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Freydoon Shahidi

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# A proof of Langlands' conjecture on Plancherel measures; Complementary series for $p$ -adic groups

By FREYDOON SHAHIDI\*

## Introduction

The purpose of this paper is to prove two general results on harmonic analysis of  $p$ -adic reductive groups. Our first result, Theorem 7.9, proves a conjecture of Langlands on normalization of intertwining operators by means of local Langlands root numbers and  $L$ -functions, at least when the group is quasi-split and the inducing representation is generic. Assuming two natural conjectures in harmonic analysis of  $p$ -adic groups, we also prove the validity of the conjecture in general (Theorem 9.5). As our second result we obtain all the complementary series and special representations of quasi-split  $p$ -adic groups coming from rank-one parabolic subgroups and generic supercuspidal representations of their Levi factors (Theorem 8.1). Both results are consequences of our main Theorem 3.5 which proves the most general result about local Langlands root numbers and  $L$ -functions at present.

More precisely, let  $\mathbf{G}$  be a quasi-split connected reductive algebraic group over a non-archimedean field  $F$  of characteristic zero. Fix a Borel subgroup  $\mathbf{B} = \mathbf{T}\mathbf{U}$  of  $\mathbf{G}$  and let  $\mathbf{P} = \mathbf{M}\mathbf{N}$  be a parabolic subgroup of  $\mathbf{G}$  such that  $\mathbf{N} \subset \mathbf{U}$ . If  $\mathbf{A}_0$  is the maximal split torus of  $\mathbf{T}$ , we let  $W(\mathbf{A}_0)$  denote its Weyl group. Finally, if  $\theta$  is a subset of simple roots  $\Delta$  of  $\mathbf{A}_0$  in  $\mathbf{U}$  such that  $\mathbf{P} = \mathbf{P}_\theta$ , we choose  $\tilde{w} \in W(\mathbf{A}_0)$  such that  $\tilde{w}(\theta) \subset \Delta$ .

Fix a non-degenerate character  $\chi$  of  $U = \mathbf{U}(F)$ . We use  $\chi$  to denote its restriction to  $U^0 = U \cap M$  as well. Let  $\sigma$  be an irreducible admissible  $\chi$ -generic representation of  $M$ . We will say  $\chi$  and  $\tilde{w}$  are *compatible* if  $\chi|_{U^0} = \chi|_{\mathbf{Ad}} \tilde{w}(U^0)$ . Either changing the splitting for  $\mathbf{G}$ , or said in other words, replacing  $\sigma$  with another generic representation in its  $L$ -packet, we may always assume  $\chi$  and  $\tilde{w}$  are compatible (cf. Section 3). This will simplify our calculation of local coefficients (cf. equation (1.2) for their definition) and makes their definition canonical. Observe that this is permissible since local factors are expected to depend only on the  $L$ -packet of  $\sigma$ .

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Given an irreducible admissible representation  $\sigma$  of  $M$ ,  $\nu \in \mathfrak{a}_\mathbb{C}^*$ , the complex dual of the real Lie algebra of the split torus  $\mathbf{A}$  of the center of  $\mathbf{M}$ , and a representative  $w$  of  $\tilde{w}$ , let  $A(\nu, \sigma, w)$  be the standard intertwining operator defined by equation (1.1) of Section 1. Let  $\gamma_{\tilde{w}}(G/P)$  and  $\gamma_{\tilde{w}^{-1}}(G/P')$  be defined as in Section 2. Then there exists a constant  $\mu(\nu, \sigma, \tilde{w})$ , which depends only on  $\nu$ , the class of  $\sigma$ , and  $\tilde{w}$ , but not the choice of  $w$ , such that

$$A(\nu, \sigma, w)A(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) = \mu(\nu, \sigma, \tilde{w})\gamma_{\tilde{w}}(G/P)\gamma_{\tilde{w}^{-1}}(G/P').$$

In analogy with the tempered case [12], [44], this is what we call the *Plancherel measure* attached to  $\nu, \sigma, \tilde{w}$ , knowledge of which is of great importance in harmonic analysis on  $G$  (cf. Theorem 8.1, for example). Finally, set  $A(\sigma, w) = A(0, \sigma, w)$  and  $\mu(\sigma, \tilde{w}) = \mu(0, \sigma, \tilde{w})$ .

On the other hand let  ${}^L M$  be the  $L$ -group of  $\mathbf{M}$  and denote by  ${}^L \mathfrak{n}_{\tilde{w}}$  the Lie algebra of the  $L$ -group of  $\mathbf{N}_{\tilde{w}} = \mathbf{U} \cap w\mathbf{N}^-w^{-1}$ . Let  $r_{\tilde{w}}$  be the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}_{\tilde{w}}$  and denote by  $r_{i, \tilde{w}}$  the restriction of  $r_{\tilde{w}}$  to the subspace  $V_i \subset {}^L \mathfrak{n}_{\tilde{w}}$ ,  $1 \leq i \leq m$ , defined in Section 1. Then  $r_{\tilde{w}} = \bigoplus_{i=1}^m r_{i, \tilde{w}}$ . If  $\mathbf{P}$  is maximal this is the decomposition of  $r_{\tilde{w}}$  to its irreducible constituents. We then set  $r = r_{\tilde{w}_0}$  and  $r_i = r_{\tilde{w}_{0,i}}$ , where  $\tilde{w}_0$  is the longest element in the Weyl group  $W(\mathbf{A}_0)$  modulo the Weyl group of  $\mathbf{A}_0$  in  $\mathbf{M}$ .

Given a complex number  $s$ , an irreducible admissible representation  $\sigma$  of  $M$ , a representation  $\rho$  of  ${}^L M$ , and a non-trivial additive character  $\psi_F$  of  $F$ , let  $L(s, \sigma, \rho)$  and  $\varepsilon(s, \sigma, \rho, \psi_F)$  be the conjectural Langlands  $L$ -function and root number attached to  $\sigma$  and  $\rho$ . Next, let

$$\gamma(s, \sigma, \rho, \psi_F) = \varepsilon(s, \sigma, \rho, \psi_F)L(1 - s, \tilde{\sigma}, \rho)/L(s, \sigma, \rho).$$

Even though neither of these factors is defined in general, they are expected to satisfy a number of properties. For example, if  $\varphi': W'_F \rightarrow {}^L M$  is a parametrization of  $\sigma$ , then  $\varepsilon(s, \sigma, \rho, \psi_F) = \varepsilon(s, \rho \cdot \varphi', \psi_F)$  and  $L(s, \sigma, \rho) = L(s, \rho \cdot \varphi')$ , where the new factors are those of Artin (cf. [26], [46], and Part 1 of Theorem 3.5 for  $\rho = r_{\tilde{w}, i}$ ). Moreover they are supposed to be inductive (Part 3 of Theorem 3.5) and when  $\sigma$  becomes the local component of a global cusp form, they must be the corresponding factors appearing in the functional equation satisfied by the cusp form (Part 4 of Theorem 3.5). A first version of Langlands' conjecture on Plancherel measures [24] can now be formulated as follows:

*The Plancherel measure  $\mu(\sigma, \tilde{w})$  satisfies:*

$$(1) \quad \mu(\sigma, \tilde{w})\gamma_{\tilde{w}}(G/P)^{-1}\gamma_{\tilde{w}^{-1}}(G/P')^{-1} = \gamma(0, \sigma, r_{\tilde{w}}, \bar{\psi}_F)\gamma(0, \tilde{\sigma}, r_{\tilde{w}}, \psi_F).$$

The first result of this paper is a proof of this conjecture in the generic case. More precisely, assume  $\sigma$  is  $\chi$ -generic and  $\chi$  and  $\tilde{w}$  are compatible. Then for each  $i$ ,  $1 \leq i \leq m$ , there exists a unique factor  $\gamma_i(s, \sigma, \psi_F) = \gamma(s, \sigma, r_{\tilde{w}, i}, \psi_F)$

satisfying Properties 1)–4) of Theorem 3.5 such that if

$$\gamma(s, \sigma, r_{\tilde{w}}, \psi_F) = \prod_{i=1}^m \gamma(s, \sigma, r_{\tilde{w}, i}, \psi_F),$$

then (1) is satisfied (Corollary 3.6).

To get the finer version of the conjecture, one needs to define root numbers  $\varepsilon(s, \sigma, r_{\tilde{w}}, \psi_F)$  and  $L$ -functions  $L(s, \sigma, r_{\tilde{w}})$ . This we have done in Section 7, by using, first  $\gamma_i$ 's to define these factors when  $\sigma$  is tempered, and then analytic continuation to extend this to arbitrary  $\sigma$  by Langlands' classification [43]. It agrees completely with the case of real groups [28] and we believe that these factors will equal their Artin counterparts as soon as a parametrization is available (cf. Lemma 3.3, Proposition 3.4, and Part 1 of Theorem 3.5). The final form of the functional equation is then formulated in terms of these root numbers and  $L$ -functions as our Theorem 7.7. Next let

$$\mathcal{A}(\sigma, w) = \varepsilon(0, \sigma, \tilde{r}_{\tilde{w}}, \psi_F) L(1, \sigma, \tilde{r}_{\tilde{w}}) L(0, \sigma, \tilde{r}_{\tilde{w}})^{-1} A(\sigma, w)$$

be the standard normalization suggested by Langlands (cf. [27], [2]). Then our Theorem 7.9 proves Langlands' conjecture as follows:

*The normalized operator  $\mathcal{A}(\sigma, w)$  satisfies:*

- a)  $\mathcal{A}(\sigma, w_1 w_2) = \mathcal{A}(\tilde{w}_2(\sigma), w_1) \mathcal{A}(\sigma, w_2)$ , and
- b)  $\mathcal{A}(\sigma, w)^* = \mathcal{A}(\tilde{w}(\sigma), w^{-1})$ , i.e.,  $\mathcal{A}(\sigma, w)$  is unitary.

We should remark that in the case of real groups this was proved in [2], while for  $G = \text{GL}(n)$ ,  $G_2$ , or  $P$  minimal, and  $F$  non-archimedean it was established in [37], [40], and [20], respectively.

When we assume two standard conjectures (Conjectures 9.2 and 9.4) in harmonic analysis this is then extended to any irreducible admissible (not necessarily generic) representation  $\sigma$  of  $M$  in Section 9 (Theorem 9.5). Since inner forms are supposed to preserve Plancherel measures [1], this must complete the conjecture in general.

Such a fine normalization is expected to be of importance in applications of the trace formula [1], [3], [20], [35], [36]. We shall only refer to the possible application of this to compatible normalization of intertwining operators for two different groups when their trace formulas are compared (cf. [1]). We hope to address this in a future paper.

Theorem 3.5 is of interest by itself since it proves a general result about local Langlands root numbers and  $L$ -functions attached to representations  $r_{i, \tilde{w}}$  which include practically every case presently known. It proves the existence and uniqueness of the factors  $\gamma(s, \sigma, \rho, \psi_F)$  satisfying a number of important

properties when  $\rho = r_{i, \bar{w}}$ . Although not as strong, it is reminiscent of the Deligne-Langlands result on the existence and uniqueness of local root numbers for Artin  $L$ -functions [8], [26], [46]. Finally part 2 of Theorem 5.3 must be considered as a factorization of local coefficients at finite places (cf. Conjecture 7.1). This was previously proved for real groups in [36].

Even though the results are local, the proof of Theorem 3.5 is global and uses a version of a result of Henniart [14] and Vignéras [47] (Proposition 5.1) on generic Poincaré series. Our experience with  $GL(n)$  shows that any local proof of these properties, especially equation (3.10), is extremely difficult in general ([37], [15]). We refer the reader to comments at the end of Section 3 on the inductive property 3) and equation (3.10) in the case of  $GL(n)$ .

The factors  $\gamma_i$  are defined inductively and the induction step, which generalizes Lemma 4.2 of [39], is established in Section 4. The proof of Theorem 3.5 which starts in Section 4 is finally completed in Section 6.

One consequence of the results of Section 4 is the removal of the technical assumption that the restriction of  $\pi$  to the center of  $M$  is trivial whenever  $m \geq 2$ , from all the results of [39] (cf. Remark 4.12 here). Thus Theorems 5.1 and 5.2, Corollary 5.3, Lemma 5.8, and Proposition 5.9 of [39] are now all valid with no assumption on  $\pi$ .

As our second result in harmonic analysis we obtain all the complementary series and special representations of  $G$  when  $\mathbf{P}$  is maximal and  $\sigma$  is an irreducible unitary supercuspidal generic representation of  $M$ . More precisely, let  $\alpha \in \Delta$  be identified with the unique reduced root of  $\mathbf{A}$  in  $\mathbf{N}$  and set  $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$ , where  $\rho$  is half the sum of positive roots in  $\mathbf{N}$ . Let  $L(s, \sigma, r_i)$  be the  $L$ -function attached to  $\sigma$  and  $r_i$  in Section 7 and denote by  $P_{\sigma, i}(q^{-s})$  its inverse which is a polynomial in  $q^{-s}$ . By Lemma 7.5,  $P_{\sigma, i} = 1$  for  $i \geq 3$ . Our Theorem 8.1 then asserts:

*Assume  $\sigma$  is ramified, i.e.,  $\tilde{w}_0(\sigma) \cong \sigma$ , and  $I(\sigma)$  is irreducible. Choose (by Corollary 7.6) a unique  $i$ ,  $i = 1, 2$ , such that  $P_{\sigma, i}(1) = 0$ . Then:*

a) *For  $0 < s < 1/i$ , the representation  $I(s\tilde{\alpha}, \sigma)$  is irreducible and in the complementary series.*

b) *The representation  $I(\tilde{\alpha}/i, \sigma)$  is reducible with a unique  $\chi$ -generic subrepresentation which is in the discrete series. Its Langlands quotient is never generic. It is a pre-unitary non-tempered representation.*

c) *For  $s > 1/i$ , the representations  $I(s\tilde{\alpha}, \sigma)$  are always irreducible and never in the complementary series.*

*If  $\sigma$  is ramified and  $I(\sigma)$  is reducible, then no  $I(s\tilde{\alpha}, \sigma)$ ,  $s > 0$ , is preunitary. They are all irreducible.*

*In particular, the edge of complementary series is always either  $\frac{1}{2}\tilde{\alpha}$  or  $\tilde{\alpha}$ .*

Since the cases of quasi-split rank-one groups are completely answered [19], our examples of Theorem 8.1 are the most interesting and difficult cases of split rank-two groups (Propositions 8.3 and 8.4, and Remark 8.5); namely:

1)  $\mathbf{G}$  is a split group of type  $G_2$  and  $\mathbf{M}$  is generated by the long root of  $\mathbf{T} = \mathbf{A}_0$ .

2)  $\mathbf{G}$  is either  $\mathrm{Sp}_4$  or  $\mathrm{GSp}_4$  and  $\mathbf{P}$  is the non-Siegel maximal parabolic subgroup of  $\mathbf{G}$ . The case  $\mathbf{G} = \mathrm{GSp}_4$  was first studied by Waldspurger [49], using an entirely different method.

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### 1. Notation and terminology

Let  $F$  be a non-archimedean field of characteristic zero. Denote by  $O$  its ring of integers and let  $P$  be its maximal ideal. Let  $q$  be the number of elements in the residue field  $O/P$ . We use  $|\cdot|_F$  to denote the absolute value in  $F$ .

Let  $\mathbf{G}$  be a connected reductive algebraic group over  $F$ . We shall further assume that  $\mathbf{G}$  is quasi-split over  $F$ . Fix a Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$  over  $F$ . Write  $\mathbf{B} = \mathbf{T}\mathbf{U}$  where  $\mathbf{T}$  is a maximal torus and  $\mathbf{U}$  denotes the unipotent radical of  $\mathbf{B}$ . Finally, let  $\mathbf{P}$  be an  $F$ -parabolic subgroup of  $\mathbf{G}$ , containing  $\mathbf{B}$ . Write  $\mathbf{P} = \mathbf{M}\mathbf{N}$ , a Levi decomposition. Then  $\mathbf{U} \supset \mathbf{N}$ .

For every algebraic group  $\mathbf{H}$  over  $F$ , let  $H = \mathbf{H}(F)$ . We then have  $G, B, T, U, P, M$ , and  $N$ .

Let  $X(\mathbf{M})_F$  be the group of  $F$ -rational characters of  $\mathbf{M}$ . We then set

$$\alpha = \mathrm{Hom}(X(\mathbf{M})_F, \mathbf{R})$$

and let  $\alpha_{\mathbf{C}}^* = \alpha^* \otimes_{\mathbf{R}} \mathbf{C}$ , where

$$\alpha^* = X(\mathbf{M})_F \otimes_{\mathbf{Z}} \mathbf{R}$$

denotes the dual of  $\alpha$ . Observe that  $\alpha$  is the real Lie algebra of the maximal split torus  $\mathbf{A}$  of the center of  $\mathbf{M}$ . Finally define the homomorphism  $H_P: M \rightarrow \alpha$  by

$$q^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F,$$

for all  $\chi \in X(\mathbf{M})_F$ .

Next, let  $\mathbf{A}_0$  be the maximal  $F$ -split torus in  $\mathbf{T}$ ,  $\mathbf{A}_0 \supset \mathbf{A}$ . Denote by  $\psi$  the set of  $F$ -roots of  $\mathbf{A}_0$ . Then  $\psi = \psi^+ \cup \psi^-$ , where  $\psi^+$  is the set of positive roots,

i.e., those generating  $U$ . Let  $\Delta \subset \psi^+$  be the set of simple roots. We use  $\theta \subset \Delta$  to denote the subset of  $\Delta$  which generates  $M$ . Finally, let  $W(A_0)$  be the Weyl group of  $A_0$  in  $G$ .

We shall always fix a special maximal compact subgroup  $Q$  of  $G$ , adapted to  $A_0 = A_0(F)$  (cf. [44]). Then  $G = PQ$ . If  $G$  is unramified, we may choose  $Q = G(O)$ .

Given an irreducible admissible representation  $\sigma$  of  $M$  and  $\nu \in \mathfrak{a}_C^*$ , let  $I(\nu, \sigma)$  be the representation of  $G$  induced from  $\sigma$  and  $\nu$ . More precisely, let

$$I(\nu, \sigma) = \text{Ind}_{MN \uparrow G} \sigma \otimes q^{\langle \nu, H_p(\cdot) \rangle} \otimes 1.$$

We use  $V(\nu, \sigma)$  to denote its space and let  $I(\sigma) = I(0, \sigma)$  and  $V(\sigma) = V(0, \sigma)$ .

Fix a  $\tilde{w} \in W(A_0)$  such that  $\tilde{w}(\theta) \subset \Delta$  and let  $w \in G$  be a representative for  $\tilde{w}$ . Let  $N_{\tilde{w}} = U \cap wN^-w^{-1}$ , where  $N^-$  is the unipotent subgroup opposed to  $N$ . Given  $f \in V(\nu, \sigma)$ , let

$$(1.1) \quad A(\nu, \sigma, w)f(g) = \int_{N_{\tilde{w}}} f(w^{-1}ng) \, dn \quad (g \in G).$$

The integral converges absolutely if  $\text{Re}\langle \nu, H_\alpha \rangle \gg 0$  for every  $\alpha \in \Delta - \theta$ , where  $H_\alpha$  is the standard coroot attached to  $\alpha$ . Moreover it extends to a meromorphic function of  $\nu$  on all of  $\mathfrak{a}_C^*$  (cf. [38], [44]), and away from its poles, it defines an intertwining map between  $I(\nu, \sigma)$  and  $I(\tilde{w}(\nu), \tilde{w}(\sigma))$ , where  $\tilde{w}(\sigma)(m') = \sigma(w^{-1}m'w)$ ,  $m' \in M' = wMw^{-1}$ . Finally let  $A(\sigma, w) = A(0, \sigma, w)$ .

Now, let  ${}^L M$  be the  $L$ -group of  $M$ . Denote by  ${}^L \mathfrak{n}$  the Lie algebra of the  $L$ -group  ${}^L N$  of  $N$ . The group  ${}^L M$  acts by adjoint action on  ${}^L \mathfrak{n}$ . If  ${}^L \mathfrak{n}_{\tilde{w}}$  denotes the Lie algebra of the  $L$ -group of  $N_{\tilde{w}}$ , it is then clear that  ${}^L \mathfrak{n}_{\tilde{w}}$  is stable under this adjoint action. We use  $r$  and  $r_{\tilde{w}}$  to denote the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}$  and  ${}^L \mathfrak{n}_{\tilde{w}}$ , respectively. Let  $\rho = \rho_{\mathbf{P}}$  be half the sum of the roots whose root spaces generate  $N$ . Then for each  $\alpha$  with  $X_{\alpha^v} \in {}^L \mathfrak{n}$ ,  $\langle 2\rho, \alpha \rangle$  is a positive integer. Let  $a_1 < a_2 < \dots < a_m$  be their distinct values. Set

$$V_i = \{X_{\alpha^v} \in {}^L \mathfrak{n}_{\tilde{w}} \mid \langle 2\rho, \alpha \rangle = a_i\}.$$

Each  $V_i$  is invariant under  $r_{\tilde{w}}$ . We let  $r_{\tilde{w}, i}$  be the restriction of  $r_{\tilde{w}}$  to  $V_i$ . Throughout this paper we shall respect the above order of  $r_i$ 's.

If  $\mathbf{P}$  is maximal, then for non-trivial  $\tilde{w}$ ,  $r_{\tilde{w}} = r$ . We then set  $r_i = r_{\tilde{w}, i}$  and therefore  $r = \bigoplus_{i=1}^m r_i$  with each  $r_i$  irreducible (cf. [39]). Moreover if  $\alpha \in \Delta$  identifies the unique reduced root of  $\mathbf{A}$  in  $N$ , we let  $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$  which is an element of  $\mathfrak{a}^*$ . We then observe that for each  $i$ ,  $1 \leq i \leq m$ ,

$$V_i = \{X_{\beta^v} \in {}^L \mathfrak{n} \mid \langle \tilde{\alpha}, \beta \rangle = i\},$$

and therefore each  $V_i$  is an eigenspace for the action of the center of  ${}^L M^0$ , the connected component of  ${}^L M$ . We shall call  $m$  the *length* of the above data.

Let  $\chi$  be a generic character of  $U$  (cf. Section 3 of [39] for its exact definition). Then by restriction  $\chi$  is also a generic character of  $U^0 = \mathbf{U}^0(F)$ , where  $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{M}$ . An irreducible admissible representation  $\sigma$  of  $M$  is called  $\chi$ -generic if it can be realized on a space of smooth functions  $W^0$  satisfying  $W^0(um) = \chi(u)W^0(m)$ ,  $m \in M$  and  $u \in U^0$ . We call this realization (which is unique), the  $\chi$ -Whittaker model  $W(\sigma)$  for  $\sigma$ .

