-
function $L_S(s, \Pi)$ can be extended to all the finite ramified primes in
such a way that the resulting L-function satisfies a functional equa-
tion. Moreover its possible poles are at $s = 0, \frac{1}{4}, \frac{3}{4},$ and 1.*

We remark that in view of Theorems 3.2.1 and 3.2.4 of Section 3
what is new and does not seem to follow from the method of Section
3 is the possible location of poles.

7. Rankin-Selberg type L-functions.

Let $\mathbf{G}$ and $\mathbf{G}'$ be two con-
nected reductive algebraic groups over a number field F. Then $^L G$
and $^L G'$ are naturally embedded in groups of type $GL_N(\mathbb{C}) \rtimes \Gamma_{\overline{F}/F}$.
Let r and r' be these embeddings. These are what we call the stan-
dard representations of $^L G$ and $^L G'$. Fix two automorphic forms π

and π' on G and G'. Let (π, π') be the form on $G \times G'$. The L-function $L_S(s, (\pi, \pi'), r \otimes r')$ is usually called the *Rankin-Selberg L-function* for the pair (π, π'). As in Section 2, we shall denote it by $L_S(s, \pi \times \pi')$. When $\mathbf{G} = GL_n$ and $\mathbf{G}' = GL_m$, these L-functions were discussed in Section 2. This is the only case where the theory is now complete (cf. the recent work of Waldspurger [71]). In every case known at present, the second group is always a GL_n.

7.1. The work of Gelbart and Piatetski-Shapiro [20]. We start with the case $\mathbf{G} \times GL_n$, where $\mathbf{G} = SO_{2n+1}$. Then $\mathbf{G}$ has a subgroup $\mathbf{H}$ isomorphic to SO_{2n} and SO_{2n} has a Levi subgroup isomorphic to GL_n. Let π' be a cusp form on $GL_n(\mathbf{A}_F)$ and choose f in the space of π'. There is an Eisenstein series defined by f which we denote by $E(s, f, h)$, $h \in H$. Now let φ be a cusp form in the space of π and consider

$$(7.1.1) \qquad Z(s, \varphi, f) = \int\limits_{\mathbf{H}(F) \backslash H} \varphi(h) E(s, f, h) dh.$$

The normalizing factor for $E(s, f, h)$, (cf. 5.1), is now more delicate and is the L-function $L_S(2s, \pi', \wedge^2 \rho_n)$, where $\wedge^2 \rho_n$ is the exterior square of the standard representation ρ_n of $GL_n(\mathbb{C})$ (cf. §3.3.3).

Now, if π is also globally generic (π' always is [66]), it can be shown that $Z(s, \varphi, f)$ is Eulerian. Moreover if E is replaced by its normalization, the local factors at $v \notin S$, are $L(s, \pi_v \times \pi'_v)$.

Using the finiteness of poles for $L_S(2s, \pi', \wedge^2 \rho_n)$ (cf. §3.3.3), it is expected that (7.1.1) leads to a proof of the finiteness of poles for $L_S(s, \pi \times \pi')$.

Similar results are expected when $\mathbf{G} = SO_{2n}$ or Sp_{2n}. In the case $\mathbf{G} = GSp_4$ and $\mathbf{G}' = GL_2$ these results are also obtained by Piatetski-Shapiro and Soudry [52, 53]. Finally, we refer to [19], where $\mathbf{G} = U(2, 1)$ defined by a quadratic extension E of F, and $\mathbf{G}' = \mathrm{Res}_{E/F} GL_1$ (cf. Sectin 8.1 below).

7.2. Examples from Euler products method. We refer to [34] and [63] for many examples of Rankin products including every case mentioned so far but:

7.3. The Case $G_2 \times GL_2$ [51]. This is a very new result just obtained by Piatetski-Shapiro, Rallis, and Schiffman. In fact, using their theory explained in Chapter 5, they have now been able to obtain an integral

representation for $L_S(s, \pi \times \pi')$, where π is a globally generic cusp form on adelic points of a split exceptional group of type G_2, and π' is an automorphic form on $PGL_2(\mathbf{A}_F)$. Taking π' equal to the trivial representation, this also gives the L-function attached to the standard representation of G_2. This is very striking since neither of these L-functions can be obtained by any other method (also see 8.1 and 8.2 below).