If  $\sigma$  is  $\chi$ -generic, then so is  $V(\nu, \sigma)$  and to it we have attached a canonical Whittaker functional  $\lambda_\chi(\nu, \sigma)$  in [38], [39]. Moreover, if  $\lambda_\chi(\tilde{w}(\nu), \tilde{w}(\sigma))$  is the canonical functional attached to  $I(\tilde{w}(\nu), \tilde{w}(\sigma))$ , then there exists a complex number  $C_\chi(\nu, \sigma, w)$ , the local coefficient attached to  $\chi, \nu, \sigma$ , and  $w$  in [38], such that

$$(1.2) \quad \lambda_\chi(\nu, \sigma) = C_\chi(\nu, \sigma, w)\lambda_\chi(\tilde{w}(\nu), \tilde{w}(\sigma))A(\nu, \sigma, w).$$

Let  $C_\chi(\sigma, w) = C_\chi(0, \sigma, w)$ . It satisfies (see [38])

$$(1.3) \quad C_\chi(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) = \overline{C_\chi(-\nu, \sigma, w)}$$

and

$$(1.4) \quad \begin{aligned} A(\nu, \sigma, w)A(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) \\ = C_\chi(\nu, \sigma, w)^{-1}C_\chi(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1})^{-1}. \end{aligned}$$

Finally we observe that if  $\sigma$  is a unitary  $\chi$ -generic representation, then its contragredient  $\tilde{\sigma}$  is  $\bar{\chi}$ -generic and moreover

$$(1.5) \quad C_{\bar{\chi}}(\tilde{\sigma}, w) = \overline{C_\chi(\sigma, w)}.$$

## 2. Plancherel measures and Langlands' conjecture

Let  $\sigma$  be an irreducible unitary representation of  $M$  and choose  $\tilde{w}$  as before. There exists a constant  $\mu(\nu, \sigma, \tilde{w})$ , which in analogy with the tempered case we shall call the Plancherel measure attached to  $\nu, \sigma$ , and  $\tilde{w}$ , such that

$$A(\nu, \sigma, w)A(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) = \mu(\nu, \sigma, \tilde{w})^{-1}\gamma_{\tilde{w}}(G/P)\gamma_{\tilde{w}^{-1}}(G/P').$$

The constants  $\gamma_{\tilde{w}}(G/P)$  and  $\gamma_{\tilde{w}^{-1}}(G/P')$  are defined as follows. We first extend  $H_P$  to a function of  $G$  by letting  $H_P(mnk) = H_P(m)$ ,  $m \in M$ ,  $n \in N$ , and  $k \in K$ . Next let

$$\begin{aligned} \bar{N}_{\tilde{w}} &= w^{-1}N_{\tilde{w}}w \\ &= N \cap w^{-1}Uw. \end{aligned}$$

We then define:

$$\gamma_{\tilde{w}}(G/P) = \int_{\tilde{N}_{\tilde{w}}(F)} q^{\langle 2\rho_F, H_P(\tilde{n}) \rangle} d\tilde{n}.$$

The measure  $d\tilde{n}$  is obtained by transporting the measure  $dn$  defining  $A(\nu, \sigma, w)$  from  $N_{\tilde{w}}(F)$  to  $\tilde{N}_{\tilde{w}}(F)$ . Similarly we define  $\gamma_{\tilde{w}^{-1}}(G/P')$ , where  $P'$  is the standard parabolic subgroup which has  $M' = wMw^{-1}$  as its Levi factor.

Clearly  $\mu(\nu, \sigma, \tilde{w})$  is independent of the choice of measures. We shall show that it is also independent of the choice of  $w$ . Let  $w = mw_1$  for some  $m \in M'$ . Define a map  $T$  from  $V(w_1(\nu), w_1(\sigma))$  into  $V(w(\nu), w(\sigma))$  by  $(Tf)(g) = f(m^{-1}g)$ . Then it is easily checked that

$$T \cdot A(\nu, \sigma, w_1) = A(\nu, \sigma, w) \cdot d(m^{-1}nm)/dn$$

and

$$A(w_1(\nu), w_1(\sigma), w_1^{-1})T^{-1} = A(w(\nu), w(\sigma), w^{-1}).$$

The factor  $d(m^{-1}nm)/dn$  appears due to the change of the measure  $dn$  on  $N_{\tilde{w}}$  and therefore the factor  $\gamma_{\tilde{w}}(G/P)$  must be corrected by it. It is now clear that  $\mu(\nu, \sigma, \tilde{w})$  remains unchanged. It clearly depends only on the equivalence class of  $\sigma$ . We finally set  $\mu(\sigma, \tilde{w}) = \mu(0, \sigma, \tilde{w})$ .

Now assume  $\tilde{w} = \tilde{w}_0$  is the element which sends every root  $\alpha \in \Delta - \theta$  to a negative root. Then  $\mu(\nu, \sigma) = \mu(\nu, \sigma, \tilde{w}_0)$  will be the generalization (from tempered  $\sigma$ ) of the Plancherel measure defined by Harish-Chandra (cf. [12], [44]). We shall then take the measures defining  $A(\nu, \sigma, w)$  and  $A(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1})$  in such a way that

$$\gamma_{\tilde{w}_0}(G/P) = \gamma_{\tilde{w}_0^{-1}}(G/P'),$$

and let  $\gamma(G/P)$  denote their common value.

Fix a nontrivial additive character  $\psi_F$  of  $F$ . Given an irreducible admissible representation  $\sigma$  of  $M$  and a representation  $\rho$  of  ${}^L M$ , let  $L(s, \sigma, \rho)$  and  $\varepsilon(s, \sigma, \rho, \psi_F)$  be the conjectural Langlands  $L$ -function and root number attached to  $\sigma$  and  $\rho$ ,  $s \in \mathbf{C}$ . In general this has been defined only if  $F$  is archimedean or  $\sigma$  is of class one; i.e.,  $\mathbf{G}$  is unramified and the space of  $\sigma$  contains a  $\mathbf{G}(O)$ -fixed vector. Next, let

$$\gamma(s, \sigma, \rho, \psi_F) = \varepsilon(s, \sigma, \rho, \psi_F)L(1 - s, \tilde{\sigma}, \rho)/L(s, \sigma, \rho),$$

where  $\tilde{\sigma}$  denotes the contragredient of  $\sigma$ . The following conjecture is due to Langlands [27].

*Conjecture (Langlands).* The Plancherel measure  $\mu(\sigma, \tilde{w})$  satisfies:

$$\begin{aligned} \mu(\sigma, \tilde{w})\gamma_{\tilde{w}}(G/P)^{-1}\gamma_{\tilde{w}^{-1}}(G/P')^{-1} &= \gamma(0, \sigma, r_{\tilde{w}}, \bar{\psi}_F)\gamma(0, \tilde{\sigma}, r_{\tilde{w}}, \psi_F) \\ &= \gamma(0, \sigma, r_{\tilde{w}}, \bar{\psi}_F)/\gamma(1, \sigma, r_{\tilde{w}}, \bar{\psi}_F). \end{aligned}$$

The equality of the last two statements is just a consequence of the conjectural relation

$$\gamma(s, \sigma, \rho, \psi_F)\gamma(1 - s, \tilde{\sigma}, \rho, \bar{\psi}_F) = 1$$

(cf. equation (3.10)).

The main result of this paper (Theorem 3.5 and its corollary) is a proof of this conjecture when  $\sigma$  is generic. The finer version of this conjecture, Theorem 7.9, is proved in Section 7. Later, in Section 9 we shall discuss how the general case can be reduced (modulo certain standard conjectures) to the generic case and prove the conjecture in general (Theorem 9.5).

### 3. The fundamental theorem

In this section we shall state the fundamental result of this paper, Theorem 3.5, from which a number of results including a proof of Langlands' conjecture on Plancherel measures will follow. We first need some preparation.

We shall always fix a nontrivial additive character  $\psi_F$  of  $F$ . Given  $\tilde{w} \in W(\mathbf{A}_0)$ , we shall always fix a representative  $w$  as in [36].

Next, given  $\alpha \in \psi^+$ , let  $\tilde{\mathbf{G}}_{\alpha, D}$  be the simply connected covering of the derived group of the rank-one subgroup of  $\mathbf{G}$  attached to  $\alpha$ . Then  $\tilde{\mathbf{G}}_{\alpha, D} = \text{Res}_{F_\alpha/F} \text{SL}(2)$  or  $\text{Res}_{F_\alpha/F} \text{SU}(2, 1)$  for some finite (separable) extension  $F_\alpha$  of  $F$ . In the second case, let  $E_\alpha/F_\alpha$  be the defining quadratic extension for  $\text{SU}(2, 1)$ .

Fix  $\tilde{w} \in W(\mathbf{A}_0)$ . Denote by  $\Delta_1(\tilde{w})$  and  $\Delta_2(\tilde{w})$  the sets of reduced roots in  $\psi^+$  with  $\tilde{w}(\alpha) \in \psi^-$  for which  $\tilde{\mathbf{G}}_{\alpha, D} = \text{Res}_{F_\alpha/F} \text{SL}(2)$  or  $\tilde{\mathbf{G}}_{\alpha, D} = \text{Res}_{F_\alpha/F} \text{SU}(2, 1)$ , respectively. We now set

$$\begin{aligned} (3.1) \quad \lambda_{\mathbf{G}}(\psi_F, w) &= \prod_{\alpha \in \Delta_1(\tilde{w})} \lambda(F_\alpha/F, \psi_F) \\ &\quad \cdot \prod_{\alpha \in \Delta_2(\tilde{w})} \lambda(E_\alpha/F, \psi_F)^2 \lambda(F_\alpha/F, \psi_F)^{-1}, \end{aligned}$$

where for each finite separable extension  $E/F$ ,  $\lambda(E/F, \psi_F)$  denotes the corresponding Langlands  $\lambda$ -function (cf. [26]). This factor is imposed on us by the main result of [20].

Having fixed  $\psi_F$ , we then define a generic character  $\chi_0$  of  $U$  as follows. Let  $\alpha \in \Delta$ . If  $E_\alpha$  is the smallest extension of  $F$  over which the rank-one subgroup of  $\mathbf{G}$  generated by  $\alpha$  splits, we let  $\chi_0|_{U_\alpha} = \psi_F \cdot \text{Tr}_{E_\alpha/F}$ . This defines  $\chi_0$

uniquely. An arbitrary character  $\chi$  of  $U$  is obtained by conjugating  $\chi_0$  by an element  $a \in \mathbf{A}_0(\bar{F})$ . More precisely  $\chi(u) = \chi_0(a^{-1}ua)$ ,  $u \in U$ , and therefore if  $\mathbf{Ad}_U(a)(u) = a^{-1}ua$ ,  $u \in U$ , then  $\mathbf{Ad}_U(a): U \rightarrow U$  must be  $F$ -rational ( $\mathbf{Ad}_U$  denotes the restriction of  $\mathbf{Ad}$  to  $U$ ).

If  $F$  is a non-archimedean local field,  $\mathbf{G}$  is unramified, and  $\psi_F$  is an unramified additive character of  $F$ , then unramified characters of  $U$  are obtained by choosing those  $a \in \mathbf{A}_0(\bar{O})$  for which  $\mathbf{Ad}_U(a)$  is  $F$ -rational, where  $\bar{O}$  is the ring of integers of  $\bar{F}$ . Finally, suppose  $F$  is archimedean. We fix  $\psi_F$  as in [36], i.e.  $\psi_{\mathbf{R}}(x) = \exp(2\pi ix)$  and  $\psi_{\mathbf{C}}(z) = \psi_{\mathbf{R}}(z + \bar{z})$ ; in other words  $\psi_{\mathbf{C}} = \psi_{\mathbf{C}/\mathbf{R}}$ . By restriction of scalars we may always assume  $F = \mathbf{R}$ . Let  $\mathfrak{g}$  be the Lie algebra of  $\mathbf{G}(\mathbf{R})$ . As in [36], denote by  $G_{\text{Max}}$  the group

$$G_{\text{Max}} = \{g \in \mathbf{G}(\mathbf{C}) \mid (\text{Ad } g)\mathfrak{g} = \mathfrak{g}\}.$$

Then it is immediate from the definition that  $\mathbf{Ad}_U(a)$  is  $\mathbf{R}$ -rational if and only if  $a \in G_{\text{Max}}$ . In the notation of [36], if  $I(\nu, \eta)$  is a principal series representation of  $G$  and  $\tilde{\eta}$  is an extension of  $\eta$  to  $M_{\text{Max},0}$  ( $A_0^0 M_0 U$  is a Langlands decomposition for  $B$ ), then  $I(\nu, \tilde{\eta})|G = I(\nu, \eta)$ . We start with the following lemma.

LEMMA 3.1. *Assume  $F$  is either local or global. Fix  $\tilde{w} \in W(\mathbf{A}_0)$  and let  $a \in \mathbf{A}_0(\bar{F})$  be such that  $\mathbf{Ad}_U(a)$  is  $F$ -rational. Then  $\tilde{w}(a)a^{-1} \in \mathbf{G}(F)$ .*

*Proof.* It is enough to show  $\sigma(\tilde{w}(a)a^{-1}) = \tilde{w}(a)a^{-1}$  for every  $\sigma \in \Gamma = \text{Gal}(\bar{F}/F)$ . Since  $\mathbf{Ad}(\tilde{w}): \mathbf{A}_0 \rightarrow \mathbf{A}_0$  is  $F$ -rational it would be enough to show  $\tilde{w}(a^{-1}\sigma(a)) = a^{-1}\sigma(a)$ . Since  $\mathbf{Ad}_U(a)$  is  $F$ -rational, then  $\sigma(\mathbf{Ad}_U(a)) = \mathbf{Ad}_U(a)$ , for all  $\sigma \in \Gamma$ , where  $\mathbf{Ad}_U(a) = \mathbf{Ad}_U(a)|U$ . On the other hand  $\sigma(\mathbf{Ad}_U(a)) = \mathbf{Ad}_U(\sigma(a))$  and therefore  $\mathbf{Ad}_U(a^{-1}\sigma(a)) = 1$ . This implies that  $a^{-1}\sigma(a)$  belongs to the center of  $\mathbf{G}$  and is therefore fixed by  $\tilde{w}$ , completing the lemma.

Now, let  $\mathbf{P} = \mathbf{MN}$  be a parabolic subgroup of  $\mathbf{G}$ ,  $\mathbf{N} \subset \mathbf{U}$ , and assume  $\mathbf{M} = \mathbf{M}_\theta$ ,  $\theta \subset \Delta$ . Fix  $\tilde{w} \in W(\mathbf{A}_0)$  such that  $\tilde{w}(\theta) \subset \Delta$ . Let  $\sigma$  be an irreducible admissible  $\chi$ -generic representation of  $M$ . It is implicit in the definition of local coefficients that  $\chi$  and  $\tilde{w}$  must be *compatible*. (This makes them canonical and will have no global effect.) More precisely

$$\chi|U^0 = \chi|\mathbf{Ad} \tilde{w}(U^0),$$

where  $U^0 = U \cap \mathbf{M}$ . This implies that if  $\chi = \chi_0 \cdot \mathbf{Ad}_U(a)$ , then since  $\chi_0$  is defined by  $\psi_F$ ,

$$\chi_0(\mathbf{Ad}_U(a)u) = \chi_0(\mathbf{Ad}_U(\tilde{w}(a))u)$$

for all  $u \in U^0$ . Consequently

$$\chi_0 \cdot \mathbf{Ad}_U(\tilde{w}(a)a^{-1}) = \chi_0$$

on  $U^0$  which immediately implies that  $\tilde{w}(a)a^{-1}$  must be in the center of  $\mathbf{M}(\bar{F})$ . But then by Lemma 3.1 it must be in fact in the center of  $\mathbf{M}(F)$ .

Given a generic representation  $\sigma$  of  $\mathbf{M}$  and  $\tilde{w} \in W(\mathbf{A}_0)$  with  $\tilde{w}(\theta) \subset \Delta$ , the splitting for  $\mathbf{G}$  (cf. [32], [41]), i.e., the choice of the basis  $\{X_\alpha\}$ ,  $\alpha \in \Delta$ , and therefore for  $\mathbf{M}$ , can be changed so that  $\sigma$  becomes generic with respect to a generic character  $\chi$ , compatible with  $\tilde{w}$ . In fact by a change of splitting  $\sigma$  can be made generic with respect to  $\chi_0$  which is compatible with every  $\tilde{w}$ .

Otherwise, i.e., if we do not want to change the splitting, then, up to conjugation by elements in  $\mathbf{A}_0(\bar{F})$ , or said in other words, up to  $L$ -indistinguishability,  $\sigma$  may be chosen to be  $\chi_0$ -generic which is then compatible with every  $\tilde{w}$  satisfying  $\tilde{w}(\theta) \subset \Delta$ .

Finally we should remark that the choice of splitting for  $\mathbf{G}$  will have no effect on the value of Plancherel measures and therefore in (1.4) we may always assume that  $\chi$  and  $\tilde{w}$  are compatible.

We shall now observe the following extension of Theorem 3.1 of [36].

**PROPOSITION 3.2.** *Assume  $F = \mathbf{R}$ . Let  $\sigma$  be an irreducible admissible  $\chi$ -generic representation of  $M$ . Choose  $a \in \mathbf{A}_0(\mathbf{C})$  such that  $\chi = \chi_0 \cdot \text{Ad}_U(\tilde{w}_l(a))$ , where  $\tilde{w}_l$  is the longest element in  $W(\mathbf{A}_0)$ . Choose  $\tilde{w} \in W(\mathbf{A}_0)$ , compatible with  $\chi$ . Let  $\varphi: W_{\mathbf{R}} \rightarrow {}^L M$  be the homomorphism attached to  $\sigma$ , where  $W_{\mathbf{R}}$  is the Weil group of  $\mathbf{C}/\mathbf{R}$ . Denote by  $\omega_s$  the central character of  $\sigma \otimes e^{2s\rho_\theta}$ ,  $s \in \mathbf{C}$ , where  $\rho_\theta$  is half the sum of the roots in  $\mathbf{N}$ . Then*

$$(3.2) \quad C_\chi(2s\rho_\theta, \sigma, w) = \lambda_{\mathbf{C}}(\psi_{\mathbf{R}}, w)^{-1} \omega_s^{-1}(\tilde{w}(a)a^{-1}) \cdot \prod_{i=1}^m \varepsilon(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi, \bar{\psi}_{\mathbf{R}}) L(1 - a_i s, r_{\tilde{w}, i} \cdot \varphi) / L(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi).$$

Here  $\varepsilon(s, r_{\tilde{w}, i} \cdot \varphi, \psi_{\mathbf{R}})$  and  $L(s, r_{\tilde{w}, i} \cdot \varphi)$  are the Artin root number and  $L$ -function attached to  $r_{\tilde{w}, i} \cdot \varphi$ .

*Proof.* In view of Theorem 3.1 of [36] we only need to study the effect of changing  $\chi_0$  to  $\chi$ . As in [36] we shall again use an embedding  $\sigma \subset I_M(\nu_0, \eta_0)$  to get  $C_\chi(\nu, \sigma, \theta, w) = C_\chi(\nu + \nu_0, \eta_0, \phi, w)$ . By the observations made before,  $a$  as well as  $\tilde{w}_0(a)$  both belong to  $G_{\text{Max}}$  and if  $\eta_0$  extends to  $\tilde{\eta}_0$ , then  $I(2s\rho_\theta + \nu_0, \tilde{\eta}_0)|G = I(2s\rho_\theta + \nu_0, \eta_0)$ . Therefore we only need compute  $C_\chi(\nu + \nu_0, \tilde{\eta}_0, \phi, w)$  for  $G_{\text{Max}}$  with  $\nu = 2s\rho_\theta$ . But a direct calculation using the definition of local coefficients and the fact that  $\tilde{w}(a)a^{-1}$  belongs to the center of  $M$  (using  $\chi = \chi'_0 \cdot \text{Ad}_U(a^{-1})$ , where  $\chi'_0$  is defined by  $\chi'_0|U_\alpha = \chi_0|U_{-\tilde{w}(\alpha)}$  for

every simple root  $\alpha$ ) implies

$$C_\chi(2s\rho_\theta + \nu_0, \tilde{\eta}_0, \phi, w) = \omega_s^{-1}(\tilde{w}(a)a^{-1})C_{\chi_0}(2s\rho_\theta + \nu_0, \tilde{\eta}_0, \phi, w),$$

completing the proof of the proposition.

Now assume  $\psi_{\mathbf{R}}$  is arbitrary, i.e.  $\psi_{\mathbf{R}}(x) = \exp(2\pi ibx)$  with some non-zero  $b \in \mathbf{R}$ . Again let  $\psi_{\mathbf{C}} = \psi_{\mathbf{R}} \cdot \text{Tr}_{\mathbf{C}/\mathbf{R}}$ . Define  $\chi$  such that  $\chi|_{U_\alpha} = \psi_{\mathbf{R}} \cdot \text{Tr}_{E_\alpha/\mathbf{R}}$  with  $E_\alpha$  as before. Choose  $a \in \mathbf{A}_0(\bar{F})$  such that  $\chi = \chi_0 \cdot \text{Ad}_U(a)$ . Then  $\chi$  is compatible with respect to every  $\tilde{w}$  with  $\tilde{w}(\theta) \subset \Delta$ . We have:

LEMMA 3.3. *Assume  $F = \mathbf{R}$ . Fix  $\chi$  as above with  $\psi_{\mathbf{R}}(x) = \exp(2\pi ibx)$ . Let  $\sigma$  be an irreducible admissible  $\chi$ -generic representation of  $M$ . Let  $\varphi: W_{\mathbf{R}} \rightarrow {}^L M$  be the homomorphism attached to  $\sigma$ . Then*

$$C_\chi(2s\rho_\theta, \sigma, w) = \lambda_{\mathbf{G}}(\psi_{\mathbf{R}}, w)^{-1} \prod_{i=1}^m \varepsilon(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi, \bar{\psi}_{\mathbf{R}}) \cdot L(1 - a_i s, r_{\tilde{w}, i} \cdot \varphi) / L(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi).$$

*Proof.* By Proposition 3.2 it is enough to show that  $\omega_s(\tilde{w}(a)a^{-1})$  cancels the effect of changing the unramified character  $\exp(2\pi ix)$  to

$$\psi_{\mathbf{R}}(x) = \exp(2\pi ibx).$$

Let  $\lambda = (\nu + \nu_0) \oplus \eta_0$ ,  $\nu = 2s\rho_\theta$ , where  $\nu_0$  and  $\eta_0$  are as in Proposition 3.2 (cf. [36]). Fix a reduced decomposition  $\tilde{w} = \tilde{w}_n \cdots \tilde{w}_1$  of  $\tilde{w}$ . Then

$$(3.3.1) \quad \omega_s(\tilde{w}(a)a^{-1}) = \prod_{j=1}^n (\tilde{w}_{j-1} \cdots \tilde{w}_1 \lambda)(\tilde{w}_j(a)a^{-1}).$$

Since the measures are self-dual the lemma will be a consequence of the relation

$$(3.3.2) \quad \prod_{i=1}^m \det(r_{\tilde{w}, i} \cdot \varphi \otimes | \cdot |^{-a_i s})(b) = \prod_{j=1}^n (\tilde{w}_{j-1} \cdots \tilde{w}_1 \lambda)(\tilde{w}_j(a)a^{-1})$$

and relation (3.6.6) of [46].

By the Langlands classification we only need prove this for  $\sigma$  in the discrete series. Given  $i$ ,  $1 \leq i \leq m$ , let  $r_{il}$  denote an irreducible component of  $r_{\tilde{w}, i} \cdot \varphi$ . Then the left-hand side of (3.3.2) is equal to

$$\prod_{i=1}^m \prod_l \det(r_{il} \otimes | \cdot |^{-a_i s})(b).$$

Every factor in the right-hand side of (3.3.2) points to a positive root which under  $\tilde{w}$  is sent to a negative one. Conversely, every such root in  $\mathbf{N}$  points to a factor in the right-hand side of (3.3.2). If we group the roots on both sides of (3.3.2) according to Lemmas 3.4, 3.6, 3.9, 3.10, 3.11, and 3.12 of [36], (3.3.2) is then a consequence of similar identities for each group of roots. When the roots

in a group on the right-hand side of (3.3.2) are not real  $\alpha_0$ -roots (Lemma 3.4 of [36]), the corresponding identity is then a consequence of equality of infinitesimal characters of  $\sigma$  and  $I_M(\nu_0, \eta_0)$  (cf. relations (3.4.2) and (3.18) of [36]). But when the group of  $\alpha_0$ -roots in the right-hand side of (3.3.1) consists of real  $\alpha_0$ -roots, then for those roots which do not restrict to real  $\alpha$ -roots, the sign of  $b$  will not matter (cf. relations (3.23) and (3.24) of [36] in the case of Lemma 3.6 of [36], for example) and the equality is again a consequence of equality of infinitesimal characters. On the other hand for each of those (real)  $\alpha_0$ -roots which restrict to real  $\alpha$ -roots, the corresponding one-dimensional representation of  $W_{\mathbf{R}}$  on the left-hand side of (3.3.2), when evaluated at  $b$ , is exactly equal to the corresponding factor  $(\tilde{w}_{j-1} \cdots \tilde{w}_1 \lambda)(\tilde{w}_j(a)a^{-1})$  of (3.3.2). This completes the proof of the lemma.

In view of Lemma 3.3 we can now remove the assumption that  $\psi_{\mathbf{R}}$  is unramified and from now on assume that the character  $\chi_0$  (in Proposition 3.2 and elsewhere) is defined by an arbitrary nontrivial character  $\psi_{\mathbf{R}}$  of  $\mathbf{R}$ . All our discussions on compatibility then carry over.

Now if  $F$  is any local field and  $\sigma$  is an irreducible admissible  $\chi$ -generic representation of  $\mathbf{M}(F)$ , we choose  $a$  in  $\mathbf{A}_0(\bar{F})$  such that  $\chi = \chi_0 \cdot \text{Ad}_U(\tilde{w}_l(a))$ . Up to elements in the center of  $\mathbf{G}(\bar{F})$ ,  $a$  is unique. Given  $s \in \mathbf{C}$ , let  $\omega_s$  be

$$\omega_s = \omega \otimes q^{\langle 2s\rho_{\theta}, H_P(\cdot) \rangle},$$

where  $\omega$  is the central character of  $\sigma$ . It is the central character of

$$\sigma \otimes q^{\langle 2s\rho_{\theta}, H_P(\cdot) \rangle}.$$

Let  $a_{\chi}$  be the trivial cocycle in  $H^1(W(\mathbf{A}_0), A_0)$  defined by  $a_{\chi}(\tilde{w}) = \tilde{w}(a)a^{-1}$ . If  $\tilde{w}$  is compatible with  $\chi$ , then  $a_{\chi}$  takes values in the center of  $M$  and  $\omega_s(a_{\chi}(\tilde{w}))$  makes sense. If  $F$  is non-archimedean and  $\mathbf{G}$ ,  $\sigma$ , and  $\chi$  are all unramified, then  $\psi_F$  is unramified and  $a \in \mathbf{A}_0(\bar{O})$  and therefore  $\omega_s(a_{\chi}(\tilde{w})) = 1$ .

Finally if  $F$  is a number field and  $\sigma$  is a  $\chi$ -generic cusp form on  $\mathbf{G}(\mathbb{A}_F)$ , where  $\chi = \otimes_v \chi_v$  is a nontrivial character of  $\mathbf{U}(F) \backslash \mathbf{U}(\mathbb{A}_F)$ , then  $\chi_0 = \otimes_v \chi_{0,v}$  will be another one (defined by a character  $\psi_F = \otimes_v \psi_{F_v}$  of  $F \backslash \mathbb{A}_F$ ), which we shall call the unramified one. Moreover, there exists  $a \in \mathbf{A}_0(\bar{F})$  such that  $\chi = \chi_0 \cdot \text{Ad}_U(\tilde{w}_l(a))$ , and if  $\tilde{w}$  is compatible with  $\chi$ , then  $a_{\chi}(\tilde{w})$  belongs to the center of every  $M_v$  and therefore the center of  $\mathbf{M}(F)$  and

$$(3.3) \quad \omega_s(a_{\chi}(\tilde{w})) = \prod_v \omega_{v,s}(a_{\chi}(\tilde{w})) = 1.$$

Similarly,

$$(3.4) \quad \lambda_{\mathbf{G}}(\psi_F, w) = \prod_v \lambda_{\mathbf{G} \times_{F_v} F_v}(\psi_{F_v}, w) = 1.$$

Therefore both factors have no global significance.

Now, if  $\mathbf{P}$  is maximal and  $\alpha \in \Delta$  is the unique simple root in  $\mathbf{N}$  and  $s \in \mathbf{C}$ , we set

$$(3.5) \quad s_1 = \langle 2\rho_\theta, \alpha \rangle^{-1}s.$$

Next, let  $F$  be non-archimedean. Assume the representation  $\sigma$  of  $M$  has a vector fixed by an Iwahori subgroup of  $M$ . Then by [7], [6], it is a constituent of an unramified principal series (not necessarily unitary) representation of  $M$ . Let  $\varphi: W_F \rightarrow {}^L T \subset {}^L M$  be the homomorphism of the Weil group  $W_F$  attached to the inducing unramified character of  $T$  of this principal series. Let  $W'_F$  be the Deligne-Weil group of  $\bar{F}/F$ . It is a group scheme over  $\mathbf{Q}$  which is a semi-direct product of  $W_F$  by  $\mathbf{G}_a$ , on which  $W_F$  acts by  $wxw^{-1} = \|w\|x$  (cf. Paragraph 4.1.1 of [46]). It is expected (cf. [17], [18]) that the representation  $\sigma$  be parametrized by an admissible homomorphism  $\varphi'$  of  $W'_F$  into  ${}^L M$  such that  $\varphi'|_{W_F} = \varphi$  (cf. Paragraph 8.2 of [5]). Then  $\varphi'(\mathbf{G}_a)$  consists of unipotent elements and if  $r$  is a representation of  ${}^L M$  on a finite-dimensional complex space  $V$ , then for every  $x \in \mathbf{G}_a$ ,  $r \cdot \varphi'(x)$  becomes a unipotent element in  $\text{GL}(V)$ . If  $\mathbf{M}$  is unramified and  $\sigma$  has a vector fixed by  $\mathbf{M}(O)$ , then  $\varphi'|_{\mathbf{G}_a}$  is trivial and therefore  $\sigma$  is only parametrized by  $\varphi$ .

Let  $r$  be a representation of  ${}^L M$  such that  $r \cdot \varphi'$  is indecomposable. Then  $r \cdot \varphi' = \rho' \otimes \text{sp}(n)$ , where  $\text{sp}(n)$  is the representation  $(\rho, N)$  of  $W'_F$  over  $\mathbf{Q}$  on

$$V = \mathbf{Q}^n = \mathbf{Q}e_0 + \mathbf{Q}e_1 + \cdots + \mathbf{Q}e_{n-1}$$

defined by

$$\begin{aligned} \rho(w)e_j &= \|w\|^j e_j & 0 \leq j \leq n-1, \\ Ne_j &= e_{j+1} & 0 \leq j < n-1, \end{aligned}$$

and

$$Ne_{n-1} = 0,$$

and  $\rho'$  is an irreducible representation of  $W_F$  satisfying  $\rho' \otimes \rho = r \cdot \varphi$  (cf. Paragraphs 4.1.2 through 4.1.5 of [46]). The contragredient of  $r \cdot \varphi'$  is  $\tilde{\rho}' \otimes \text{sp}(n)$ , where  $\text{sp}(n) = (\rho^{-1}, \tilde{N})$  is considered on the same space  $V = \mathbf{Q}^n$  such that

$$\tilde{N}e_j = e_{j-1}, \quad 1 \leq j \leq n-1,$$

while  $\tilde{N}e_0 = 0$ .

Artin  $L$ -functions and root numbers are defined by Deligne [8] as follows (also cf. Paragraph 4.1.6 of [46]):

$$(3.6) \quad L(s, r \cdot \varphi') = \det(I - (\rho' \otimes \rho)(\phi)q^{-s}|_{V_{\rho'} \otimes e_{n-1}})^{-1},$$

$$(3.7) \quad \varepsilon(s, r \cdot \varphi', \psi_F) = \varepsilon(s, r \cdot \varphi, \psi_F) \det(-(\rho' \otimes \rho)(\phi)|_{V/V_{\rho'} \otimes e_{n-1}}) \cdot q^{-s(\dim V - \dim V_{\rho'})},$$

and

$$(3.8) \quad L(s, \tilde{r} \cdot \varphi') = \det(I - (\tilde{\rho}' \otimes \rho^{-1})(\phi)q^{-s}|V_{\tilde{\rho}'} \otimes e_0)^{-1}.$$

Here  $\phi$  denotes an inverse Frobenius (cf. Paragraph 1.4.1 of [46]), and  $V_{\rho'}$  and  $V_{\tilde{\rho}'}$  are the spaces of  $\rho'$  and its contragredient. Observe that  $\varphi$  is unramified and therefore  $(V_{\rho'} \otimes e_{n-1})^I = V_{\rho'} \otimes e_{n-1}$ . We shall now prove:

PROPOSITION 3.4. *Assume  $F$  is non-archimedean and  $\sigma$  has an Iwahori-fixed vector. Moreover suppose  $\sigma$  is  $\chi$ -generic and  $a \in \mathbf{A}_0(\bar{F})$  is such that  $\chi = \chi_0 \cdot \text{Ad}_U(\tilde{w}_l(a))$ . Choose  $\tilde{w} \in \mathfrak{w}(\mathbf{A}_0)$ , compatible with  $\chi$ . Let  $\varphi': W'_F \rightarrow {}^L M$  be the homomorphism of  $W'_F$  attached to  $\sigma$ . Then*

$$C_\chi(2s\rho_\theta, \sigma, w) = \lambda_{\mathbb{C}}(\psi_F, w)^{-1} \omega_s^{-1}(\tilde{w}(a)a^{-1}) \cdot \prod_{i=1}^m \varepsilon(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi', \bar{\psi}_F) L(1 - a_i s, r_{\tilde{w}, i} \cdot \varphi') | L(a_i s, \tilde{r}_{\tilde{w}, i} \cdot \varphi').$$

*Proof.* Fix  $i$ ,  $1 \leq i \leq m$ . Let  $V$  be an indecomposable constituent of  $\tilde{r}_{\tilde{w}, i} \cdot \varphi'$ . Then

$$\tilde{r}_{\tilde{w}, i} \cdot \varphi | V = \bigoplus_{j=0}^{n-1} \rho' \otimes \rho_j,$$

where  $\rho_j = \rho|e_j$ . Consequently

$$(3.4.1) \quad \gamma(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \bar{\psi}_F) = \varepsilon(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \bar{\psi}_F) \cdot \prod_{0 \leq j \leq n-1} \det(I - (\tilde{\rho}' \otimes \rho_j^{-1})(\phi)q^{-(1-s)})^{-1} / \det(I - (\rho' \otimes \rho_j)(\phi)q^{-s})^{-1},$$

where  $\gamma(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \psi_F)$  has its usual meaning. But then for each  $j$ ,  $0 \leq j < n - 1$ ,

$$\det(I - (\rho' \otimes \rho_j)(\phi)q^{-s}) = q^{-s \dim \rho'} \det(-(\rho' \otimes \rho)(\phi)|V_{\rho'} \otimes e_j) \cdot \det(I - (\tilde{\rho}' \otimes \rho_{j+1}^{-1})(\phi)q^{-(1-s)}).$$

Substitution in (3.4.1) then implies that up to the factor  $\varepsilon(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \bar{\psi}_F)$ , the factor  $\gamma(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \bar{\psi}_F)$  equals

$$q^{-s(\dim V - \dim \rho')} \det(-(\rho' \otimes \rho)(\phi)|V/V_{\rho'} \otimes e_{n-1}) \cdot \det(I - (\tilde{\rho}' \otimes \rho_0^{-1})(\phi)q^{-(1-s)})^{-1} / \det(I - (\rho' \otimes \rho_{n-1})(\phi)q^{-s})^{-1}.$$

But by definitions (3.6), (3.7), and (3.8),

$$\varepsilon(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi' | V, \bar{\psi}_F) = \varepsilon(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi | V, \bar{\psi}_F) \det\left(-(\rho' \otimes \rho)(\phi) | V/V_{\rho'} \otimes e_{n-1}\right) \cdot q^{-s(\dim V - \dim \rho')}$$

$$L(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi' | V) = \det(I - (\rho' \otimes \rho_{n-1})(\phi) q^{-s})^{-1},$$

and

$$L(s, r_{\tilde{w}, i} \cdot \varphi' | V) = \det(I - (\tilde{\rho}' \otimes \rho_0^{-1})(\phi) q^{-s})^{-1}.$$

Consequently

$$\gamma(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi' | V, \bar{\psi}_F) = \gamma(s, \tilde{r}_{\tilde{w}, i} \cdot \varphi' | V, \bar{\psi}_F).$$

Now the lemma is a consequence of Proposition 3.4 of [20].

*Remark 1.* As is clear from the proof of Proposition 3.4, while the  $L$ -functions and root numbers may depend on all of  $\varphi'$ , the factors  $\gamma(s, r_{\tilde{w}, i} \cdot \varphi', \psi_F)$  only depend on  $\varphi$ , the restriction of  $\varphi'$  to  $W_F$ . Consequently, the extent of our use of any parametrization (cf. [17], [18]) is the fact that the inducing character  $\lambda$  is the one attached to  $\varphi$  by the main local theorem of [31] (also cf. [24]).

*Remark 2.* Assume  $\sigma$  is an irreducible class-one representation of  $M$  with respect to a special maximal compact subgroup  $Q$  of  $M$  adapted to  $\mathbf{A}_0(F)$ . Then as remarked on page 45 of [5], there is no reason in general that the parametrization  $\varphi'$  of  $\sigma$  be trivial on  $\mathbf{G}_a$ , unless  $Q$  is hyperspecial. But this then requires  $\mathbf{M}$  to be unramified and the use of Proposition 5.1 (which is one of our basic tools in proving Theorem 3.5) becomes quite limited in general. It is for this reason that we have stated and proved Proposition 3.4 (cf. Part 1 of Theorem 3.5).

Another remark to be made is as follows: Let  $\sigma$  be an irreducible admissible representation of  $M$ . Suppose  $\sigma \subset \text{Ind}_{M_\theta N_\theta \uparrow M} \sigma_1 \otimes 1$ , where  $\mathbf{M}_\theta \mathbf{N}_\theta$ ,  $\theta \in \Delta$ , is a parabolic subgroup of  $\mathbf{M}$  and  $\sigma_1$  is an irreducible admissible representation of  $M_\theta$ . Let  $\theta' = \tilde{w}(\theta) \in \Delta$  and fix a reduced decomposition  $\tilde{w} = \tilde{w}_{n-1} \cdots \tilde{w}_1$  of  $\tilde{w}$  as in Lemma 2.1.1 of [38]. Then for each  $j$ , there exists a unique root  $\alpha_j \in \Delta$  such that  $\tilde{w}_j(\alpha_j) < 0$ . For each  $j$ ,  $2 \leq j \leq n - 1$ , let  $\bar{w}_j = \tilde{w}_{j-1} \cdots \tilde{w}_1$ . Set  $\bar{w}_1 = 1$ . Also, let  $\Omega_j = \theta_j \cup \{\alpha_j\}$ , where  $\theta_1 = \theta$ ,  $\theta_n = \theta'$ , and  $\theta_{j+1} = \tilde{w}_j(\theta_j)$ ,  $1 \leq j \leq n - 1$ . Then the group  $\mathbf{M}_{\Omega_j}$  contains  $\mathbf{M}_{\theta_j} \mathbf{N}_{\theta_j}$  as a maximal parabolic subgroup and  $\bar{w}_j(\sigma_1)$  is a representation of  $M_{\theta_j}$ . The  $L$ -group  ${}^L M_\theta$  acts on  $V_i$ . Given an irreducible constituent of this action, there exists a unique  $j$ ,  $1 \leq j \leq n - 1$ , which under  $\bar{w}_j$  is equivalent to an irreducible constituent of the action of  ${}^L M_{\theta_j}$  on the Lie algebra of  ${}^L N_{\theta_j}$  ( $\mathbf{M}_{\theta_j} \mathbf{N}_{\theta_j}$  is a maximal parabolic subgroup of

$\mathbf{M}_{\Omega_j}$ ). We denote by  $i(j)$  the index of this subspace of the Lie algebra of  ${}^L N_{\theta_j}$ . Finally let  $S_i$  denote the set of all such  $j$ 's where  $S_i$ , in general, is a proper subset of  $1 \leq j \leq n - 1$ .

There is one last matter which we should attend to before stating our main results.

Let  $\mathbf{G}$  be a reductive group over a number field  $K$ . Fix a maximal parabolic subgroup  $\mathbf{P} = \mathbf{M}\mathbf{N}$  of  $\mathbf{G}$ . Let  $\alpha$  denote the unique simple root in  $\mathbf{N}$ . By a quasi-cusp form on  $M = \mathbf{M}(\mathbb{A}_K)$ , we shall mean a representation of  $M$  of the form  $\pi_0 \otimes \exp\langle s_0 \tilde{\alpha}, H_P(\ ) \rangle$ , where  $\pi_0$  is a cusp form on  $M$  and  $s_0 \in \mathbf{C}$  (notation as in Section 1 of [39]). Observe that every cusp form is a quasi-cusp form.

We shall now state our main theorem.

**THEOREM 3.5.** *Given a local field  $F$  of characteristic zero and a quasi-split connected reductive algebraic group  $\mathbf{G}$  over  $F$ , containing an  $F$ -parabolic subgroup  $\mathbf{P} = \mathbf{M}\mathbf{N}$ ,  $\mathbf{N} \subset \mathbf{U}$ , let  $r_{\tilde{w}} = \bigoplus_{i=1}^m r_{\tilde{w},i}$  be the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}_{\tilde{w}}$  as in Section 1. Let  $\chi$  be a non-degenerate character of  $U$  defined by  $\psi_F$  and  $\tilde{w}_l(a)$  with  $a \in \mathbf{A}_0(\bar{F})$ , and compatible with  $\tilde{w}$ . Given an irreducible admissible  $\chi$ -generic representation  $\sigma$  of  $M$ , there exist  $m$  complex functions  $\gamma_i(s, \sigma, \psi_F, \tilde{w})$ ,  $s \in \mathbf{C}$ , satisfying the following properties:*

1). *If  $F$  is archimedean or  $\sigma$  has an Iwahori fixed vector, let  $\varphi': W'_F \rightarrow {}^L M$  be the homomorphism attached to  $\sigma$ , where  $W'_F$  is the Deligne-Weil group of  $F$ . Denote by  $\varepsilon(s, r_{\tilde{w},i} \cdot \varphi', \psi_F)$  and  $L(s, r_{\tilde{w},i} \cdot \varphi')$  the Artin root number and  $L$ -function attached to  $r_{\tilde{w},i} \cdot \varphi'$ , respectively. Then*

$$(3.9) \quad \gamma_i(s, \sigma, \psi_F, \tilde{w}) = \varepsilon(s, r_{\tilde{w},i} \cdot \varphi', \psi_F) L(1 - s, \tilde{r}_{\tilde{w},i} \cdot \varphi') / L(s, r_{\tilde{w},i} \cdot \varphi').$$

2). *For each  $i$ ,  $1 \leq i \leq m$ ,*

$$(3.10) \quad \gamma_i(s, \sigma, \psi_F, \tilde{w}) \gamma_i(1 - s, \bar{\sigma}, \bar{\psi}_F, \tilde{w}) = 1$$

*and moreover if  $\mathbf{P}$  is maximal, then*

$$(3.11) \quad C_\chi(s \tilde{\alpha}, \sigma, w_0) = \lambda_G(\psi_F, w_0)^{-1} \omega_{s_1}^{-1}(\tilde{w}_0(a) a^{-1}) \prod_{i=1}^m \gamma_i(is, \bar{\sigma}, \bar{\psi}_F, \tilde{w}_0),$$

*where  $\alpha$  is the unique simple root in  $\mathbf{N}$ ,  $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$ , and  $s_1$  is defined by (3.5). If  $F$  is non-archimedean, then each  $\gamma_i(s, \sigma, \psi_F, \tilde{w})$  is a rational function of  $q^{-s}$ . Finally assume  $\sigma = \sigma_0 \otimes q^{\langle s_0 \tilde{\alpha}, H_P(\ ) \rangle}$ ,  $s_0 \in \mathbf{C}$ . Then*

$$(3.12) \quad \gamma_i(s, \sigma, \psi_F, \tilde{w}) = \gamma_i(s + s_0, \sigma_0, \psi_F, \tilde{w}).$$

3). *Inductive property: Suppose  $\sigma \subset \text{Ind}_{\mathbf{M}_\theta \mathbf{N}_\theta \uparrow M} \sigma_1 \otimes 1$ , where  $\mathbf{M}_\theta \mathbf{N}_\theta$  is a parabolic subgroup of  $\mathbf{M}$  and  $\sigma_1$  is an irreducible admissible  $\chi$ -generic repre-*

sentation of  $M_\theta$ . Write  $\tilde{w} = \tilde{w}_{n-1} \cdots \tilde{w}_1$  and for each  $j$ ,  $2 \leq j \leq n - 1$ , let  $\bar{w}_j = \tilde{w}_{j-1} \cdots \tilde{w}_1$  and  $\bar{w}_1 = 1$ . Then for each  $j$ ,  $\bar{w}_j(\sigma_1)$  is a representation of  $M_{\theta_j}$ . If for each  $j \in S_i$ ,  $\gamma_{i(j)}(s, \bar{w}_j(\sigma_1), \psi_F, \tilde{w}_j)$ ,  $1 \leq i \leq m$ , denotes the corresponding factor, then

$$(3.13) \quad \gamma_i(s, \sigma, \psi_F, \tilde{w}) = \prod_{j \in S_i} \gamma_{i(j)}(s, \bar{w}_j(\sigma_1), \psi_F, \tilde{w}_j).$$

4). *Functional equations:* Let  $K$  be a number field and  $\mathbf{G}$  a quasi-split connected reductive algebraic group over  $K$ . Let  $\mathbf{P} = \mathbf{M}\mathbf{N}$ ,  $\mathbf{N} \subset \mathbf{U}$ , be a maximal  $K$ -parabolic subgroup of  $\mathbf{G}$ . Denote by  $G = \mathbf{G}(\mathbb{A}_K)$ , where  $\mathbb{A}_K$  is the ring of adèles of  $K$ . Fix a non-degenerate character  $\chi = \otimes_v \chi_v$  of  $U = \mathbf{U}(\mathbb{A}_K)$ , trivial on  $\mathbf{U}(K)$ , compatible with  $\tilde{w}_0$  and defined by a non-trivial character  $\psi = \otimes_v \psi_v$  of  $K \backslash \mathbb{A}_K$ . Let  $\pi = \otimes_v \pi_v$  be a globally  $\chi$ -generic quasi-cusp form on  $M = \mathbf{M}(\mathbb{A}_K)$ . Finally if  $r$  is the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}$ , write  $r = \bigoplus_{i=1}^m r_i$ . Then  $r_v = \bigoplus_{i=1}^m r_{i,v}$  is the adjoint action of  ${}^L M_v$ , where  $r_v = r \cdot \eta_v$ ,  $r_{i,v} = r_i \cdot \eta_v$ , and  $\eta_v: {}^L M_v \rightarrow {}^L M$  is the natural map. Let  $S$  be a finite set of places of  $K$  such that for  $v \notin S$ ,  $\mathbf{G} \times_K K_v$ ,  $\pi_v$ , and  $\chi_v$  are all unramified. Set

$$L_S(s, \pi, r_i) = \prod_{v \notin S} L(s, \pi_v, r_{i,v}).$$

Then

$$(3.14) \quad L_S(s, \pi, r_i) = \prod_{v \in S} \gamma_i(s, \pi_v, \psi_v) L_S(1 - s, \pi, \tilde{r}_i),$$

for every  $i$ ,  $1 \leq i \leq m$ , where  $\gamma_i(s, \pi_v, \psi_v) = \gamma_i(s, \pi_v, \psi_v, \tilde{w}_0)$ .

Moreover, conditions 1, 3, and 4, determine  $\gamma_i$ 's uniquely.

COROLLARY 3.6. *Let*

$$(3.15) \quad \gamma(s, \sigma, r_{\tilde{w}}, \bar{\psi}_F) = \prod_{i=1}^m \gamma_i(s, \sigma, \psi_F, \tilde{w}).$$

Then

$$(3.16) \quad \begin{aligned} \mu(\sigma, \tilde{w}) \gamma_{\tilde{w}}(G/P)^{-1} \gamma_{\tilde{w}^{-1}}(G/P')^{-1} \\ = \gamma(0, \sigma, r_{\tilde{w}}, \bar{\psi}_F) \gamma(0, \tilde{\sigma}, r_{\tilde{w}}, \psi_F) \\ = \gamma(0, \sigma, r_{\tilde{w}}, \bar{\psi}_F) / \gamma(1, \sigma, r_{\tilde{w}}, \bar{\psi}_F). \end{aligned}$$

There are two results in Theorem 3.5 whose importance must be pointed out.

The first one is the inductive property 3) whose local proof is only available when  $\mathbf{G} = \text{GL}(n)$  (Part i of Theorem 3.1 of [15]) and even in that case requires an elaborate proof.

For the second, assume  $\mathbf{P}$  is maximal,  $m = 1$  and  $a = 1$ ; i.e.,  $\sigma$  is  $\chi_0$ -generic. Then by (3.10)

$$(3.17) \quad C_{\chi_0}(s\tilde{\alpha}, \sigma, w_0)C_{\tilde{\chi}_0}((1-s)\tilde{\alpha}, \tilde{\sigma}, w_0) = \lambda_{\mathbf{G}}(\psi_F, w_0)^{-2}.$$

This is again a very deep identity since all the elementary relations for local coefficients (Propositions 3.1.1, 3.1.2, and 3.1.3 of [38]) are a reflection of those of intertwining operators and always appear as a change from  $s$  to  $-s$ , while in (3.17) there is a change of  $s$  to  $1 - s$ . In Section 8, this will be used to study reducibility of representations induced from supercuspidal representations. Finally, we should remark that the identity

$$C_{\chi}(s, \pi_1 \otimes \tilde{\pi}_2)C_{\chi}(1-s, \tilde{\pi}_1 \otimes \pi_2) = \omega_1^m \omega_2^n (-1)$$

(in the notation of [37]) stated at the end of the introduction of [37] agrees with (3.17) if the representative for  $w_{m,n}$  is chosen as in [36].