8. Functoriality principle and L-functions. Going back to the general conjecture, let $\mathbf{G}$ be as in Section 1. A representation $r :$ $^LG \rightarrow GL_N(\mathbf{C})$ is in fact a homomorphism from ^LG into ^LGL_N and therefore by Langlands' Functoriality Principle [36, 39], there must exist a map r_* from the space of automorphic forms on G into those on $GL_N(\mathbf{A}_F)$ such that

$$L(s, \pi, r) = L(s, r_*(\pi), \rho_n)$$

and

$$\varepsilon(s, \pi, r) = \varepsilon(s, r_*(\pi), \rho_n),$$

where the factors on the right are the standard L-function and root number for $r_*(\pi)$ which is an automorphic form on $GL_N(\mathbf{A}_F)$ [26]. Since Conjecture 1.2 is in fact proved for the standard L-functions for GL_N, the conjecture for $L(s, \pi, r)$ now follows. For briefness, we shall restrict ourselves to only two cases of functoriality (also see 4.1.2).

8.1. The unitary group in 3-variables. In [56], Rogawski has proved the existence of θ_*, where

$$\theta : {}^LU(2,1) = GL_3(\mathbf{C}) \rtimes \Gamma_{E/F} \rightarrow (GL_3(\mathbf{C}) \times GL_3(\mathbf{C})) \rtimes \Gamma_{E/F}$$
$$= {}^L(\mathrm{Res}_{E/F} GL_3),$$

sends $g \rtimes \tau$ to $(g, g) \rtimes \tau$. Now let r be a representation of $(GL_3(\mathbf{C}) \times GL_3(\mathbf{C})) \rtimes \Gamma_{E/F}$, then

(8.1.1) $$L_S(s, \pi, r \cdot \theta) = L_S(s, \theta_*(\pi), r).$$

Choosing r from the examples in [63] and using the theory of Section 3 must then lead to new L-functions for the unitary group $U(2,1)$ which can not be found by any other method.

8.2. Base change for GL_n. It is proved in [3] that if

$$\theta : {}^L GL_n$$
$$= GL_n(\mathbb{C}) \times \Gamma_{E/F} \to (GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C})) \rtimes \Gamma_{E/F}$$
$$= {}^L(\mathrm{Res}_{E/F} GL_n)$$

sends (g, τ) to $(g, \ldots, g) \rtimes \tau$, then θ_* exists (cf. [12] and [38] for $n = 3$ and 2, respectively). Here E/F is a cyclic extension. If r is a representation of ${}^L(\mathrm{Res}_{E/F} GL_n)$, then again (8.1.1) holds. It is intriging to see what new L-functions can be obtained, if one combines this with possibilities in [63].

9. Concluding remarks.

It is clear that each of the methods discussed above has its advantages and limitations. While the method of Section 3 is powerful in establishing the functional equation with Artin factors at every place where the representation can be parametrized (Theorem 3.2.1), and even a proof of the finiteness of poles in many cases, it is the use of integral representations which has proved more useful in locating the poles. On the other hand when it comes to local analysis at the archimedean places, the method of integral representations has often been very cumbersome and unsuccessful. It may well turn out that, at least for those L-functions which appear in the constant terms of Eisenstein series (cf. Chapter 3), the most efficient way of obtaining complete results is in mixing the two methods. It is for this reason that the recent work of Waldspuger [71] on $GL(n) \times GL(m)$ (cf. §2.2.2) must be considered a breakthrough.

As experience has shown [3, 56], the use of analytic properties of those L-functions which appear in the constant terms of Eisenstein series, if not absolutely necessary, has greatly simplified any use of the trace formula in establishing the principle of functoriality. Whether this is the extent of which the analytic properties of automorphic L-functions can be used in establishing the principle of functoriality (cf. Section 8) in general remains to be seen.

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