In the next three sections we shall prove Theorem 3.5.

#### 4. The induction step

In this section we shall state and prove our necessary induction results. We start with:

**PROPOSITION 4.1.** *Let  $\mathbf{G}$  be a quasi-split connected reductive algebraic group over a number field  $F$ . Fix a maximal parabolic subgroup  $\mathbf{P} = \mathbf{MN}$  of  $\mathbf{G}$ . Decompose the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}$ , the Lie algebra of the  $L$ -group of  $\mathbf{N}$  as  $r = \bigoplus_{i=1}^m r_i$ . Let  $\pi = \bigotimes_v \pi_v$  be an automorphic form on  $M = \mathbf{M}(\mathbb{A}_F)$ . Fix a finite set  $S$  of places of  $F$  such that every  $v$  which is not in  $S$  is unramified. Then the following statement is true for all except possibly for one  $i$ ,  $1 \leq i \leq m$ : Given  $i$ , there exist a quasi-split connected reductive algebraic group  $\mathbf{G}_i$  over  $F$ , a maximal parabolic  $F$ -subgroup  $\mathbf{P}_i = \mathbf{M}_i \mathbf{N}_i$ , both unramified for every  $v \notin S$ , and an automorphic form  $\pi'$  of  $M_i = \mathbf{M}_i(\mathbb{A}_F)$ , unramified for every  $v \notin S$ , such that if the adjoint action  $r'$  of  ${}^L M_i$  on  ${}^L \mathfrak{n}_i$  decomposes as  $r' = \bigoplus_{j=1}^{m'} r'_j$ , then:*

$$L_S(s, \pi, r_i) = L_S(s, \pi', r'_1),$$

and moreover  $m' < m$ .

We shall first prove the following local analogue of this result:

**PROPOSITION 4.2.** *Let  $\mathbf{G}$  be an unramified quasi-split connected reductive algebraic group over a non-archimedean local field  $F$ . Fix a maximal parabolic subgroup  $\mathbf{P} = \mathbf{MN}$  of  $\mathbf{G}$ . Decompose the adjoint action of  ${}^L M$  on  ${}^L \mathfrak{n}$ , the Lie algebra of the  $L$ -group of  $\mathbf{N}$  as  $r = \bigoplus_{i=1}^m r_i$ . Let  $\pi$  be a class one representation of  $M = \mathbf{M}(F)$ . Then the following statement is true for all except possibly for one*

$i, 1 \leq i \leq m$ : Given  $i$ , there exist an unramified quasi-split connected reductive algebraic group  $\mathbf{G}_i$  over  $F$ , a maximal parabolic  $F$ -subgroup  $\mathbf{P}_i = \mathbf{M}_i \mathbf{N}_i$ , and a class one representation  $\pi'$  of  $M_i = \mathbf{M}_i(F)$  such that if the adjoint action  $r'$  of  ${}^L M_i$  on  ${}^L \mathfrak{n}_i$  decomposes as  $r' = \bigoplus_{j=1}^{m'} r'_j$ , then

$$L(s, \pi, r_i) = L(s, \pi', r'_1),$$

and moreover  $m' < m$ .

We start with the following lemma.

LEMMA 4.3. Let  $\hat{G}$  be a complex connected reductive algebraic group of adjoint type. Denote by  $\hat{P}$  a parabolic subgroup of  $\hat{G}$ . Write  $\hat{P} = \hat{M}\hat{N}$ . Let  $\hat{\mathfrak{n}}$  be the Lie algebra of  $\hat{N}$  and let  $r$  be a faithful subrepresentation of the adjoint action of  $\hat{M}$  on  $\hat{\mathfrak{n}}$ . Let  $\hat{M}'$  be a complex connected reductive algebraic group whose connected center is a torus isomorphic to the center  $\hat{A}$  of  $\hat{M}$ . Moreover assume that  $\hat{M}'$  has a faithful complex analytic action  $r'$  on a subspace of  $\hat{\mathfrak{n}}$  such that if  $\alpha: \widetilde{\hat{M}}_D \rightarrow \hat{M}_D \rightarrow \{1\}$  and  $\alpha': \widetilde{\hat{M}'_D} \rightarrow \hat{M}'_D \rightarrow \{1\}$  are the corresponding surjections ( $\hat{M}_D$  is the derived group of  $\hat{M}$  and  $\widetilde{\hat{M}}_D$  is its simply connected covering; similarly  $\hat{M}'_D$  and  $\widetilde{\hat{M}'_D}$ ),  $\widetilde{\hat{M}}_D = \widetilde{\hat{M}'_D}$ , and  $r_D = r|_{\hat{M}_D}$  and  $r'_D = r'|_{\hat{M}'_D}$ , then  $r_D \cdot \alpha = r'_D \cdot \alpha'$ . Then  $\hat{M} \cong \hat{M}'$  and there exists a system  $\rho$  of complex analytic characters of  $\hat{M}$  such that  $r' = r \otimes \rho$ .

*Proof.* Let  $K \subset \widetilde{\hat{M}}_D$  be the kernel of  $r_D \cdot \alpha = r'_D \cdot \alpha'$ . If  $x \in \ker(\alpha)$ , then  $r'_D(\alpha'(x)) = 0$  which implies  $\alpha'(x) = 0$  since  $r'_D$  is faithful. Consequently  $\ker(\alpha) \subset \ker(\alpha')$ . Changing the roles now implies  $K = \ker(\alpha) = \ker(\alpha')$ . Thus  $\widetilde{\hat{M}}_D = \widetilde{\hat{M}'_D}/K \cong \widetilde{\hat{M}'_D}$  and  $r_D = r'_D$ . But now

$$\hat{M} \cong \hat{A} \times \hat{M}_D / \hat{A} \cap \hat{M}_D \cong \hat{M}'.$$

Next let  $\omega_i$  and  $\omega'_i$  be the action of  $\hat{A}$  on an irreducible component of  $r_D = r'_D$ , under  $r$  and  $r'$ , respectively. Then  $\omega'_i \omega_i^{-1}|_{\hat{A} \cap \hat{M}_D}$  is trivial and consequently  $\omega'_i \omega_i^{-1}$  extends to a character  $\rho_i$  of  $\hat{M}$ . Set  $\rho = \otimes_i \rho_i$ . Then  $r' = r \otimes \rho$ .

LEMMA 4.4. Let  $F$  be a non-archimedean local field and  $M$  the group of  $F$ -points of an unramified quasi-split connected reductive algebraic group over  $F$ . Assume  $\pi$  is a class-one representation of  $M$ . Let  $r_0$  and  $r'_0$  be two  $\Gamma$ -invariant,  $\Gamma = \text{Gal}(\bar{F}/F)$ , complex analytic representations of  ${}^L M^0$ , restricting representations  $r$  and  $r'$  of  ${}^L M$ , respectively, for which  $r|_{W_F} \cong r'|_{W_F}$ . Moreover assume that there exists a  $\Gamma$ -invariant character  $\rho = \otimes_{i=1}^n \rho_i$  of  ${}^L M^0 \times \cdots \times {}^L M^0$ ,  $n$  being the number of irreducible components of  $r_0$  (or  $r'_0$ ), such that  $r'_0 = r_0 \otimes \rho$ . Finally, carry the action of  $\Gamma$  on  $\rho$  to  $(\mathbf{C}^*)^n$ ; i.e.  $\sigma((x_i)) = (x_{\sigma(i)})$ ,  $(x_i) \in (\mathbf{C}^*)^n$ ,

where  $\sigma(i)$  is defined by  $\rho_i^\sigma = \rho_{\sigma(i)}$ . Assume that there exists a complex analytic  $\Gamma$ -morphism  $\hat{\alpha}: (\mathbb{C}^*)^n \rightarrow \hat{Z}$ , center of  ${}^L M^0$ , such that  $r_0 \cdot \hat{\alpha}((x_i)) = (x_i)$ ,  $(x_i) \in (\mathbb{C}^*)^n$ . Then there exists a character  $\eta$  of  $M$  such that

$$L(s, \pi, r') = L(s, \pi \otimes \eta, r),$$

*Proof.* Let  $\varphi_0 \in H^1(W_F, {}^L T^0)$  be attached to  $\pi$  (cf. [31], [24]). Then for  $w_1$  and  $w_2$  in  $W_F$ ,

$$\begin{aligned} \rho \cdot \varphi_0(w_1 w_2) &= \rho(\varphi_0(w_1)) \rho^{w_1}(\varphi_0(w_2)) \\ &= \rho(\varphi_0(w_1)) w_1(\rho(\varphi_0(w_2))), \end{aligned}$$

where the action of  $W_F$  is always through  $\Gamma$ . Consequently  $\rho \cdot \varphi_0 \in H^1(W_F, (\mathbb{C}^*)^n)$ . Since  $\hat{\alpha}$  is a  $\Gamma$ -map, we see immediately that  $\hat{\alpha} \cdot \rho \cdot \varphi_0 \in H^1(W_F, \hat{Z})$ . Let  $\theta = \hat{\alpha} \cdot \rho \cdot \varphi_0$ . By [5], [28], there exists a character  $\chi_\theta$  of  $M$  such that if  $\pi'$  is the class-one representation attached to  $\varphi\theta$ , where  $\varphi(w) = (\varphi_0(w), w)$  is the corresponding homomorphism of  $W_F$  into  ${}^L T$ , then  $\pi' = \pi \otimes \chi_\theta$ . Moreover,

$$\begin{aligned} r'_0 \cdot \varphi_0 &= (r_0 \otimes \rho) \cdot \varphi_0 \\ &= (r_0 \cdot \varphi_0) \otimes (\rho \cdot \varphi_0). \end{aligned}$$

But since  $\rho \cdot \varphi_0 = r_0 \cdot \hat{\alpha} \cdot \rho \cdot \varphi_0$ , we have:

$$\begin{aligned} r'_0 \cdot \varphi_0 &= (r_0 \cdot \varphi_0) \otimes (r_0 \cdot \theta) \\ &= r_0 \cdot (\varphi_0 \theta), \end{aligned}$$

which implies

$$r' \cdot \varphi = r \cdot (\varphi\theta).$$

Thus

$$\begin{aligned} L(s, \pi, r') &= L(s, \pi', r) \\ &= L(s, \pi \otimes \chi_\theta, r). \end{aligned}$$

Now, set  $\eta = \chi_\theta$ . This completes the proof of the lemma.

Let  $\rho$  be a  $\Gamma$ -invariant element of  $X^*({}^L T^0)$  which extends to a character of  ${}^L M^0$ . Then  $\rho$  is perpendicular to every root of  ${}^L M^0$  and therefore as an element of  $X_*(T)$ ,  $\rho(a)$  must belong to the center of  $\mathbf{M}(\bar{F})$ ,  $a \in \bar{F}^*$ . We thus have:

**LEMMA 4.5.** *Let  $\rho$  be a  $\Gamma$ -invariant element of  $X^*({}^L T^0)$  which extends to a character of  ${}^L M^0$ . Then for each  $a \in \bar{F}^*$  for which  $\rho(a) \in M = \mathbf{M}(F)$ ,  $\rho(a)$  belongs to the center of  $M = \mathbf{M}(F)$ .*

**LEMMA 4.6.** *Let  $\rho$  be a  $\Gamma$ -invariant element of  $X^*({}^L T^0)$  which extends to a character of  ${}^L M^0$ . Given an irreducible class-one representation  $\pi$  of  $M = \mathbf{M}(F)$ ,*

let  $\lambda \in \text{Hom}(\mathbf{T}(F), \mathbf{C}^*)$  and  $\varphi_0 \in H^1(W_F, {}^L T^0)$  be the corresponding parameters. Finally let  $\omega$  denote the central character of  $M$ . Then for each  $a \in \bar{F}^*$  with  $\rho(a) \in M$ ,

$$\rho(\varphi_0(a)) = \omega(\rho(a)),$$

where on the left-hand side  $\rho \in X^*({}^L T^0)$  while on the right-hand side  $\rho$  denotes an element of  $X_*(\mathbf{T})$ .

*Proof.* By [24], [31],

$$\lambda \cdot \rho = \rho \cdot \varphi_0.$$

Now, take  $a$  as above; then  $\lambda(\rho(a)) = \rho(\varphi_0(a))$ . But by Lemma 4.5,  $\rho(a)$  belongs to the center of  $M$  and therefore  $\lambda(\rho(a)) = \omega(\rho(a))$ .

LEMMA 4.7. Let  $\beta \in X_*(\mathbf{T})^\Gamma$  and fix  $\hat{\alpha} \in X_*({}^L T^0)$  as in Lemma 4.4. Consider  $\hat{\beta}$  where  $\beta$  is an element in  $X^*({}^L T^0)$ . Then for every  $t \in \bar{F}^*$  with  $\beta(t) \in M$ ,  $\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}$  belongs to the center of  $M$  and

$$\eta(\beta(t)) = \omega(\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}),$$

where  $\omega$  is the central character of  $\pi$  and  $\eta$  is attached to  $\theta = \hat{\alpha} \cdot \rho \cdot \varphi_0 \in H^1(W_F, \hat{Z})$ .

*Proof.* If  $\varphi_0$  is attached to  $\lambda \in \text{Hom}(\mathbf{T}(F), \mathbf{C}^*)$ , then  $\varphi'_0 = \varphi_0 \theta$  will be the one for  $\lambda \eta$ . Consequently

$$(\lambda \eta) \cdot \beta = \hat{\beta} \cdot (\varphi_0 \theta).$$

But then  $\lambda \cdot \beta = \hat{\beta} \cdot \varphi_0$  implies  $\eta \cdot \beta = \hat{\beta} \cdot \theta$ . Now assume  $t \in \bar{F}^*$  is such that  $\sigma(\beta(t)) = \beta(t)$  for all  $\sigma \in \Gamma$ . We must show that  $\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}$  belongs to the center of  $M$ . By Lemma 4.5, we need only show its rationality. Since  $\rho$  is  $\Gamma$ -invariant

$$\sigma(\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}) = \rho(\sigma(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle})$$

for all  $\sigma \in \Gamma$ . But considering  $\alpha \in X^*(\mathbf{T})$ , representing  $\hat{\alpha} \in X_*({}^L T^0)$ , and  $\beta \in X_*(\mathbf{T})^\Gamma$ , we have

$$\begin{aligned} \sigma(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle} &= \alpha(\beta(\sigma(t))) \\ &= \alpha(\sigma(\beta(t))) \\ &= t^{\langle \hat{\beta}, \hat{\alpha} \rangle}, \end{aligned}$$

as desired. Now by Lemma 4.6

$$\begin{aligned} \eta(\beta(t)) &= \hat{\beta}(\hat{\alpha}(\rho(\varphi_0(t)))) \\ &= \omega(\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}). \end{aligned}$$

LEMMA 4.8. *Let  $\hat{G}$  be a complex connected reductive algebraic group of adjoint type. Denote by  $\hat{P}$  a maximal parabolic subgroup of  $\hat{G}$ . Write  $\hat{P} = \hat{M}\hat{N}$ . Let  $\hat{\mathfrak{n}}$  be the Lie algebra of  $\hat{N}$ . Let  $r$  be the adjoint action of  $\hat{M}$  on  $\hat{\mathfrak{n}}$ . Write  $r = \bigoplus_i r_i$  as before. Then  $r_1$  is faithful.*

*Proof.* Let  $\hat{M}_D$  be the derived group of  $\hat{M}$ . Then

$$\tilde{M}_D \xrightarrow{\pi} \hat{M}_D \rightarrow \{1\},$$

where  $\tilde{M}_D$  is the simply connected covering of  $\hat{M}_D$ . Let  $\hat{K}_D = \ker(r_1) \cap \hat{M}_D$  and denote by  $\tilde{K}_D$  its preimage in  $\tilde{M}_D$ . Then  $\tilde{K}_D$  is a normal subgroup of  $\tilde{M}_D$ . If  $\tilde{K}_D$  is not central, let  $\tilde{S}$  be an almost simple factor of  $\tilde{M}_D$  contained in  $\tilde{K}_D$ . Let  $\tilde{r}_1 = r_1 \cdot \pi$ . Then  $\tilde{r}_1(\tilde{s}) = r_1(\pi(\tilde{s}))$  will act trivially for every  $\tilde{s} \in \tilde{S}$ . But by the tables in [29] and [39] this cannot happen. Thus  $\tilde{K}_D$  must be in the center of  $\tilde{M}_D$  and therefore  $\hat{K}_D$  must be central as well. Since  $\hat{M}$  acts on  $\hat{\mathfrak{n}}$  faithfully,  $\hat{G}$  being of adjoint type,  $r|_{\hat{A}}$  must also be faithful. But now if  $r_1|_{\hat{A}}$  is not faithful, then every other  $r_i$  will act trivially for the elements in the kernel of  $r_1|_{\hat{A}}$ . This completes the proof.

LEMMA 4.9. *Let  $G$  be a complex algebraic group whose connected component  $G^0$  is reductive and let  $\varphi: G \rightarrow G_1$  be a  $\mathbb{C}$ -rational surjection. Let  $A^0$  be the connected center of  $G^0$ . Then  $G_1^0$  is reductive and  $\varphi(A^0)$  is the connected center of  $G_1^0$ .*

*Proof.* Let  $G^0 = A^0 \cdot G_D^0$ , where  $G_D^0$  is the derived group of  $G^0$ . Let  $\widetilde{G}_D^0$  be the simply connected covering of  $G_D^0$ . Then  $\varphi(G_D^0)$  is isomorphic to a quotient of  $\widetilde{G}_D^0$  and is therefore semi-simple and thus of finite center. Consequently  $\varphi(A^0)$  which is connected must be the connected center of  $G_1^0 = \varphi(A^0)\varphi(G_D^0)$ , making  $G_1^0$  a reductive group.

*Proof of Proposition 4.2.* We shall first assume that  $\mathbf{G}$  is not of type  ${}^2A_{2k}$  (cf. the tables in [39]). Fix  $i$ ,  $1 \leq i \leq m$ . Let  $\hat{M}'$  be the connected component of  ${}^L M / \ker(r_i)$ . Then  $\hat{M}'$  is a complex connected algebraic group which is a quotient of  ${}^L M^0$  and thus by Lemma 4.9 is a connected reductive group whose connected center is the image of the connected center of  ${}^L M^0$ . Moreover it acts on the space of  $r_i$  faithfully. But now by Lemma 4.2 of [39], except possibly for one  $i$ ,  $1 \leq i \leq m$ , there exists a complex connected reductive algebraic group  $\hat{G}$  of adjoint type containing a maximal parabolic subgroup  $\hat{P} = \hat{M}\hat{N}$  such that if  $\hat{r} = \bigoplus_j \hat{r}_j$  is the adjoint action of  $\hat{M}$  on the Lie algebra  $\hat{\mathfrak{n}}$  of  $\hat{N}$ , then  $\hat{r}_{1,D} \cdot \alpha = r'_i \cdot \alpha'$ , where  $\hat{r}_{1,D} = \hat{r}_1|_{\hat{M}_D}$  while  $r'_i$  is the restriction of  $r_i$ , which factors through  ${}^L M / \ker(r_i)$ , to  $\hat{M}'$ . Here  $\alpha$  and  $\alpha'$  are as in Lemma 4.3. Moreover, since the connected center of  $\hat{M}'$  is the image of the connected center of  ${}^L M^0$

under the surjection  ${}^L M^0 \rightarrow \hat{M}' \rightarrow \{1\}$ , action of the connected center of  $\hat{M}'$  on irreducible components of the restriction of  $r_i$  to  ${}^L M^0$  is defined by linearly independent roots of the connected center of  ${}^L M^0$  in  ${}^L \mathfrak{n}$ . Since the action is also faithful, the dimension of this connected center will be equal to the number of irreducible components of  $r_i|{}^L M^0$ . This will also be true for the dimension of the center of  $\hat{M}$  (which is connected), showing that it is isomorphic to the connected center of  $\hat{M}'$ . But now by Lemma 4.8,  $\hat{r}_1$  is faithful and Lemma 4.3 applies, implying that  $\hat{M} \cong \hat{M}'$ , while  $r_i^0 = \hat{r}_1 \otimes \rho$ , where  $\rho$  is a system of complex analytic characters of  $\hat{M}$ . Here  $r_i^0 = r_i|{}^L M^0$  is being considered as one of  $\hat{M}$ .

The Weil group  $W_F$  acts on  ${}^L M$  and leaves  $\ker(r_i)$  invariant. It therefore acts on  $\hat{M} = ({}^L M/\ker(r_i))^0$  and moreover,  $\hat{M} \rtimes W_F \cong {}^L M/\ker(r_i)$ , where it is understood that on the left-hand side  $W_F \cap \ker(r_i)$  acts trivially on  $\hat{M}$ . It is implicit in Lemma 4.2 of [39] that the action of  $W_F$  on  $\hat{M}$  can be extended to one on  $\hat{G}$  in such a way that the action of  $W_F \subset \hat{M} \rtimes W_F$  on the space of  $\hat{r}_1$  is isomorphic to that of  $r_i$ .

Now let  $\mathbf{G}_i$  be an unramified quasi-split connected reductive algebraic  $F$ -group, containing a maximal parabolic  $F$ -subgroup  $\mathbf{P}_i = \mathbf{M}_i \mathbf{N}_i$  for which  ${}^L \mathbf{G}_i = \hat{G} \rtimes W_F$  and  ${}^L \mathbf{M}_i = \hat{M} \rtimes W_F$ . It is easily checked (by the tables in [39]) that  $\hat{r}_1$  is irreducible unless  $\mathbf{M}_i = \text{Res}_{E/F} \tilde{\mathbf{M}}_i$ ,  $[E : F] = n$ , in which case it will have  $n$  irreducible components. At any rate, let  $\hat{\alpha}$  be the  $\Gamma$ -map identifying  $(\mathbf{C}^*)^n$  with  $\hat{Z}$ , the center of  ${}^L M^0 = \hat{M}$ . Then, since by Lemma 4.8,  $\hat{r}_1$  is faithful, we have  $\hat{r}_1 \cdot \hat{\alpha}((x_i)) = (x_i)$ ,  $(x_i) \in (\mathbf{C}^*)^n$ . Now Lemma 4.4 applies, implying

$$L(s, \pi, r_i) = L(s, \pi_1 \otimes \eta, r'_i).$$

Here  $r'_i$  is the action of  ${}^L M_i$  on the space of  $\hat{r}_1$ ,  $\eta$  is the character of  $M_i = \mathbf{M}_i(F)$  attached to  $\rho$  and  $\hat{\alpha}$  by Lemma 4.4, and finally  $\pi_1$  is obtained by restricting  $\pi$  to  $M_i$ . Observe that  ${}^L M_i = {}^L M/\ker(r_i)$  allows us to consider  $\mathbf{M}_i$  as a normal subgroup of  $\mathbf{M}$  and therefore similarly for  $M_i$  and  $M$ . Now, let  $\pi' = \pi_1 \otimes \eta$ , to complete the proposition unless  $\mathbf{G}$  is of type  ${}^2A_{2k}$ .

*Proof of Proposition 4.1.* Proof of Proposition 4.2 applies, showing the existence of  $\mathbf{G}_i$  and  $\mathbf{P}_i = \mathbf{M}_i \mathbf{N}_i$  for which  ${}^L \mathbf{G}_i = \hat{G} \rtimes W_F$  and  ${}^L \mathbf{M}_i = \hat{M} \rtimes W_F$ , where  $\hat{M} = ({}^L M/\ker(r_i))^0$ . Moreover, the action of  $W_F \subset \hat{M} \rtimes W_F$  on the space of  $\hat{r}_1$  is isomorphic to that of  $r_i$ . The proof also goes through to imply the existence of a  $\Gamma$ -coroot  $\hat{\alpha}: (\mathbf{C}^*)^n \rightarrow \hat{Z}$ , the center of  $\hat{M}$ , satisfying  $\hat{r}_1 \cdot \hat{\alpha}((x_i)) = (x_i)$ ,  $(x_i) \in (\mathbf{C}^*)^n$ .

Now let  $v \notin S$  be an unramified place of  $F$ . Choose  $\varphi_{0,v} \in H^1(W_{F_v}, {}^L T^0)$  attached to  $\pi_v$ . Then  $\theta_v = \hat{\alpha} \cdot \rho \cdot \varphi_{0,v} \in H^1(W_{F_v}, \hat{Z})$ . Let  $\eta_v$  be the corresponding character of  $\mathbf{M}_i(F_v)$ . Denote by  $\pi_1$  the restriction of  $\pi$  to  $M_i = \mathbf{M}_i(\mathbb{A}_F)$ . It is cuspidal if  $\pi$  is. Choose  $\beta \in X_*(\mathbf{T})^\Gamma$ . Denote  $\beta$  by  $\hat{\beta}$  as an element in

$X^*({}^L T^0)$ . Then by Lemma 4.7,

$$(4.1.1) \quad \eta_v(\beta(t)) = \omega_v(\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}),$$

whenever  $t \in \bar{F}^*$  is such that  $\beta(t) \in \mathbf{M}(F)$ . Here  $\omega = \otimes_v \omega_v$  is the central character of  $\pi$ . We use (4.1.1) to define  $\eta_v$  for every  $v \in S$ . Let  $\eta = \otimes_v \eta_v$ . Assume  $t \in \bar{F}^*$  is such that  $\beta(t)$  is in  $\mathbf{M}(F)$ . Then

$$\eta(\beta(t)) = \prod_v \omega_v(\rho(t)^{\langle \hat{\beta}, \hat{\alpha} \rangle}) = 1.$$

Thus  $\eta$  is trivial on  $\mathbf{T}(F)$ .

By construction (cf. [28]), each  $\eta_v$  is trivial on  $\mathbf{U}^0(F_v)$  as well as on  $\mathbf{U}^{0,-}(F_v)$ , where  $\mathbf{U}^0$  is the unipotent radical of  $\mathbf{B} \cap \mathbf{M}$  and  $\mathbf{U}^{0,-}$  is its opposite. Now Bruhat decomposition for  $\mathbf{M}(F)$  implies that  $\eta$  is in fact trivial on  $\mathbf{M}(F)$  and therefore is a character of  $\mathbf{M}(F) \setminus \mathbf{M}(\mathbb{A}_F)$ . Now again simply set  $\pi' = \pi_1 \otimes \eta$  and use Proposition 4.2.

*Proof of both propositions when  $\mathbf{G}$  is of type  ${}^2A_{2k}$ .* We shall treat the case  ${}^2A_{2k} - 3$ , since two other cases are similar.

Here  $m = 2$  and the restriction of  $r_2$  to  ${}^L M^0$  is irreducible. The argument at the beginning of the proof of Proposition 2 applies and consequently for  $i = 2$ ,  $(\mathbf{G}_i, \mathbf{M}_i)$  may be chosen as in case  ${}^2A_{2k-1} - 2$ , with  $\mathbf{G}_{2,1} = \mathbf{SL}_{2k}$  simply connected, where  $\mathbf{G}_{2,1}$  is the Chevalley group isomorphic to  $\mathbf{G}_2$  over  $L$ . Here  $L/F$  is the quadratic extension defining  $\mathbf{G}$  as well as  $\mathbf{G}_2$ . The Levi factor  $\mathbf{M}_2$  is defined by:

$$\mathbf{M}_2 = \{(g_1, g_2) \in \mathbf{GL}_k \times \mathbf{GL}_k \mid \det(g_1 g_2) = 1\}$$

together with the twisting

$$\tau_\sigma((g_1, g_2)) = (\tau'_\sigma(g_2), \tau'_\sigma(g_1))$$

where  $\tau'_\sigma$  is the twisting of  $\mathbf{GL}_k$  which defines  $U_k$ , the quasi-split unitary group in  $k$  variables and  $\sigma$  is the non-trivial element of  $\text{Gal}(L/F)$ . If  $F$  is local, the  $F$ -points  $M_2$  of  $\mathbf{M}_2$  are

$$\{g \in \mathbf{GL}_k(L) \mid \det g \in F^*\},$$

while for global  $F$ ,

$$\mathbf{M}_2(\mathbb{A}_F) = \{g \in \mathbf{GL}_k(\mathbb{A}_L) \mid \det g \in \mathbb{A}_F^*\}.$$

Let  $\pi_1$  be an irreducible component of the restriction of  $\pi$  to  $M_2$ . Denote by  $\nu$  the character of  $F^*$  ( $\mathbb{A}_F^*$  if  $F$  is global) attached by class field theory to  $L/F$ . Use  $\nu$  also to denote the character  $\nu \cdot \det$  of  $M_2$ . Finally let  $\eta$  be the character

of  $M_2$  as in Lemma 4.4. Now set  $\pi' = \pi_1 \otimes \eta\nu$ . Then

$$L(s, \pi, r_2) = L(s, \pi', r'_1),$$

which completes the proof of the proposition.

*Remark.* Unless  $(\mathbf{G}, \mathbf{M})$  is obtained by restriction of scalars from the case  $F_4 - 1$ , it must be remarked that in both propositions the exceptional  $i$  for which the statement of the corresponding proposition may not be true can be chosen to be  $i = 1$ .

We finally need:

LEMMA 4.10. *Assume  $\pi = \otimes_v \pi_v$  is a globally  $\chi$ -generic cusp form. Then  $\pi' = \otimes_v \pi'_v$  can also be chosen to be a globally  $\chi$ -generic cusp form. Moreover, if  $\mathbf{G}$  as a group over  $F_v$  and  $\pi_v$  (with respect to  $\mathbf{M}(O_v)$ ) are both unramified, then so is  $\pi'_v$ .*

Proof of the lemma is a consequence of the following discussion in which  $F$  is either local or global.

We start by explaining the restriction of  $\pi$  to  $M_i$ . Let  $\tilde{M}_D$  be the simply connected covering of the derived group of  $\mathbf{M}$ . Lift  $\pi$  to an irreducible  $\chi$ -generic representation  $\tilde{\pi}$  of  $A \times \tilde{M}_D$ . Let  $\tilde{M}_i \subset \tilde{M}_D \times \mathbf{A}$  be the full preimage of  $\mathbf{M}_i$ . Since  $\mathbf{M}_i$  is normal,  $\mathbf{A} \times \tilde{M}_D$  becomes a product of  $\tilde{M}_i$  and another connected reductive group. Consequently  $\tilde{\pi}$  determines a unique irreducible  $\chi$ -generic representation  $\tilde{\pi}_1$  of  $\tilde{M}_i$ . Let  $\theta_i: \tilde{M}_i \rightarrow M_i$  be the natural map. Then  $\theta_i(\tilde{M}_i) \setminus M_i$  is at most compact (finite if  $F$  is local). Let  $\tilde{\pi}_1$  also denote the representation of  $\theta_i(\tilde{M}_i)$  obtained from  $\tilde{\pi}_1$ .

Let  $\pi_1$  be an irreducible constituent of  $\pi|M_i$ . Then  $\pi_1|_{\theta_i(\tilde{M}_i)}$  is equivalent to a multiple (possibly infinite) of  $\tilde{\pi}_1$ . Thus  $\pi_1$  must appear in  $\text{Ind}_{\theta_i(\tilde{M}_i) \uparrow M_i} \tilde{\pi}_1$ . Consequently, distinct irreducible constituents of  $\pi|M_i$  are among those of  $\text{Ind}_{\theta_i(\tilde{M}_i) \uparrow M_i} \tilde{\pi}_1$ . In particular, distinct components of  $\pi|M_i$  are finite in number, if  $F$  is local, and appear discretely otherwise.

Assume  $F$  is local and  $\pi$  is  $\chi$ -generic. Then so is  $\tilde{\pi}_1$ ; i.e.,  $\tilde{\pi}_1 \subset \text{Ind}_{U_i \uparrow \theta_i(\tilde{M}_i)} \chi$ , where  $U_i$  is the maximal unipotent radical of  $M_i$  which is the same as that of  $\theta_i(\tilde{M}_i)$ . Consequently  $\text{Ind}_{\theta_i(\tilde{M}_i) \uparrow M_i} \tilde{\pi}_1 \subset \text{Ind}_{U_i \uparrow M_i} \chi$  and therefore each component of  $\text{Ind}_{\theta_i(\tilde{M}_i) \uparrow M_i} \tilde{\pi}_1$  is  $\chi$ -generic and appears with multiplicity one.

Next assume  $F$  is global. If  $\varphi$  is a cusp form in the space of  $\pi$ , then  $\varphi|M_i$  becomes one in the space of  $\pi|M_i$ . It is again a cusp form. Moreover if  $\pi$  is globally  $\chi$ -generic, then there must exist a constituent in this restriction which is globally  $\chi$ -generic; otherwise the  $\chi$ -Fourier coefficient of  $\pi$  would vanish as well. This is a consequence of the fact that the maximal unipotent radical of  $\mathbf{M}_i$

can be identified with a normal subgroup of that of  $\mathbf{M}$  in such a way that the simple roots in the first one are among those of the second one.

If  $F$  is non-archimedean and  $\pi$  is of class-one with respect to some special maximal compact subgroup, the homomorphism  $\varphi': W'_F \rightarrow {}^L M$  (cf. Section 3), attached to  $\pi$ , when composed with the natural projection  $\rho_i$  of  ${}^L M$  onto  ${}^L M_i$ , defines one of  $W'_F$  into  ${}^L M_i$ . All the distinct components of  $\pi|M_i$  are attached to the parameter  $\rho_i \cdot \varphi'$  and therefore they will all appear in the same principal series which contains only one  $\chi$ -generic component. In particular  $\pi|M_i$  must have only one distinct component. Thus  $\pi_1$ , as well as  $\pi'$ , are both  $\chi$ -generic. Local coefficients for  $\pi'$ ,  $M_i$ , and  $G_i$  are then the Artin factors defined by  $\rho_i \cdot \varphi'$  (cf. Proposition 3.4 here). If  $\mathbf{G}$  (and thus  $\nu$  in case  ${}^2A_2$ ) and  $\pi$  (with respect to  $\mathbf{M}(O)$ ) are both unramified, then so is  $\pi_1$ . By Lemma 4.7, the character  $\eta$  is unramified and therefore so is  $\pi'$ .

Finally assume  $F$  is archimedean. Then all the distinct components of  $\pi|M_i$  will belong to the  $L$ -packet defined by  $\rho_i \cdot \varphi$ , where  $\varphi: W_F \rightarrow {}^L M$  is the homomorphism attached to  $\pi$ . It follows from a result of Vogan (cf. page 95 of the proof of Theorem 6.2 of [48]) that in every  $L$ -packet of  $M_i$ , there is at most one  $\chi$ -generic representation. Thus again  $\pi|M_i$  must have only one distinct component, and consequently each  $\pi$  determines a unique  $\pi_1$ , as well as  $\pi'$ .

The following proposition must now be clear. It can be used as a reference to replace Proposition 4.2.

PROPOSITION 4.11. *Assume  $F$  is a local field and  $W'_F$  is the Deligne-Weil group of  $\bar{F}/F$ . Suppose there exists a homomorphism  $\varphi: W'_F \rightarrow {}^L M$  which is attached to  $\pi$ . Choose  $\mathbf{G}_i$  and  $\mathbf{M}_i\mathbf{N}_i$  as before. Let  $\varphi_i$  denote the composition of  $\varphi$  with the natural projection  ${}^L M \rightarrow {}^L M_i$ . Finally if  $\theta$  is the 1-cocycle of  $W_F$  into the center of  ${}^L M^0$  as in Lemma 4.4, let  $\theta_i$  be again the corresponding composition. Then*

$$L(s, r_i \cdot \varphi) = L(s, r'_i \cdot \varphi_i \theta_i)$$

and

$$\varepsilon(s, r_i \cdot \varphi, \psi_F) = \varepsilon(s, r'_i \cdot \varphi_i \theta_i, \psi_F),$$

$\psi_F \in \hat{F}$ ,  $\psi_F \neq 1$ , where the factors are those of Artin.

Remark 4.12. By Propositions 4.1 and 4.2, it is easy to see that the technical assumption that the restriction of  $\pi$  to the center of  $M$  is trivial whenever  $m \geq 2$  in all the results of [39] can now be removed. Thus Theorems 5.1 and 5.2, Corollary 5.3, Lemma 5.8, and Proposition 5.9 of [39] are now valid with no assumption on  $\pi$ .

### 5. On a result of Henniart and Vignéras

Let  $F$  be a non-archimedean local field and let  $\mathbf{G}$  be a quasi-split connected reductive algebraic group over  $F$ . Fix a Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$ . Write  $\mathbf{B} = \mathbf{T}\mathbf{U}$ , where  $\mathbf{U}$  is the unipotent radical of  $\mathbf{B}$ . Let  $\mathbf{A}_0$  be a maximal split torus in  $\mathbf{T}$ . Fix a non-degenerate character  $\chi$  of  $U = \mathbf{U}(F)$ . In this section, we shall modify the proof of Theorem 2.2 of [47] to obtain the following version of a result of Henniart [14] and Vignéras [47].

**PROPOSITION 5.1.** *Let  $\sigma$  be an irreducible  $\chi$ -generic supercuspidal representation of  $G = \mathbf{G}(F)$ . Then there exist a number field  $K$  with ring of integers  $O$ , a quasi-split connected reductive algebraic group  $\mathbf{H}$  over  $K$ , a non-degenerate character  $\tilde{\chi} = \otimes_v \tilde{\chi}_v$  of  $\mathbf{U}_H(K) \setminus \mathbf{U}_H(\mathbb{A}_K)$ , and a globally  $\tilde{\chi}$ -generic quasi-cusp form  $\pi = \otimes_v \pi_v$  on  $H = \mathbf{H}(\mathbb{A}_K)$  such that:*

- a)  $K_{v_0} = F$  for some place  $v_0$  of  $K$ ,
- b)  $\tilde{\chi}_{v_0} = \chi$ ,
- c)  $\mathbf{H} \times_K F = \mathbf{G}$ ; i.e.  $\mathbf{H}$ , as a group over  $F$ , is equal to  $\mathbf{G}$ ,
- d)  $\pi_{v_0} = \sigma$ , and finally
- e) for every other finite place  $v$  of  $K$ ,  $v \neq v_0$ ,  $\pi_v$  is of class-one with respect to a special maximal compact subgroup  $Q_v$  of  $\mathbf{H}(K_v)$ . Here  $\mathbf{U}_H$  is the unipotent radical of a Borel subgroup of  $\mathbf{H}$  for which  $\mathbf{U}_H \times_K F = \mathbf{U}$ .

*Proof.* We may assume  $\sigma$  is unitary and  $\pi$  is a cusp form. Let  $K$  be a number field that has  $F$  as its  $v_0$ -completion. Clearly we can choose a quasi-split group  $\mathbf{H}$  over  $K$  such that  $\mathbf{H} \times_K F = \mathbf{G}$ . Let  $\omega = \otimes_v \omega_v$  be a character of the center  $Z = \mathbf{Z}(\mathbb{A}_K)$  of  $H$ , trivial on  $\mathbf{Z}(K)$ , such that  $\omega_{v_0}$  is equal to the central character of  $\sigma$  and moreover,  $\omega_v | \mathbf{Z}(K_v) \cap \mathbf{T}_H(K_v)^0 = 1$ , where  $\mathbf{T}_H$  is a maximal torus of  $\mathbf{H}$  such that  $\mathbf{T} = \mathbf{T}_H \otimes_K F$  and  $\mathbf{T}_H(K_v)^0$  is the largest compact subgroup of  $\mathbf{T}_H(K_v)$ .

Next let  $\chi' = \otimes_v \chi'_v$  be an unramified character of  $\mathbf{U}_H(K) \setminus \mathbf{U}_H(\mathbb{A}_K)$ . Fix a maximal split torus  $\mathbf{A}_{H,0}$  of  $\mathbf{H}$  such that  $\mathbf{A}_{H,0} \subset \mathbf{T}_H$  and  $\mathbf{A}_{H,0} \times_K F = \mathbf{A}_0$ . Choosing  $a \in \mathbf{A}_{H,0}(\bar{K})$ , with  $\text{Ad}(a)$  defined over  $\bar{K}$ , we can obtain a character  $\chi'' = \chi' \cdot \text{Ad}(a)$  such that for every finite place  $v$  of  $K$  for which  $\mathbf{G}$  is ramified,

$$(5.1.1) \quad \chi''|_{\mathbf{U}_H(K_v) \cap Q'_v} = 1,$$

where  $Q'_v$  is a special maximal compact subgroup of  $\mathbf{H}(K_v)$ , adapted to  $\mathbf{A}_{H,0}(K_v)$  (cf. [44]; in particular  $\mathbf{T}_H(K_v) \cap Q'_v = \mathbf{T}_H(K_v)^0$ ). The element  $a$  can be chosen so that (5.1.1) holds for every finite place of  $K$ . Now, changing  $\chi'$  if necessary, we can find a  $c \in \mathbf{A}_{H,0}(\bar{K})$ , with  $\text{Ad}(c)$  defined over  $K$ , such that  $\chi = \chi''_{v_0} \cdot \text{Ad}(c)$ . Set  $\tilde{\chi} = \chi'' \cdot \text{Ad}(c) = \otimes_v \tilde{\chi}_v$ . At each place  $v$  of  $K$ , let  $Q_v = \text{Ad}(c^{-1})Q'_v$ . Then unless either  $\mathbf{H} \times_K K_v$  or  $\tilde{\chi}_v$  is ramified, or  $c \notin \mathbf{A}_{H,0}(O_{\bar{K}_v})$ ,

$Q_v = \mathbf{H}(O_{K_v})$ . Since  $\text{Ad}(c^{-1})$  is defined over  $K$ ,  $Q_v \subset \mathbf{H}(K_v)$  and is again a special maximal compact subgroup adapted to  $\mathbf{A}_{\mathbf{H},0}(K_v)$ . Moreover, if  $f_v$  and  $f'_v$  are functions supported on  $Z_v Q_v$  and  $Z_v Q'_v$ , respectively, satisfying  $f_v|_{Q_v} = f'_v|_{Q'_v} = 1$ , then

$$(5.1.2) \quad \int_{\mathbf{U}_{\mathbf{H}}(K_v)} f_v(u) \tilde{\chi}_v(u) du = \alpha(a) \int_{\mathbf{U}_{\mathbf{H}}(K_v)} f'_v(u) \chi''_v(u) du \neq 0$$

by (5.1.1), where  $\alpha(a) = d(a u a^{-1})/du$ .

As in [47], given a pair of smooth functions  $f^0 = \otimes_v f_v^0$  and  $f^1 = \otimes_v f_v^1$  on  $H$ , of compact support modulo  $Z$ , which transform on  $Z$  according to  $\omega$ , we shall consider

$$I = \int_{Z \backslash H} f^1(g^{-1}) W_{\tilde{\chi}}(P f^0)(g) dg,$$

where  $P f^0$  is the Poincaré series attached to  $f^0$  (cf. [47]) and

$$W_{\tilde{\chi}}(P f^0)(g) = \int_{\mathbf{U}_{\mathbf{H}}(K) \backslash \mathbf{U}_{\mathbf{H}}(\mathbb{A}_K)} \tilde{\chi}^{-1}(u) P f^0(ug) du.$$

Let  $C$  be a compact subset of  $H$  such that

$$\mathbf{U}_{\mathbf{H}}(\mathbb{A}_K) \subset \mathbf{U}_{\mathbf{H}}(K) C^{-1}.$$

Assume that  $f^0$  and  $f^1$  are chosen so that

$$(5.1.3) \quad \mathbf{H}(K) \cap \text{Supp}(f^1 * f^0) C \subset Z(K),$$

where

$$(f^1 * f^0)(g) = \int_{Z \backslash H} f^1(h^{-1}) f^0(gh) dh.$$

Then as explained in [47],

$$I = \prod_v I_v,$$

where

$$(5.1.4) \quad I_v = \int_{\mathbf{U}_{\mathbf{H}}(K_v)} (f_v^1 * f_v^0)(u) \tilde{\chi}_v(u) du.$$

We shall now show that it is possible to choose  $f^0$  and  $f^1$  in such a way that:

- 1) (5.1.3) is valid;
- 2) For each  $v$ ,  $I_v$ , given by (5.1.4), is non-zero; thus the automorphic form given by the Poincaré series  $P$  is globally  $\tilde{\chi}$ -generic;
- 3) For every  $v < \infty$ ,  $v \neq v_0$ ,  $\pi_v$  is of class-one with respect to  $Q_v$ ; and
- 4) At  $v = v_0$ ,  $\pi_{v_0} = \sigma$ ; in particular  $\pi$  is cuspidal.

Fix an infinite place  $v_1$  of  $K$ . First if  $v < \infty$ ,  $v \neq v_0$ , let  $f_v^0 = f_v^1$  be supported on  $Z_v Q_v$  and equal to 1 on  $Q_v$ . Then by (5.1.2)  $I_v \neq 0$  and moreover  $\pi_v$  is of class-one with respect to  $Q_v$ . Next for  $v = v_0$ , let  $f_{v_0}^0 = f_{v_0}^1$  be a matrix coefficient of  $\sigma$  modulo the center of  $G$ . As argued in [47], again  $I_{v_0} \neq 0$  and  $\pi_{v_0} = \sigma$ . Finally if  $v = \infty$ ,  $v \neq v_1$ , choose  $f_v^0$  and  $f_v^1$  so that  $I_v \neq 0$ .

We may assume  $C$  is of the form  $C = C^1 \times C_1$ , where

$$C^1 \subset H^{v_1} = \{g = (g_v) \in H \mid g_{v_1} = 1\}$$

and  $C_1 \subset H_{v_1}$  are both compact. Moreover, we shall assume  $C_1^{-1}$  is the closure of a precompact neighborhood of identity in  $H_{v_1}$ .

Let

$$D^1 = \prod_{v \neq v_1} \text{Supp}(f_v^1 * f_v^0) C^1(\text{mod } Z).$$

Then  $D^1$  is a compact subset of  $H^{v_1}$ .

Lemma 2.7 of [47] goes through word by word to imply the existence of a precompact open neighborhood  $K_1$  of the identity in  $H_{v_1}$  such that

$$\mathbf{H}(K) \cap \mathbf{Z}(\mathbb{A}_K) D^1 K_1 \subset \mathbf{Z}(K).$$

We shall now assume that

$$(5.1.5) \quad \text{Supp}(f_{v_1}^1 * f_{v_1}^0)(\text{mod } Z_{v_1}) \subset K_1 C_1^{-1}.$$

This then implies the validity of (5.1.3) for  $f^i = \otimes f_v^i$ ,  $i = 0, 1$ , and consequently  $I = \prod_v I_v$ . The following lemma allows us to choose  $f_{v_1}^0$  and  $f_{v_1}^1$ , satisfying (5.1.5), in such a way that  $I_{v_1} \neq 0$ , completing the proposition.

LEMMA 5.2. *Fix a character  $\omega_{v_1}$  of the center  $Z_{v_1}$  of  $H_{v_1}$ . Let  $\varphi$  be a  $C^\infty$ -function of compact support modulo  $Z_{v_1}$  for which  $\varphi(zg) = \omega_{v_1}(z)\varphi(g)$ ,  $z \in Z_{v_1}$ , and  $g \in H_{v_1}$ . Then there exist two functions  $f_{v_1}^0$  and  $f_{v_1}^1$  in  $C^\infty(H_{v_1})$ , both of compact support modulo  $Z_{v_1}$ , and satisfying  $f_{v_1}^i(zg) = \omega_{v_1}(z)f_{v_1}^i(g)$ ,  $i = 0, 1$ , such that*

$$\varphi = f_{v_1}^1 * f_{v_1}^0.$$

*More precisely*

$$\varphi(g) = \int_{Z_{v_1} \backslash H_{v_1}} f_{v_1}^1(h^{-1}) f_{v_1}^0(gh) dh.$$

*Proof.* Write  $H_{v_1} = Z_{v_1} H'_{v_1}$ , where  $H'_{v_1}$  is a semisimple (not necessarily connected) real Lie group with finite center equal to  $Z_{v_1} \cap H'_{v_1}$ . Let  $\varphi' = \varphi|_{H'_{v_1}}$ . Then  $\varphi'(zg) = \omega_{v_1}(z)\varphi'(g)$  for  $z \in Z_{v_1} \cap H'_{v_1}$  and  $g \in H'_{v_1}$ .

Choose  $\varphi'_1 \in C_c^\infty(H'_{v_1})$  such that

$$\varphi'(g) = \sum_{Z_{v_1} \cap H'_{v_1}} \varphi'_1(zg)\omega_{v_1}(z^{-1}).$$

By the main result of [9], choose  $\varphi''_0$  and  $\varphi''_1$  in  $C_c^\infty(H'_{v_1})$  such that  $\varphi'_1 = \varphi''_1 * \varphi''_0$ . Let

$$\tilde{\varphi}_i(g) = \sum_{Z_{v_1} \cap H'_{v_1}} \varphi''_i(zg)\omega_{v_1}(z^{-1}),$$

where  $i = 0, 1$ . Then it is easily checked that

$$\begin{aligned} \varphi'(g) &= (\tilde{\varphi}_1 * \tilde{\varphi}_0)(g) \\ &= \int_{Z_{v_1} \cap H'_{v_1} \backslash H'_{v_1}} \tilde{\varphi}_1(h^{-1})\tilde{\varphi}_0(gh) dh, \quad (g \in H'_{v_1}). \end{aligned}$$

Now define

$$f_{v_1}^i(zg) = \omega_{v_1}(z)\tilde{\varphi}_i(g) \quad (z \in Z_{v_1}, g \in H'_{v_1}),$$

$i = 0, 1$ . Then for  $z \in Z_{v_1}$  and  $g \in H'_{v_1}$

$$\begin{aligned} \varphi(zg) &= \omega_{v_1}(z)\varphi'(g) \\ &= \int_{Z_{v_1} \backslash H'_{v_1}} f_{v_1}^1(h^{-1})f_{v_1}^0(zgh) dh, \end{aligned}$$

completing the proof of the lemma.

### 6. Proof of Theorem 3.5

We shall first assume  $\mathbf{P} = \mathbf{MN}$  is maximal. While proving parts 2 and 4 of the theorem we shall prove the existence of  $\gamma_i$ 's. Their uniqueness will be proved later.

*Existence of  $\gamma_i$ 's and proofs of (3.11) and part 4.* Let  $\mathbf{G}'$  be a connected reductive quasi-split algebraic group over a local field  $F$ . Assume  $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$  is an  $F$ -parabolic subgroup of  $\mathbf{G}'$ . Suppose that  $\mathbf{P}'$  is either maximal or more generally, there exists a quasi-split connected reductive group  $\mathbf{H}'$  over a number field  $K$  with  $K_{v_0} = F$  for some place  $v_0$  of  $K$ , and a maximal parabolic subgroup  $\mathbf{P}_{\mathbf{H}'}$  of  $\mathbf{H}'$  such that  $\mathbf{P}' = \mathbf{P}_{\mathbf{H}'} \times_K F$ . Let  $\rho'$  be half the sum of positive roots in  $\mathbf{N}'$ . Then  $\langle \rho', \alpha' \rangle$  is the same for every simple root  $\alpha'$  in  $\mathbf{N}'$ . Let  $\tilde{\alpha}' = \langle \rho', \alpha' \rangle^{-1} \rho'$  for some simple root  $\alpha'$  in  $\mathbf{N}'$ . Finally let  $r'$  be the adjoint action of  ${}^L M'$  on  ${}^L \mathfrak{n}'$ . Write  $r' = \sum_{j=1}^{m'} r'_j$  as before.

We shall now define  $\gamma_i$ 's inductively. This will be done by defining  $\gamma_i$ 's so that (3.11) holds. Our induction statement is: For every  $\mathbf{G}'$  and  $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$  of

length  $m'$  as above, and an irreducible admissible  $\chi'$ -generic representation  $\sigma'$  of  $M'$ , there exist  $m'$  functions  $\gamma_j(s, \sigma', \psi_F)$ , each satisfying conditions 1 and 4 of Theorem 3.5, such that

$$C_{\chi'}(s\tilde{\alpha}', \sigma', w'_0) = \lambda_{\mathbf{G}'}(\psi_F, w'_0)^{-1} \omega'_{s'_1}(\tilde{w}'_0(a)a^{-1})^{-1} \prod_{j=1}^{m'} \gamma_j(js, \tilde{\sigma}', \bar{\psi}_F),$$

where  $\tilde{w}'_0$  and  $\chi'$  are compatible.

Assume this is true for every  $\mathbf{G}'$ ,  $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ , and  $\sigma'$ , where the length  $m' < m$ . Then it follows from the results of Section 4 that for all except possibly one  $i$ , say  $i_0$ , there exists a group  $\mathbf{G}_i$  over  $F$  and an  $F$ -parabolic subgroup  $\mathbf{P}_i = \mathbf{M}_i\mathbf{N}_i$  of  $\mathbf{G}_i$  such that  $r_i^0 = \hat{r}_1 \otimes \rho$  (notation as in Propositions 4.1 and 4.2). We shall choose  $i_0$  arbitrarily if for every  $i$ ,  $1 \leq i \leq m$ , this is always the case. Let  $\sigma' = \sigma_1 \otimes \eta$  ( $\sigma' = \sigma_1 \otimes \eta\nu$  if  $\mathbf{G}$  is of type  ${}^2A_{2k}$ ), where  $\sigma_1$  is a restriction of  $\sigma$  to  $M_i$  and  $\eta$  is as in Lemma 4.4. For  $i \neq i_0$ , define

$$\gamma_i(s, \sigma, \psi_F) = \gamma_1(s, \sigma', \psi_F),$$

where the factor on the right is defined by  $\mathbf{G}' = \mathbf{G}_i$ . Let

$$\begin{aligned} \gamma_{i_0}(s, \sigma, \psi_F) &= C_{\tilde{\chi}}(s\tilde{\alpha}, \tilde{\sigma}, w_0) \lambda_{\mathbf{G}}(\psi_F, w_0) \omega_{s_1}(\tilde{w}_0(a)a^{-1}) \\ &\quad \cdot \prod_{i \neq i_0} \gamma_i(is, \sigma, \psi_F)^{-1}. \end{aligned}$$

This proves (3.11). We finally observe that if  $\sigma$  is as in part 1 of the theorem, then (3.11) is just Lemma 3.3 or Proposition 3.4, according to whether  $F$  is archimedean or not.

Now, if  $\pi$  is a cusp form on  $\mathbf{M}(\mathbb{A}_K)$ , where  $\mathbf{M}$  is a Levi factor in a quasi-split connected reductive group  $\mathbf{G}$  over a number field  $K$ , then by relation (3.6) of [39],

$$(6.1) \quad \prod_{i=1}^m L_S(is, \pi, r_i) = \prod_{v \in S} C_{\tilde{\chi}_v}(s\tilde{\alpha}, \tilde{\pi}_v, w_0) \prod_{i=1}^m L_S(1 - is, \pi, \tilde{r}_i),$$

where  $\pi = \otimes_v \pi_v$  and  $r_i$ ,  $1 \leq i \leq m$ , is as in part 4 of the theorem. Again by induction we may assume that (3.14) holds for all  $i$ ;  $1 \leq i \leq m$ , except possibly one, say  $i = i_0$ , where for each  $v \in S$ , we take a fixed decomposition (3.11) of  $C_{\tilde{\chi}_v}(s\tilde{\alpha}, \tilde{\pi}, w_0)$ , using the global group  $\mathbf{G}_i$  attached to  $\mathbf{G}$ , to define  $\gamma_i$ 's for  $i \neq i_0$ . The functional equation (3.14) for  $i_0$  now follows from (6.1), completing the induction. The case of a quasi-cusp form is just a shift in  $s$ .

Equation (3.12) is now a simple consequence of our induction applied to

$$C_{\chi}(s\tilde{\alpha}, \sigma, w_0) = C_{\chi}((s + s_0)\tilde{\alpha}, \sigma_0, w_0).$$

Given  $\mathbf{G}$ ,  $\mathbf{M}$ , and  $\sigma$  it is clear that the choices of  $\mathbf{G}_i$ ,  $\mathbf{M}_i$ , and  $\sigma_1$  are not, in general, unique. But the following lemma proves the uniqueness of  $\gamma_i$ 's for supercuspidal representations.

LEMMA 6.1. *Assume  $F$  is non-archimedean and  $\sigma$  is supercuspidal. Then conditions 1 and 4 of Theorem 3.5 determine each  $\gamma_i(s, \sigma, \psi_F)$  uniquely. In particular each  $\gamma_i$  is independent of the possible choices of the group  $\mathbf{G}_i$ , the character  $\chi'$ , and the representation  $\sigma'$  used in its definition.*

*Proof.* By Proposition 5.1 we can choose a number field  $K$ , a quasi-split connected reductive group  $\tilde{\mathbf{M}}$  over  $K$ , a non-degenerate character  $\tilde{\chi} = \otimes_v \tilde{\chi}_v$  of  $\mathbf{U}_{\tilde{\mathbf{M}}}(K) \setminus \mathbf{U}_{\tilde{\mathbf{M}}}(\mathbb{A}_K)$ , and finally a globally  $\tilde{\chi}$ -generic quasi-cusp form  $\pi = \otimes_v \pi_v$  on  $\tilde{\mathbf{M}}$  such that for some place  $v_0$  of  $K$ ,  $K_{v_0} = F$ ,  $\mathbf{M} = \tilde{\mathbf{M}} \times_K F$ ,  $\tilde{\chi}_{v_0} = \chi$ , and  $\pi_{v_0} = \sigma$ . Moreover, if  $v < \infty$ ,  $v \neq v_0$ , then  $\pi_v$  is of class-one with respect to some special maximal compact subgroup  $Q_v$  of  $\tilde{\mathbf{M}}(K_v)$  (equal to  $\tilde{\mathbf{M}}(O_v)$  for almost all  $v$ ). Choose a connected reductive quasi-split group  $\tilde{\mathbf{G}}$  over  $K$  which has  $\tilde{\mathbf{M}}$  as its Levi factor such that  $\tilde{\mathbf{G}} \times_K F = \mathbf{G}$ . We may assume each  $Q_v$ ,  $v < \infty$ ,  $v \neq v_0$ , is the restriction of a special maximal compact subgroup of  $\tilde{\mathbf{G}}(K_v)$ , adapted to  $\mathbf{A}_0(K_v)$ . If  $\tilde{r}$  is the adjoint action of  ${}^L\tilde{\mathbf{M}}$  on  ${}^L\tilde{\mathfrak{n}}$ , write  $\tilde{r} = \bigoplus_{i=1}^m \tilde{r}_i$ . Fix  $i$ ,  $1 \leq i \leq m$ . Then  $r_i = \tilde{r}_i \cdot \eta_{v_0}$ , where  $r$  is the adjoint action of  ${}^L\mathbf{M}$  on  ${}^L\mathfrak{n}$ ,  $r = \bigoplus_{i=1}^m r_i$ . Assume  $\gamma_i(s, \sigma, \psi_F)$  and  $\gamma'_i(s, \sigma, \psi_F)$  are two factors attached to  $\sigma$  and  $r_i$ . Then by part 4 of Theorem 3.5 the functional equation for  $\pi$  and  $\tilde{r}_i$  must be satisfied with both factors appearing as the factor at  $v_0$ . But by part 1 of the theorem the factors defined at all other places are Artin factors attached to  $\pi_v$  and  $\tilde{r}_i \cdot \eta_v$  and are therefore uniquely determined. Comparison of the two functional equations now implies  $\gamma_i(s, \sigma, \psi_F) = \gamma'_i(s, \sigma, \psi_F)$ , which completes the lemma.

Now, assume  $F$  is non-archimedean and  $\sigma$  is an irreducible admissible  $\chi$ -generic representation of  $M$ , where  $\mathbf{P} = \mathbf{M}\mathbf{N}$  is a parabolic subgroup of  $\mathbf{G}$  over  $F$ . Let  $r$  be a subrepresentation of the adjoint action of  ${}^L\mathbf{M}$  on  ${}^L\mathfrak{n}$ . Define

$$\gamma(s, \sigma, r, \psi_F) = \prod_j \gamma^j(s, \sigma, \psi_F),$$

where the product runs over factors attached to the irreducible components of  $r$ . Observe that each such component comes from a maximal parabolic and the factors on the right are all defined. We call  $\gamma(s, \sigma, r, \psi_F)$  a factor attached to  $\sigma$  and  $r$ . Then  $\gamma(s, \sigma, r, \psi_F)$  will satisfy condition 4 of the theorem and it follows from Lemma 6.1 that if  $\sigma$  is supercuspidal the factors attached to  $\sigma$  and  $r$  are unique, which we call the factor attached to  $\sigma$  and  $r$ . We shall now prove the inductive property (property 3) of Theorem 3.5.

*Proof of inductive property 3 when  $\sigma_1$  is supercuspidal.* We first observe that

$$\lambda_{\mathbf{G}}(\psi_F, w_0) = \prod_{j=1}^{n-1} \lambda_{\mathbf{M}_{\Omega_j}}(\psi_F, w_j),$$

and moreover  $m = \text{Max}_{1 \leq j \leq n-1} m_j$ . A similar identity holds for  $\omega_s(\tilde{w}_0(a)a^{-1})$ , where  $\chi = \chi_0 \cdot \text{Ad}_U(\tilde{w}_l(a))$ . If  $m = 1$ , then (3.13) is just Proposition 3.2.1 of [38]. Otherwise for all but possibly one  $i$ , say  $i_0$ , we can choose  $\mathbf{G}_i, \mathbf{M}_i \mathbf{N}_i$ , and a  $\chi'$ -generic representation  $\sigma'$  of  $M_i$  such that if

$$C_{\chi'}(s\tilde{\alpha}_i, \sigma', w^i) = \lambda_{\mathbf{G}_i}(\psi_F, w^i)^{-1} \omega'_{s_i}(\tilde{w}^i(a')a'^{-1})^{-1} \prod_{l=1}^{m'} \gamma_l(ls, \sigma', \psi_F),$$

then  $\gamma_i(s, \sigma, \psi_F) = \gamma_1(s, \sigma', \psi_F)$  and  $m' < m$ . Here  $\tilde{w}^i$  and  $\tilde{\alpha}_i$  have the obvious meaning for  $\mathbf{G}_i$  and  $\mathbf{M}_i$ , and  $\chi'$  and  $\tilde{w}^i$  are compatible, where  $\chi' = \chi_0 \cdot \text{Ad}_U(\tilde{w}_i^i(a'))$ .

The natural projection from  ${}^L M^0$  onto  ${}^L M_i^0$ , when restricted to the  $L$ -group of  $\mathbf{U} \cap \mathbf{M}$ , splits and therefore the simple roots of  ${}^L M_i^0$  may be identified with a subset of those of  ${}^L M^0$ . The action of  $W_F$  on the roots of  ${}^L M_i^0$  also agrees with this identification. In this fashion we may choose  $\mathbf{M}_i$ , as we in fact do, so that its simple roots are a subset of those of  $\mathbf{M}$ .

The group  $\overline{\mathbf{M}} = \mathbf{M}_i \cap \mathbf{M}_\theta$  becomes a Levi factor for  $\mathbf{M}_i$  and thus  $\mathbf{G}_i$ . Observe that  $\sigma'$  was obtained by restricting  $\sigma$  to  $M_i$ . Consequently  $\sigma' \subset \text{Ind}_{\overline{\mathbf{M}\mathbf{N}} \uparrow M_i}(\sigma_1 | \overline{\mathbf{M}})$ . Let  $\bar{\sigma}$  be a component of  $\sigma_1 | \overline{\mathbf{M}} = \overline{\mathbf{M}}(F)$  for which  $\sigma' \subset \text{Ind}_{\overline{\mathbf{M}\mathbf{N}} \uparrow M_i} \bar{\sigma} \otimes 1$ . We also observe that every unipotent radical of  $\overline{\mathbf{M}}$  is in fact one of  $\mathbf{M}_\theta$  and therefore if  $\sigma_1$  is supercuspidal, then so is  $\bar{\sigma}$ .

Now by induction on groups with smaller  $m$ ,

$$(6.2) \quad \begin{aligned} \gamma_i(s, \sigma, \psi_F) &= \gamma_1(s, \sigma', \psi_F) \\ &= \prod_{j=1}^{n_i-1} \gamma_1(s, (\overline{w^i})_j(\bar{\sigma}), \psi_F). \end{aligned}$$

Observe that  ${}^L M_\theta$  is a Levi factor of  ${}^L M$  as well as  ${}^L G$ . It is easy to see that for a given  $i, 1 \leq i \leq m$ , the factors  $\gamma_{i(j)}(s, \overline{w}_j(\sigma), \psi_F, \tilde{w}_j)$  are so defined that their product over  $S_i$  becomes a factor attached to  $\sigma_1$  and  $r_i \cdot \kappa_\theta$ , where  $\kappa_\theta$  is the embedding of  ${}^L M_\theta$  into  ${}^L M$ . Moreover by Propositions 3.4 and 4.11, Lemma 3.3, and the discussion after Lemma 4.10, the right-hand side of (6.2) is also a factor attached to the same data. Since  $\sigma_1$  and  $\bar{\sigma}$  are both supercuspidal, by the uniqueness Lemma 6.1, these two factors, i.e., the right-hand sides of (3.13) and (6.2) are equal. We now apply Proposition 3.2.1 of [38] to  $C_\chi(s\tilde{\alpha}, \sigma, \tilde{w}_0)$  to conclude (3.1) for  $i = i_0$ , completing the induction.

LEMMA 6.2. *Assume  $\mathbf{P}$  is maximal. Then properties 1, 3, and 4 of Theorem 3.5 determine each  $\gamma_i(s, \sigma, \psi_F, \tilde{w}_0)$  uniquely, and therefore they are independent of all the choices made in their definition.*

*Proof.* We may assume  $F$  is non-archimedean. By Jacquet's submodule theorem, given an irreducible admissible representation  $\sigma$  of  $M$ , there exist a

parabolic subgroup  $\mathbf{M}_\theta \mathbf{N}_\theta$  of  $\mathbf{M}$  and an irreducible supercuspidal representation  $\sigma_1$  of  $M_\theta$  such that  $\sigma \subset \text{Ind}_{M_\theta \mathbf{N}_\theta \uparrow M} \sigma_1 \otimes 1$ . Now the lemma is a consequence of Lemma 6.1 and the inductive property 3 for supercuspidal  $\sigma_1$  which was just proved.

*Proof of inductive property 3 for arbitrary  $\sigma_1$ .* By Jacquet's submodule theorem and induction in stages, there exist a parabolic subgroup  $\mathbf{M}'\mathbf{N}'$  of  $\mathbf{M}_\theta$  and an irreducible supercuspidal representation  $\sigma'$  of  $M'$  such that  $\sigma \subset \text{Ind}_{M'\mathbf{N}' \uparrow M} \sigma' \otimes 1$  and  $\sigma_1 \subset \text{Ind}_{M'\mathbf{N}' \uparrow M_\theta} \sigma' \otimes 1$ . Then by property 3 applied to  $\sigma'$ ,  $\gamma_i(s, \sigma, \psi_F, \tilde{w}_0)$  becomes equal to the factor attached to  $\sigma'$  and the representation  $r_i \cdot \kappa'$  of  ${}^L M'$ , where  $\kappa'$  is the embedding of  ${}^L M'$  in  ${}^L M$  ( ${}^L M'$  is a Levi factor of  ${}^L M$  and  ${}^L G$ ). A similar statement is also true about the right-hand side of (3.13). Property 3 is now again a consequence of the uniqueness Lemma 6.1 applied to  $\sigma'$ .

Now assume  $\mathbf{P}$  is arbitrary. Write a reduced decomposition  $\tilde{w} = \tilde{w}_{n-1} \cdots \tilde{w}_1$  for  $\tilde{w}$ , and then define

$$\gamma_i(s, \sigma, \psi_F, \tilde{w}) = \prod_{j=1}^{n-1} \gamma_i(s, \bar{w}_j(\sigma), \psi_F, \tilde{w}_j),$$

where the factors on the right are attached to maximal parabolic subgroups which were defined before. It is independent of the decomposition of  $\tilde{w}$  and is easily checked to satisfy all the required properties.

*Proof of (3.10).* This is just a consequence of Jacquet's submodule theorem, property 3 of Theorem 3.5, and a double application of the functional equation (3.14).

We conclude this section with the following proposition.

**PROPOSITION 6.3.** *Assume the  $\gamma_i$ 's are defined, satisfying properties 1, 3, and 4 of Theorem 3.5. Then they satisfy (3.11).*

*Proof.* This is a consequence of property 1, equations (6.1) and (3.14), Jacquet's submodule theorem, and inductive property 3.

### 7. Local factors and Langlands' conjecture

In this section we shall define the local  $L$ -function  $L(s, \sigma, r_i)$  and root number  $\varepsilon(s, \sigma, r_i, \psi_F)$  for each  $i$ ,  $1 \leq i \leq m$ . Thus, let  $\mathbf{P}$  be maximal, and first fix an irreducible tempered  $\chi$ -generic representation  $\sigma$  of  $M$ . We assume that  $\chi$  is compatible with  $\tilde{w}_0$ . From now on we use  $\gamma(s, \sigma, r_i, \psi_F)$  to denote  $\gamma_i(s, \sigma, \psi_F)$ . We observe that if  $\theta$  is a graph automorphism of  ${}^L M$ , then  $r_i \cdot \theta$  is also a representation of  ${}^L M$ . Choosing  $\theta$  appropriately, we can consider  $\gamma(s, \sigma, \tilde{r}_i, \psi_F)$ . Moreover,  $\gamma(s, \sigma, \tilde{r}_i, \psi_F) = \gamma(s, \tilde{\sigma}, r_i, \psi_F)$ .

For each  $i$ , let  $P_{\sigma,i}(t)$  be the unique polynomial satisfying  $P_{\sigma,i}(0) = 1$  such that  $P_{\sigma,i}(q^{-s})$  is the numerator of  $\gamma(s, \sigma, r_i, \psi_F)$ , i.e., the polynomial in  $q^{-s}$  satisfying  $P_{\sigma,i}(0) = 1$  which has the same zeros as  $\gamma(s, \sigma, r_i, \psi_F)$ . Define the  $L$ -functions attached to  $\sigma$ ,  $r_i$ , and  $\tilde{r}_i$  as

$$(7.1) \quad L(s, \sigma, r_i) = P_{\sigma,i}(q^{-s})^{-1}$$

and

$$(7.2) \quad L(s, \sigma, \tilde{r}_i) = P_{\tilde{\sigma},i}(q^{-s})^{-1}.$$

Then by (3.10)

$$(7.3) \quad \gamma(s, \sigma, r_i, \psi_F)L(s, \sigma, r_i)/L(1-s, \sigma, \tilde{r}_i)$$

is a monomial in  $q^{-s}$  which we denote by  $\varepsilon(s, \sigma, r_i, \psi_F)$ , the root number attached to  $\sigma$  and  $r_i$ . Thus

$$(7.4) \quad \gamma(s, \sigma, r_i, \psi_F) = \varepsilon(s, \sigma, r_i, \psi_F)L(1-s, \sigma, \tilde{r}_i)/L(s, \sigma, r_i).$$

Observe that we must show that  $L(s, \sigma, r_i)$  does not depend on  $\psi_F$ . First assume  $\sigma$  is supercuspidal. Then the zeros of each local coefficient which one has to employ to define the factors  $\gamma_i(s, \sigma, \psi_F)$  are exactly the poles of the corresponding intertwining operators and therefore independent of  $\psi_F$  (Corollary 5.4.2.3 of [44] and Proposition 3.3.1.b of [38]). The polynomials  $P_{\sigma,i}$ ,  $1 \leq i \leq m$ , are defined inductively in terms of these local coefficients and consequently they are also independent of  $\psi_F$ . The independence for tempered  $\sigma$  is now a consequence of (3.13).

Having defined  $L(s, \sigma, r_i)$  for tempered representations, we now use Langlands' classification ([43], also due to Borel and Wallach) to define the factors for arbitrary  $\chi$ -generic representation  $\sigma$ , exactly as in the case of real groups [28]. In fact, we first extend the  $L$ -functions by analytic continuation to include  $L$ -functions for quasi-tempered representations. The  $L$ -function for an arbitrary  $\sigma$  is then a product of these  $L$ -functions according to a reduced decomposition of  $\tilde{w}_0$  with respect to simple reflections of the parabolic subgroup by means of which  $\sigma$  is defined as a Langlands quotient. Inducing in stages and the fact that local coefficients only depend on the class of  $\sigma$  imply that (7.3) is a monomial in  $q^{-s}$ . We then define  $\varepsilon(s, \sigma, r_i, \psi_F)$  to satisfy (7.4).

Now fix  $\psi_F$  and define  $\chi_0$  as in Section 3. Let  $\sigma$  be  $\chi_0$ -generic. If  $\sigma'$  is another generic representation which is in the same  $L$ -packet as  $\sigma$ , then the factors  $\gamma_i(s, \sigma', \psi_F)$ ,  $1 \leq i \leq m$ , are defined to be equal to  $\gamma_i(s, \sigma, \psi_F)$  (Theorem 3.5). Consequently the  $L$ -functions, and by (7.4) the root numbers, depend only on the  $L$ -packet of  $\sigma$ .

In the case of  $\mathbf{G} = \mathrm{GL}(m+n)$  and  $\mathbf{M} = \mathrm{GL}(n) \times \mathrm{GL}(m)$ , Theorem 5.1 of [37] and Propositions 8.3 and 9.4 of [15] imply that our  $L$ -functions and root

numbers are those of Rankin-Selberg defined by Jacquet, Piatetski-Shapiro, and Shalika in [15]. For  $\mathbf{G} = G_2$ , we refer to [40] for the equality of our  $L$ -functions with those of Artin.

Going back to the tempered case, the following conjecture is then expected (cf. [15] for  $GL(n)$ ) for which we shall produce ample evidence (Propositions 7.2 and 7.3). In fact if it is true, (3.11) becomes a genuine factorization of local coefficients (also see Remark 7.10).

CONJECTURE 7.1. *Assume  $\sigma$  is tempered. Then each  $L(s, \sigma, r_i)$  is holomorphic for  $\text{Re}(s) > 0$ .*

Our first evidence is:

PROPOSITION 7.2. *Assume  $\sigma$  is tempered.*

- a) *If  $m = 1$ , then  $L(s, \sigma, r)$  is holomorphic for  $\text{Re}(s) > 0$ .*
- b) *Assume  $m = 2$  and*

$$(7.5) \quad L(s, \sigma, r_2) = \prod_j (1 - \alpha_j q^{-s})^{-1},$$

*possibly an empty product, where each  $\alpha_j \in \mathbf{C}$  is of absolute value one (in particular if  $r_2$  is one dimensional). Then for  $\text{Re}(s) > 0$ ,  $L(s, \sigma, r_1)$  is holomorphic.*

*Proof.* a) The zeros of  $C_\chi(s\tilde{\alpha}, \sigma, w_0)$  are those of  $P_{\sigma,1}(q^{-s})$ . Thus by Proposition 3.3.1.b of [38], the poles of  $L(s, \sigma, r_1)$  are among those of intertwining operators. Since  $\sigma$  is tempered, Theorem 5.3.5.4 of [44] implies that these operators are holomorphic for  $\text{Re}(s) > 0$ , proving part a).

b) Using the local coefficient for the pair  $(\mathbf{G}_2, \mathbf{M}_2)$  and Lemma 3.1 of [39] we see that

$$L(s, \tilde{\sigma}, r_2) = \prod_j (1 - \bar{\alpha}_j q^{-s})^{-1}.$$

The argument of Lemma 6.4 of [39] (for the case  $m = 2$ ) now applies which shows that the zeros of  $P_{\sigma,1}(q^{-s})$  (i.e., the poles of  $L(s, \sigma, r_1)$ ) are among the zeros of  $C_\chi(s\tilde{\alpha}, \sigma, w_0)$ . But since  $\sigma$  is tempered, the argument at the end of part a) implies that  $L(s, \sigma, r_1)$  is holomorphic for  $\text{Re}(s) > 0$ . This completes the proof of the proposition.

Our next evidence is:

PROPOSITION 7.3. *Assume  $\sigma$  is an irreducible unitary supercuspidal  $\chi$ -generic representation of  $M$ . Then for  $\text{Re}(s) > 0$ , each  $L(s, \sigma, r_i)$  is holomorphic,  $1 \leq i \leq m$ . In fact each  $L(s, \sigma, r_i)$ ,  $1 \leq i \leq m$ , is a product (possibly empty) as in (7.5) with  $|\alpha_j| = 1$ .*

We need a definition and two lemmas. Let  $\mathbf{P} = \mathbf{P}_\theta$  be a maximal parabolic subgroup of  $\mathbf{G}$ . We shall say  $\mathbf{P}$  is *self-conjugate* if  $\tilde{w}_0(\theta) = \theta$ .

LEMMA 7.4. *Let  $\sigma$  be an irreducible supercuspidal  $\chi$ -generic representation of  $M$ . Suppose  $P$  is not self-conjugate. Then  $C_\chi(s\tilde{\alpha}, \sigma, \tilde{w}_0)$  never vanishes.*

*Proof.* This is just a consequence of the corollary to Lemma 2.2.5 of [38] and Proposition 3.3.1.b of [38].

LEMMA 7.5. *Suppose  $\sigma$  is as in Proposition 7.3 and  $m \geq 3$ . Then, for  $3 \leq i \leq m$ ,  $L(s, \sigma, r_i) = 1$ .*

*Proof.* We may assume  $\mathbf{G}$  is simple. Checking every case with  $m \geq 3$  in [29], [39], we easily see that except for representation  $r_3$  of the case  $(E_8 - 1)$  of [39], each  $(\mathbf{G}_i, \mathbf{M}_i)$ ,  $3 \leq i \leq m$ , may be chosen to have length one with  $P_i$  not self-conjugate in  $\mathbf{G}_i$ . The lemma is then a consequence of Lemma 7.4. The representation  $r_3$  of  $(E_8 - 1)$  appears as  $r'_1$  in the case  $(E_6 - 2)$  for which  $L(s, \sigma', r'_2) = 1$  for every irreducible supercuspidal  $\chi$ -generic representation  $\sigma'$  of  $M_3$ . Moreover,  $\mathbf{P}_3$  in  $\mathbf{G}_3$  ( $(\mathbf{G}_3, \mathbf{M}_3)$  is the case  $(E_6 - 2)$ ) is not self-conjugate. Again the lemma is a consequence of Lemma 7.4.

*Proof of Proposition 7.3.* We shall show that each  $L(s, \sigma, r_i)$  is a product as in (7.5) with  $|\alpha_j| = 1$ . The first assertion then follows. By Lemma 7.5 we may assume that, up to a monomial in  $q^{-s}$ ,  $C_\chi(s\tilde{\alpha}, \sigma, w_0)$  is equal to

$$(7.3.1) \quad L(1 - s, \tilde{\sigma}, r_1)L(1 - 2s, \tilde{\sigma}, r_2)/L(s, \sigma, r_1)L(2s, \sigma, r_2).$$

If  $L(s, \sigma, r_2)$  is of the form (7.5) with  $|\alpha_j| = 1$ , then Proposition 7.2.b, when applied to (7.3.1) implies the same for  $L(s, \sigma, r_1)$ . We now apply our general induction of Section 4, together with the induction deduced from (7.3.1), to reduce to the case where there is only the ratio of one  $L$ -function present in (7.3.1). To this we now apply Theorem 2.2.1 of [38]. This proves the proposition.

COROLLARY 7.6. *Let  $\sigma$  be an irreducible supercuspidal  $\chi$ -generic representation of  $M$ . Then up to a monomial in  $q^{-s}$ ,  $C_\chi(s\tilde{\alpha}, \sigma, w_0)$  is equal to*

$$(7.6) \quad P_{\sigma, 1}(q^{-s})P_{\sigma, 2}(q^{-2s})/P_{\tilde{\sigma}, 1}(q^{-(1-s)})P_{\tilde{\sigma}, 2}(q^{-(1-2s)}).$$

*The numerator and the denominator have no common factors. Moreover, if  $\sigma$  is unitary, then the following statements are equivalent:*

- a)  $\sigma$  is a pole of  $A(\sigma, w_0)$ .
- b)  $P_{\sigma, i}(1) = 0$  for either  $i = 1$  or  $2$  and only for one of them.
- c)  $I(\sigma)$  is irreducible and  $\sigma$  is ramified, i.e.,  $\tilde{w}_0(\sigma) \cong \sigma$ .

*Proof.* In view of Proposition 7.3 we need only verify the last three equivalent statements. But these are consequences of Corollary 5.4.2.3 of [44],

Proposition 3.3.1.b of [38], and the fact that the Plancherel measure is positive if  $\sigma$  is unramified.

*Remark.* We conclude our discussion of Conjecture 7.1 with two comments. First, by inductive property 3 of Theorem 3.5 we only need to prove the conjecture for  $\sigma$  in the discrete series. Observe that, if one assumes Conjecture 7.1, then by the same property both root numbers and  $L$ -functions are also inductive for tempered representations. Second, by Proposition 7.2 and tables in [29] and [39], we only need consider those cases where the simply connected covering of the derived group of  $\mathbf{M}$  is a product of classical groups and  $\sigma$  is a non-supercuspidal discrete series representation of  $M$ . For example if  $\mathbf{G} = \mathrm{Sp}(2(m+n))$  and  $\mathbf{M} = \mathrm{GL}(n) \times \mathrm{Sp}(2m)$ , it is not hard to see that if  $\sigma$  is only supercuspidal on one of the factors then Conjecture 7.1 is valid.

With notation as in part 4 of Theorem 3.5, let  $\pi = \otimes_v \pi_v$  be a  $\chi$ -generic cusp form on  $G = \mathbf{G}(\mathbb{A}_K)$ . Assume  $\chi$  is defined by means of  $\psi = \otimes_v \psi_v$ . For each  $i$ ,  $1 \leq i \leq m$ , define  $L(s, \pi_v, r_{i,v})$  and  $\varepsilon(s, \pi_v, r_{i,v}, \psi_v)$  as before. Set

$$(7.7) \quad L(s, \pi, r_i) = \prod_v L(s, \pi_v, r_{i,v})$$

and

$$(7.8) \quad \varepsilon(s, \pi, r_i) = \prod_v \varepsilon(s, \pi_v, r_{i,v}, \psi_v).$$

Observe that  $\varepsilon(s, \pi, r_i)$  does not depend on  $\psi$ . The functional equation (3.14) can now be stated as follows:

**THEOREM 7.7.** *With notation as above, the following functional equation holds for each  $i$ ,  $1 \leq i \leq m$ ,*

$$(7.9) \quad L(s, \pi, r_i) = \varepsilon(s, \pi, r_i) L(1-s, \pi, \tilde{r}_i).$$

Langlands' conjecture on normalization of intertwining operators by means of the  $L$ -functions and root numbers defined in this section can now be proved using Corollary 3.6. We only need:

**PROPOSITION 7.8.** *Let  $\sigma$  be an irreducible unitary generic representation of  $M$ . Then for each  $i$ ,  $1 \leq i \leq m$ ,*

$$\overline{L(s, \sigma, r_i)} = L(\bar{s}, \sigma, \tilde{r}_i)$$

and

$$\overline{\varepsilon(s, \sigma, r_i, \psi_F)} = \varepsilon(\bar{s}, \sigma, \tilde{r}_i, \bar{\psi}_F).$$

*Proof.* By Lemma 3.1 of [39] and the inductive definition of  $\gamma(s, \sigma, r_i, \psi_F)$ , it is clear that

$$(7.8.1) \quad \begin{aligned} \overline{\gamma(s, \sigma, r_i, \psi_F)} &= \gamma(\bar{s}, \bar{\sigma}, r_i, \bar{\psi}_F) \\ &= \gamma(\bar{s}, \sigma, \tilde{r}_i, \bar{\psi}_F). \end{aligned}$$

From this the proposition follows if  $\sigma$  is tempered. Otherwise, let  $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$  be a parabolic subgroup of  $\mathbf{M}$ ,  $\mathbf{N}' \subset \mathbf{U}'$ , such that  $\sigma = J(\mathbf{P}', \nu, \sigma')$ . Here  $\sigma'$  is a tempered representation of  $M'$ ,  $\nu \in (\alpha')_{\mathbb{C}}^*$  is in the positive Weyl chamber, and  $J(\mathbf{P}', \nu, \sigma')$  denotes the corresponding Langlands quotient. Then  $\bar{\sigma} = J(\bar{\mathbf{P}}', -\bar{\nu}, \sigma')$ , where  $\bar{\mathbf{P}}'$  is the parabolic subgroup opposed to  $\mathbf{P}'$ . Then

$$\overline{L(\bar{s}, \sigma, r_i)} = \overline{L(\bar{s}, \sigma'_\nu, r_i)},$$

where  $\sigma'_\nu = \sigma' \otimes q^{\langle \nu, H_{\mathbf{P}'(\cdot)} \rangle}$ . Assume  $\nu$  is pure imaginary. Then by the tempered case

$$\begin{aligned} \overline{L(\bar{s}, \sigma'_\nu, r_i)} &= L(s, \bar{\sigma}'_\nu, r_i) \\ &= L(s, \sigma'_{-\bar{\nu}}, r_i). \end{aligned}$$

But by definition (in terms of analytic continuation) this last  $L$ -function is the  $L$ -function for  $J(\bar{\mathbf{P}}', -\bar{\nu}, \sigma')$  and  $r_i$ ; i.e., it is equal to  $L(s, \bar{\sigma}, r_i)$ . The identity for root numbers now follows from (7.8.1).

Now assume  $\mathbf{P} = \mathbf{M}\mathbf{N}$ ,  $\mathbf{N} \subset \mathbf{U}$ , is any standard parabolic subgroup of  $\mathbf{G}$ . Fix  $\tilde{w} \in W(\mathbf{A}_0)$  such that  $\tilde{w}(\theta) \subset \Delta$ ,  $\mathbf{P} = \mathbf{P}_\theta$ . Let  $\sigma$  be an irreducible unitary  $\chi$ -generic representation of  $M$ . We may assume that  $\tilde{w}$  and  $\chi$  are compatible. We fix a reduced decomposition  $\tilde{w} = \tilde{w}_{n-1} \cdots \tilde{w}_1$  and set

$$(7.10) \quad L(s, \sigma, r_{\tilde{w}}) = \prod_{j=1}^{n-1} \prod_{i=1}^{m_j} L(s, \bar{w}_j(\sigma), r_{\tilde{w}_{j,i}})$$

and

$$(7.11) \quad \varepsilon(s, \sigma, r_{\tilde{w}}, \psi_F) = \prod_{j=1}^{n-1} \prod_{i=1}^{m_j} \varepsilon(s, \bar{w}_j(\sigma), r_{\tilde{w}_{j,i}}, \psi_F).$$

They are both independent of the decomposition of  $\tilde{w}$ .

We shall now normalize the intertwining operator  $A(\sigma, w)$  in the way conjectured by Langlands. Let

$$(7.12) \quad \mathcal{A}(\sigma, w) = \varepsilon(0, \sigma, \tilde{r}_{\tilde{w}}, \psi_F) L(1, \sigma, \tilde{r}_{\tilde{w}}) L(0, \sigma, \tilde{r}_{\tilde{w}})^{-1} A(\sigma, w),$$

where the right-hand side is determined as a limit. We then have:

**THEOREM 7.9** (Langlands' conjecture). *The normalized operator  $\mathcal{A}(\sigma, w)$  satisfies:*

- a)  $\mathcal{A}(\sigma, w_1 w_2) = \mathcal{A}(\tilde{w}_2(\sigma), w_1) \mathcal{A}(\sigma, w_2)$ , and
- b)  $\mathcal{A}(\sigma, w)^* = \mathcal{A}(\tilde{w}(\sigma), w^{-1})$ , i.e.,  $\mathcal{A}(\sigma, w)$  is unitary.

*Proof.* This is a consequence of Corollary 3.6 and Proposition 7.8.

*Remark 7.10.* If one assumes Conjecture 7.1, then our normalizing factor will also satisfy condition  $R_7$  (and thus all the conditions) of Theorem 2.1 of Arthur [2]. Observe that condition  $R_7$  of Theorem 2.1 of [2] was not among the original conditions conjectured by Langlands [27].

### 8. Complementary series and special representations for $p$ -adic groups

In this section we shall determine all the complementary series and special representations coming from generic supercuspidal representations of Levi factors of maximal parabolic subgroups of any quasi-split  $p$ -adic group. While the proof (of Theorem 8.1) is standard, the results are quite sharp and deep and are only possible due to our precise knowledge of Plancherel measures (Corollaries 3.6 and 7.6). In particular our formula for Plancherel measures determines the only important unknown factors in the formula obtained by Silberger [45], namely  $\alpha_j$ 's (page 579 of [45]). We should remark that in many cases the Levi subgroups are products of A-type groups for which supercuspidal representations are always generic and therefore our assumption on genericity is in many cases naturally satisfied. Our notation is as in the previous sections.

**THEOREM 8.1.** *Let  $P = MN$  be a maximal parabolic subgroup of  $G$ , where  $G$  is a quasi-split connected reductive algebraic group over a  $p$ -adic field  $F$ . Let  $\sigma$  be an irreducible unitary supercuspidal  $\chi$ -generic representation of  $M$ . Assume  $\sigma$  is ramified, i.e.,  $\tilde{w}_0(\sigma) \cong \sigma$ , and  $I(\sigma)$  is irreducible. Choose (by Corollary 7.6) a unique  $i$ ,  $i = 1$  or  $2$ , such that  $P_{\sigma, i}(1) = 0$ . Then:*

- a) *For  $0 < s < 1/i$ , the representation  $I(s\tilde{\alpha}, \sigma)$  is irreducible and in the complementary series.*
- b) *The representation  $I(\tilde{\alpha}/i, \sigma)$  is reducible with a unique  $\chi$ -generic subrepresentation which is in the discrete series (a special representation). Its Langlands quotient is never generic. It is a pre-unitary non-tempered representation.*
- c) *For  $s > 1/i$ , the representations  $I(s\tilde{\alpha}, \sigma)$  are always irreducible and never in the complementary series.*

*If  $\sigma$  is ramified and  $I(\sigma)$  is reducible, then no  $I(s\tilde{\alpha}, \sigma)$ ,  $s > 0$ , is pre-unitary. They are all irreducible.*

*Proof.* Given  $f \in V(s\tilde{\alpha}, \sigma)$  and  $f' \in V(-\bar{s}\tilde{\alpha}, \sigma)$ , let

$$(f, f') = \oint_G (f(g), f'(g))_0 d\mu(g),$$

where  $(\ , \ )_0$  is the unitary pairing on the space of  $\sigma$  and  $d\mu(g)$  is defined by

$$\mu(h) = \oint_G h(g) d\mu(g),$$

$h \in V(\rho_P, 1)$ , and  $\mu$  is a relatively bounded linear form on  $V(\rho_P, 1)$  defined by

$$\mu(h_\phi) = \int_G \phi(g) dg \quad (\phi \in C_c^\infty(G))$$

and

$$h_\phi(g) = \int_{M \times N} \phi(mng) q^{\langle -2\rho_P, H_P(m) \rangle} dm dn$$

(cf. page 303 of [38]).

Given  $f$  and  $g$  in  $V(s\tilde{\alpha}, \sigma)$  we set

$$\langle f, g \rangle = (\mathcal{A}(\sigma_{s\tilde{\alpha}}, w_0)f, g),$$

where  $\mathcal{A}(\sigma_{s\tilde{\alpha}}, w_0)$  is defined by analytic continuation of the normalized operator which was originally defined for pure imaginary  $s$  in the previous section.

If  $s$  is so that  $\mathcal{A}(\sigma_{s\tilde{\alpha}}, w_0)$  is non-zero and pole-free, and is positive or negative semi-definite, then  $\langle \ , \ \rangle$  defines a unitary pairing on the Langlands quotient of  $I(s\tilde{\alpha}, \sigma)$  (Theorem 16.6 of [21]; also cf. Section 6 of [19]). Here we use  $\tilde{w}_0(\sigma) \cong \sigma$  and the fact that since  $s$  is real,  $\tilde{w}_0(s\tilde{\alpha}) = -\bar{s}\tilde{\alpha}$ .

First assume  $I(\sigma) = I(0, \sigma)$  is irreducible. Then  $\mathcal{A}(\sigma, w_0)$  acts like a non-zero scalar. In fact  $\mathcal{A}(\sigma, w_0)$  is unitary and we may assume  $\mathcal{A}(\sigma, w_0)$  is positive definite. By continuity  $\mathcal{A}(\sigma_{s\tilde{\alpha}}, w_0)$  remains positive definite in a neighborhood of  $s = 0, s > 0$ , until  $I(s\tilde{\alpha}, \sigma)$  becomes reducible, i.e., the first pole of the Plancherel measure (Lemma 5.4.2.4 of [44]). By the above discussion all of these representations are pre-unitary. Thus what is deep and new is the location of this pole. We continue as follows:

By Corollary 7.6 choose  $i, i = 1, 2$ , uniquely, such that  $P_{\sigma, i}(1) = 0$ . In fact by simplicity of the poles of intertwining operators and Corollary 5.4.2.3 of [44], such an  $i$  exists uniquely. To get the edge of complementary series we must find this first pole. By Lemma 1.2 of [45], given the initial data  $\sigma$ , this pole (on the positive real axis) will be unique. By Corollary 7.6 we need to choose  $s, s > 0$ , such that

$$P_{\tilde{\sigma}, i}(q^{-(1-is)}) = 0.$$

Observe that  $P_{\sigma, i}(1) = 0$  if and only if  $P_{\tilde{\sigma}, i}(1) = 0$ . Thus the first pole is at

$s = 1/i$ . The representation  $I(\tilde{\alpha}/i, \sigma)$  is reducible and by Lemma 5.4.5.2 of [44] has a subrepresentation, the kernel of  $A(\tilde{\alpha}/i, \sigma, w_0)$ , which belongs to the discrete series.

Now, by Corollary 7.6 the corresponding local coefficient has a pole as well, and therefore by its definition the corresponding Langlands quotient cannot be  $\chi$ -generic. By Rodier's theorem it cannot be generic with respect to any other character with respect to which  $\sigma$  is not. Thus it is never generic. The Hermitian form is now positive definite on this quotient and therefore this quotient is pre-unitary. Moreover for  $s > 1/i$ , the normalized operator  $\mathcal{A}(s\tilde{\alpha}, \sigma, w_0)$  when considered as an operator on  $K$ -types, will be negative on those which lie in the kernel of  $\mathcal{A}(\tilde{\alpha}/i, \sigma, w_0)$ , while it will be positive on the others. Since there are no other changes of sign for  $\mathcal{A}(s\tilde{\alpha}, \sigma, w_0)$  after  $s = 1/i$ , this operator can never be positive or negative semi-definite. Now by Theorem 16.6 of [21], if  $I(s\tilde{\alpha}, \sigma)$  is pre-unitary, its form must be proportional to  $\langle \cdot, \cdot \rangle$ . This is a contradiction.

Finally assume  $I(\sigma)$  is reducible. Then by the theory of  $R$ -groups (cf. [20], for example)  $\mathcal{A}(\sigma, w_0)$  will have different signs on irreducible components of  $I(\sigma)$ . Again the same argument applies to show that for  $s > 0$ , the representation  $I(s\tilde{\alpha}, \sigma)$  is never pre-unitary.

*Remark 8.2.* The variable  $z$  used by Silberger [45] is not equal to  $q^{-s}$  in general, but rather to an integral power of it. But the polynomials  $P_{\sigma,i}$ ,  $i = 1, 2$ , are in fact rational functions of  $z$ . As an example, take  $\mathbf{G} = \mathrm{GL}_{2n}$  and  $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_n$ . Then  $m = 1$ . Let  $\sigma$  be an irreducible unitary supercuspidal representation of  $\mathrm{GL}_n(F)$ . Let  $\eta$  be an unramified character of  $F^*$ . Then its composition with determinant becomes a character of  $\mathrm{GL}_n(F)$  which we still denote by  $\eta$ . Assume  $\sigma \otimes \eta \cong \sigma$ . Then  $\eta^n = 1$ . The set of all such characters is a cyclic group of order  $r$ ,  $r|n$ . Then  $P_{\sigma,1}(q^{-s}) = 1 - q^{-rs}$  and  $z = q^{-rs}$  (cf. the definition of  $z$  in [45]).

We conclude this section by giving two examples. Both groups are split of rank two. In fact the case of rank-one quasi-split groups has already been answered by Keys [19]. Our examples comprise the most interesting and difficult cases of rank-two groups.

*Example 1.* Let  $\mathbf{G}$  denote the split group of type  $G_2$ . Fix the parabolic subgroup  $\mathbf{P} = \mathbf{M}\mathbf{N}$  of  $\mathbf{G}$  whose Levi factor is generated by the long root of  $\mathbf{A}_0 = \mathbf{T}$ . Then  $\mathbf{M} = \mathrm{GL}_2$  (cf. [40]). Let  $\sigma$  be an irreducible unitary supercuspidal representation of  $M$ . It is always generic. In Proposition 6.3 of [40] we studied the reducibility of  $I(\sigma)$ . We now apply Theorem 8.1 to obtain the complementary series and special representations of  $G$  coming from  $P$ . In what follows, we shall refer to Proposition 6.3 of [40] for details.

Let  $\omega$  be the central character of  $\sigma$ . The fact that  $\sigma$  is ramified implies  $\sigma \cong \tilde{\sigma}$  or equivalently  $\sigma \cong \sigma \otimes \omega$ . This implies  $\omega^2 = 1$ . If  $\sigma$  is extraordinary, the two polynomials are

$$P_{\sigma,1}(q^{-s}) = 1$$

and

$$P_{\sigma,2}(q^{-s}) = L(s, \omega)^{-1},$$

where  $L(s, \omega)$  is the Hecke  $L$ -function attached to  $\omega$ . Otherwise  $\sigma$  comes from an irreducible representation  $\tau = \text{Ind}(W_F, W_K, \chi)$ ,  $[K : F] = 2$ ,  $\chi \in \hat{K}^*$ . Let  $\eta \in \hat{F}^*$  be the quadratic character attached to  $K/F$ , i.e.,  $\ker(\eta) = N_{K/F}(K^*)$ . The central character  $\omega$  is equal to  $\omega = \chi|_{F^*} \cdot \eta$  and

$$P_{\sigma,1}(q^{-s}) = L_K(s, \chi^2 \chi'^{-1})^{-1},$$

and

$$P_{\sigma,2}(q^{-s}) = L(s, \omega)^{-1}.$$

Here  $\chi' = \chi \cdot \sigma$ ,  $\sigma \in \text{Gal}(K/F)$ ,  $\sigma \neq 1$ . We have:

**PROPOSITION 8.3.** *Suppose  $\sigma \cong \tilde{\sigma}$ ; i.e.,  $\sigma$  is ramified.*

a) *Assume  $\sigma$  is extraordinary. Then  $\omega = 1$  and therefore  $I(\sigma)$  is irreducible. Moreover, for such  $\sigma$  the representation  $I(s\tilde{\alpha}, \sigma)$  is in the complementary series only for  $0 < s < 1/2$ .*

b) *If  $\sigma = \pi(\tau)$ ,  $\tau = \text{Ind}(W_F, W_K, \chi)$ ,  $[K : F] = 2$ ,  $\chi \in \hat{K}^*$ , then  $I(\sigma)$  is irreducible if and only if either  $\chi|_{F^*} = \eta$ , or  $\chi|_{F^*} = 1$  and  $\chi^3 = 1$ . If  $\chi|_{F^*} = \eta$ , then  $I(s\tilde{\alpha}, \sigma)$  is in the complementary series for only  $0 < s < 1/2$ . Otherwise, i.e., if  $\chi|_{F^*} = 1$  and  $\chi^3 = 1$ , then  $I(s\tilde{\alpha}, \sigma)$  is in the complementary series only for  $0 < s < 1$ .*

*Proof.* In view of Proposition 6.3 of [40] and Theorem 8.1 we only need to check part b). The condition  $\sigma \cong \tilde{\sigma}$  implies that either  $\chi\chi' = 1$  or  $\chi^2 = 1$ . In the first case either  $\chi|_{F^*} = \eta$  or 1. If  $\chi|_{F^*} = \eta$ , then  $P_{\sigma,2}(1) = 0$  and therefore the edge of complementary series becomes  $\tilde{\alpha}/2$ . Notice that  $\chi^3 \neq 1$  and therefore  $P_{\sigma,1}(1) \neq 0$  as expected. If  $\chi|_{F^*} = 1$ , then  $\omega = \eta$  and  $P_{\sigma,2}(q^{-s})$  has no real zero for  $s > 0$ . On the other hand  $P_{\sigma,1}(1) = 0$  since  $\chi^3 = 1$ . Consequently  $I(s\tilde{\alpha}, \sigma)$  is in the complementary series for  $0 < s < 1$ , and only there. Finally assume  $\chi^2 = 1$ . Then  $P_{\sigma,1}(q^{-s}) = L_K(s, \chi'^{-1})^{-1}$ . But  $P_{\sigma,1}(1)$  can never be zero since  $\sigma$  is supercuspidal. If  $\chi|_{F^*} = \eta$ , then  $I(\sigma)$  is irreducible and  $P_{\sigma,2}(1) = 0$ . Again the edge of complementary series is  $\tilde{\alpha}/2$ .

*Example 2.* Next assume  $\mathbf{G} = \text{Sp}_4$  and  $\mathbf{P}$  is the non-Siegel maximal parabolic subgroup, i.e.,  $\mathbf{M} = \text{SL}_2 \times \text{GL}_1$ . Then  $\sigma = \sigma_1 \otimes \theta$ , where  $\sigma_1$  is a super-

cuspidal representation of  $SL_2(F)$  and  $\theta \in \hat{F}^*$ . By [25], there exists an irreducible unitary supercuspidal representation  $\tilde{\sigma}_1$  of  $GL_2(F)$  such that  $\sigma_1 \subset \tilde{\sigma}_1|_{SL_2(F)}$ . Let  $\Sigma_1$  be the Gelbart-Jacquet [10] lift of  $\tilde{\sigma}_1$  from  $GL_2(F)$  to  $GL_3(F)$ . It is unique and does not depend on the choice of  $\tilde{\sigma}_1$ . Then  $m = 1$  and

$$P_{\sigma,1}(q^{-s}) = L(s, \Sigma_1 \otimes \theta)^{-1},$$

where the  $L$ -function is the principal  $L$ -function for  $\Sigma_1 \otimes \theta$ . If  $\tilde{\sigma}_1$  is extraordinary, then  $\Sigma_1$  is supercuspidal and  $P_{\sigma,1}(q^{-s}) = 1$ . Otherwise  $\tilde{\sigma}_1 = \pi(\tau)$ ,  $\tau = \text{Ind}(W_F, W_K, \chi)$ ,  $[K : F] = 2$ ,  $\chi \in \hat{K}^*$ . Then  $\Sigma_1$  is the representation induced from  $\sigma' \otimes \eta \otimes 1$ , where  $\sigma' = \pi(\tau')$ ,  $\tau' = \text{Ind}(W_F, W_K, \chi\chi'^{-1})$  and  $\eta$  is as in Example 1; i.e.,  $\ker(\eta) = N_{K/F}(K^*)$  (cf. [10]). Consequently

$$P_{\sigma,1}(q^{-s}) = \prod_{\eta} L(s, \eta\theta)^{-1}.$$

The Levi subgroup  $\mathbf{M}$  is generated by the long root  $\beta = 2\varepsilon_2$  and  $\tilde{w}_0 = \tilde{w}_\alpha \tilde{w}_\beta \tilde{w}_\alpha$ , where  $\alpha = \varepsilon_1 - \varepsilon_2$  is the short root. If the representation  $\sigma$  is ramified, then  $\theta^2 = 1$ . By the general theory of  $R$ -groups,  $I(\sigma)$  is reducible if and only if  $P_{\sigma,1}(1) \neq 0$  and  $\theta^2 = 1$ . Thus we have:

PROPOSITION 8.4. *Suppose  $\sigma$  is ramified. Then  $\theta^2 = 1$ .*

a) *First, assume  $\tilde{\sigma}_1$  is extraordinary. Then  $I(\sigma)$  is reducible and there are no complementary series.*

b) *Next, let  $\tilde{\sigma}_1 = \pi(\tau)$ ,  $\tau = \text{Ind}(W_F, W_K, \chi)$ ,  $[K : F] = 2$ ,  $\chi \in \hat{K}^*$ . Then  $I(\sigma)$  is irreducible if and only if  $\theta = \eta$ , where  $\eta$  is the character of  $F^*$  attached by local class field theory to  $K/F$ . The representations  $I(s\tilde{\alpha}, \sigma)$  are in the complementary series only for  $0 < s < 1$ .*

Remark 8.5. Suppose  $\mathbf{G} = \text{GSp}_4$  and  $\mathbf{M} = \text{GL}_2 \times \text{GL}_1$ . Let  $\sigma = \sigma_1 \otimes \theta$ , where  $\sigma_1$  is an irreducible unitary supercuspidal representation of  $GL_2(F)$  and  $\theta \in \hat{F}^*$  is unitary. Then  $\tilde{w}_0(\sigma) \cong \sigma$  implies  $\theta^2 = 1$  and  $\sigma_1 \cong \sigma_1 \otimes \theta$ , and therefore either  $\theta = 1$  or  $\theta \neq 1$  and  $\sigma_1 \cong \sigma_1 \otimes \theta$ . But by [25],  $\sigma_1 \cong \sigma_1 \otimes \theta$  with  $\theta \neq 1$  if and only if  $\sigma_1 = \pi(\tau)$ ,  $\tau = \text{Ind}(W_K, W_F, \chi)$ ,  $\chi \in \hat{K}^*$ ,  $[K : F] = 2$ , and  $\ker \theta = N_{K/F}(K^*)$ . It follows from the proof of Proposition 8.4 that  $I(\sigma)$  is reducible if and only if  $\theta = 1$ . Moreover, if  $\sigma_1 \cong \sigma_1 \otimes \theta$ ,  $\theta \neq 1$ ,  $I(\sigma)$  is irreducible and  $I(s\tilde{\alpha}, \sigma)$  is in the complementary series only for  $0 < s < 1$ . The representation  $I(\tilde{\alpha}, \sigma)$  is reducible. This is Proposition 5.1 of [49] which was proved by Waldspurger by an entirely different method.

Remark 8.6. In view of the previous remark, it is clear that Proposition 8.4 can be formulated as follows: *Suppose  $\sigma$  is ramified. Then  $\theta^2 = 1$ . The induced representation  $I(\sigma)$  is reducible unless  $\tilde{\sigma}_1 \cong \tilde{\sigma}_1 \otimes \theta$ ,  $\theta \neq 1$ , in which case the representations  $I(s\tilde{\alpha}, \sigma)$  are in the complementary series for  $0 < s < 1$ . The representation  $I(\tilde{\alpha}, \sigma)$  is reducible.*

### 9. The conjecture in general

In this section we shall study the conjecture in general, i.e., when the representation  $\sigma$  is not necessarily generic. We shall show that assuming two natural conjectures in harmonic analysis, the conjecture whose origins are in number theory can be positively answered. Since Plancherel measures are expected to be the same for inner forms (cf. [1], for example), this should answer the conjecture in general. In what follows, we shall freely use definitions, notation, and results from [32].

Let  $\mathbf{G}$  be a quasi-split connected reductive algebraic group over a  $p$ -adic field  $F$  of characteristic zero. Given  $f \in C_c^\infty(G)$  and  $\gamma$  a regular semisimple element of  $G$ , let  $\Phi(\gamma, f)$  be the corresponding orbital integral. Now, assuming  $\gamma$  is strongly regular, i.e., its centralizer in  $\mathbf{G}$  is connected, set

$$(9.1) \quad \Phi^{\text{st}}(\gamma, f) = \sum_{\gamma'} \Phi(\gamma', f),$$

where the sum is over representatives  $\gamma'$  of the conjugacy classes inside the stable conjugacy class of  $\gamma$  (cf. [32] and [22]).

Let  $(\mathbf{H}, {}^L H, s, \xi)$  be an endoscopic datum attached to  $\mathbf{G}$ , where  $\xi: {}^L H \rightarrow {}^L G$  is an  $L$ -homomorphism. If  $\hat{H}$  ( ${}^L H^0$  in our notation) is the connected component of the  $L$ -group  ${}^L H$  of  $\mathbf{H}$ , then  $\hat{H} \stackrel{\xi}{\cong} \text{Cent}(s, \hat{G})^0$ , where  $s \in \hat{G}$  is semisimple. If  $\mathbf{T}_\mathbf{H}$  and  $\mathbf{T}_\mathbf{G}$  are maximal tori of  $\mathbf{H}$  and  $\mathbf{G}$ , respectively, then  $\hat{T}_\mathbf{H} \cong \hat{T}_\mathbf{G}$  will induce an  $F$ -map  $\mathcal{A}_{\mathbf{H}/\mathbf{G}}$  from semisimple conjugacy classes of  $\mathbf{H}(\bar{F})$  into those of  $\mathbf{G}(\bar{F})$ . An element  $\gamma_H \in H$  is called strongly  $\mathbf{G}$ -regular if the image of its conjugacy class under  $\mathcal{A}_{\mathbf{H}/\mathbf{G}}$  is strongly regular. Finally a strongly  $\mathbf{G}$ -regular  $\gamma_H$  is called an *image* of  $\gamma_G \in \mathbf{G}$  if  $\gamma_G \in \mathcal{A}_{\mathbf{H}/\mathbf{G}}(\{\gamma_H\})$  [32].

Now, let  $\Delta$  be a complex function of two variables whose first variable  $\gamma_H$  belongs to the set of strongly  $\mathbf{G}$ -regular elements of  $H$ , while its second variable  $\gamma$  lies in the set of strongly regular elements of  $G$ , satisfying:

- 1)  $\Delta(\gamma_H, \gamma)$  depends only on the conjugacy class of  $\gamma$  and the stable conjugacy class of  $\gamma_H$ , and
- 2)  $\Delta(\gamma_H, \gamma) = 0$  unless  $\gamma_H$  is an image of  $\gamma$ .

Following [32],  $\Delta$  is called a *transfer factor*, if given  $f \in C_c^\infty(G)$ , there exists a function  $f^H \in C_c^\infty(H)$  such that

$$(9.2) \quad \Phi^{\text{st}}(\gamma_H, f^H) = \sum_{\gamma} \Delta(\gamma_H, \gamma) \Phi(\gamma, f)$$

for all strongly  $\mathbf{G}$ -regular  $\gamma_H$ . The sum is over representatives for conjugacy classes of strongly regular elements in  $G$ . The functions  $f$  and  $f^H$  are then said to have  *$\Delta$ -matching orbital integrals*. A transfer factor has now been defined by Langlands and Shelstad in [32] as a product of five different factors. We shall

now fix their transfer factor and we shall make the following assumption:

ASSUMPTION 9.1. *The map  $f \mapsto f^H$  from  $C_c^\infty(G)$  into the quotient of  $C_c^\infty(H)$  by the subspace generated by all those  $g \in C_c^\infty(H)$  for which  $\Phi^{\text{st}}(\gamma_H, g) = 0$ , for all  $\gamma_H \in H$  strongly  $\mathbf{G}$ -regular, exists.*

This assumption is by now very much accepted and is the basis for stabilization of the trace formula by Langlands [30].

By a stable distribution on a group  $G$  we shall mean a distribution which vanishes on every function in  $C_c^\infty(G)$  whose stable orbital integrals (defined by (9.1)) are all zero.

Our next attempt is to list conjectural properties of tempered  $L$ -packets. For a connected reductive quasi-split group  $\mathbf{H}$  over  $F$ , let  $E_2(H)$  be the set of equivalence classes of discrete series representations of  $H$ . We shall assume that the set  $E_2(H)$  is partitioned to finite subsets with the property that if  $\Sigma$  is one such subset, then for every  $\sigma \in \Sigma$ , there exists a positive integer  $m(\sigma)$  such that

$$\chi_\Sigma = \sum_{\sigma \in \Sigma} m(\sigma) \chi_\sigma$$

is stable, where  $\chi_\sigma$  is the character of  $\sigma$ . More generally, if  $\mathbf{H}$  is endoscopic to another quasi-split group  $\mathbf{G}$ , and if  $\theta$  is a stable distribution on  $H$  and  $\theta \mapsto \theta^G$  is the dual map to  $f \mapsto f^H$ , then  $\chi_\Sigma^G(f_\pi) \neq 0$  if and only if  $\pi$  belongs to one and only one such partition  $\Pi$  of  $E_2(G)$ . Here  $f_\pi$  is the pseudo-coefficient of  $\pi$  (cf. [16]), i.e., a function in  $C_c^\infty(G)$  such that  $\chi_\pi(f_\pi) = 1$ , while  $\chi_{\pi'}(f_\pi) = 0$  for every other tempered representation  $\pi'$  of  $G$ . These finite sets are called *discrete series  $L$ -packets* for  $\mathbf{G}$  and  $\mathbf{H}$ . We denote their collections for  $\mathbf{G}$  and  $\mathbf{H}$  by  $\bar{E}_2(G)$  and  $\bar{E}_2(H)$ , respectively. The  $L$ -packet  $\Pi$  is said to be obtained by *endoscopic transfer* from  $\Sigma$ .

Tempered  $L$ -packets are then defined by inducing from discrete series  $L$ -packets on Levi factors. Observe that if  $\{\pi_1, \dots, \pi_n\}$  is a discrete series  $L$ -packet on a Levi factor  $M$  of  $G$  and  $\hat{\pi}_i = \text{Ind}_{MN \uparrow G} \pi_i$ , then

$$\sum_{i=1}^n m(\pi_i) \chi_{\hat{\pi}_i}$$

is stable. This follows from the fact that, up to a discriminant function,  $\Phi_{\mathbf{G}}^{\text{st}}(\gamma, f)$  is equal to  $\Phi_M^{\text{st}}(\gamma, \tilde{f}^P)$ , where

$$\tilde{f}(g) = \int_K f(k^{-1}gk) dk$$

and

$$(9.3) \quad \tilde{f}^P(m) = \delta_P^{1/2}(m) \int_N \tilde{f}(mn) dn$$

(cf. the equality  $F_f^{G/\Gamma}(\gamma) = \Gamma_{\tilde{f}^P}^{M/\Gamma}(\gamma)$ , page 204 of [44] and page 398 of [41]).

We shall call  $\sum_{i=1}^n m(\pi_i) \chi_{\hat{\pi}_i}$  a stable tempered character.

We start with the following conjecture whose validity for real groups has been verified by Shelstad in [42] (cf. Definition 1.2.1 of [3]).

CONJECTURE 9.2. *The space of stable distributions is the pointwise closed span of all the stable tempered characters.*

In the next proposition  $\mathbf{G}$  is a quasi-split connected reductive algebraic group over  $F$ . Let  $\mathscr{M}$  be a complete set of representatives of conjugacy classes of Levi subgroups of  $\mathbf{G}$  defined over  $F$ . Then given  $f \in C_c^\infty(G)$ , the Plancherel formula [12] implies:

$$(9.4) \quad f(1) = \sum_{\mathbf{M} \in \mathscr{M}} c(\mathbf{M}) \int_{\sigma \in E_2(\mathbf{M})} \mu(\sigma) d(\sigma) \chi_\sigma(f) d\sigma.$$

Here  $c(\mathbf{M})$  is a positive constant depending on  $\mathbf{M}$  and certain measures,  $\mu(\sigma) = \mu(\sigma, \tilde{w}_0)$  is the Plancherel measure defined in Section 2,  $d(\sigma)$  is the formal degree of  $\sigma$ ,  $\chi_\sigma$  is the character of  $\hat{\sigma} = \text{Ind}_{P \uparrow \mathbf{G}} \sigma$ , and  $d\sigma$  is the Euclidean measure explained in Section 2 of [12].

PROPOSITION 9.3. *Assume Conjecture 9.2 is valid. Let  $\Sigma \in \bar{E}_2(\mathbf{M})$  be a discrete series  $L$ -packet. Then*

- a) *For  $\sigma \in \Sigma$ ,  $d(\sigma)/m(\sigma)$  is a constant  $\lambda_{\mathbf{M}}(\Sigma)$  which depends only on the  $L$ -packet  $\Sigma$ , where  $m(\sigma)$ 's are the weights which make  $\sum_{\sigma \in \Sigma} m(\sigma) \chi_\sigma$  stable.*
- b) *The Plancherel measure  $\mu(\sigma)$  depends only on the  $L$ -packet  $\Sigma$  of  $\sigma$ .*

*Proof.* By Proposition 3.1 of [23], the distribution  $f \mapsto f(1)$  is stable. Thus by Conjecture 9.2,

$$(9.3.1) \quad f(1) = \sum_{\mathbf{M} \in \mathscr{M}} \int_{\Sigma \in \bar{E}_2(\mathbf{M})} \lambda_{\mathbf{G}}(\Sigma) \left( \sum_{\sigma \in \Sigma} m(\sigma) \chi_\sigma(f) \right) d\sigma,$$

where  $\lambda_{\mathbf{G}}(\Sigma) \in \mathbf{C}$  and  $f \in C_c^\infty(G)$ . Consequently (9.4) implies

$$(9.3.2) \quad \begin{aligned} & \sum_{\mathbf{M} \in \mathscr{M}} \int_{\Sigma \in \bar{E}_2(\mathbf{M})} \lambda_{\mathbf{G}}(\Sigma) \left( \sum_{\sigma \in \Sigma} m(\sigma) \chi_\sigma(f) \right) d\sigma \\ &= \sum_{\mathbf{M} \in \mathscr{M}} c(\mathbf{M}) \int_{\sigma \in E_2(\mathbf{M})} \mu(\sigma) d(\sigma) \chi_\sigma(f) d\sigma. \end{aligned}$$

a) Fix  $\sigma \in \Sigma \in \bar{E}_2(\mathbf{G})$ . With  $\mathbf{M}$  equal to  $\mathbf{G}$  and  $f$  equal to a pseudo-coefficient of  $\sigma$  (Proposition 5.3 of [16]), i.e., such that  $\chi_\sigma(f) = 1$  but  $\chi_{\sigma'}(f) = 0$  if  $\sigma' \in E_l(\mathbf{G})$ , the set of equivalence classes of irreducible tempered representations of  $G$ , and  $\sigma' \neq \sigma$ , it is clear from (9.3.2) that  $d(\sigma)/m(\sigma)$  is a constant  $\lambda_{\mathbf{M}}(\Sigma)$ , depending only on  $\Sigma$ .

b) Fix  $\sigma \in \Sigma \in \overline{E}_2(M)$ . Let  $M^0$  be the kernel of  $H_p$ . Then  $M = M^0A$ , where  $A$  is the split component of the center of  $\mathbf{M}$ .

Fix a complex function  $h$  defined on the equivalence classes of discrete series representations of Levi subgroups of  $G$  satisfying:

- 1)  $h(\tau) = 0$  unless the Levi subgroup is conjugate to  $M$  and  $\tau$  is conjugate to  $\sigma_\nu$ ,  $\nu \in i\alpha^*$ ;
- 2)  $h$  depends only on the conjugacy class of  $M$ ; and finally
- 3) the map  $\nu \mapsto h(\sigma_\nu)$  is regular.

Let  $\sigma^0 = \sigma|M^0$  and denote by  $\omega_\sigma$  the central character of  $\sigma$ . By the abelian Paley-Wiener theorem, choose  $\eta \in C_c^\infty(A)$  such that

$$\int_A \eta(a)\omega_\sigma(a)q^{\langle \nu, H_p(a) \rangle} da = h(\sigma_\nu).$$

Let  $\varphi^0 \in C_c^\infty(M^0)$  be a pseudo-coefficient of  $\sigma^0$ . We may assume  $\eta(am) = \omega_\sigma^{-1}(m)\eta(a)$ ,  $m \in A \cap M^0$ , as well as  $\varphi^0(am) = \omega_\sigma^{-1}(a)\varphi^0(m)$ ,  $a \in A \cap M^0$ . Consequently  $\varphi = \varphi^0 \otimes \eta$  belongs to  $C_c^\infty(M)$  and

$$\chi_{\sigma_\nu}(\varphi) = h(\sigma_\nu),$$

while

$$\chi_\tau(\varphi) = 0$$

if  $\tau \neq \sigma_\nu$  for any  $\nu \in i\alpha^*$ . Thus we have the following Paley-Wiener result:

$$\chi_\tau(\tilde{\varphi}) = \chi_\tau(\varphi) = h(\tau)$$

for every equivalence class of discrete series representations  $\tau$  on a Levi subgroup of  $G$ , where  $\tilde{\varphi}(m) = \int_{K \cap M} \varphi(k^{-1}mk) dk$  with measure of  $K \cap M$  equal to 1.

Now choose  $f \in C_c^\infty(G)$  such that  $\tilde{\varphi} = \tilde{f}^P$ . Then  $\chi_{\tilde{\tau}}(f) = h(\tau)$  for every  $\tau \in E_2(M)$ ,  $\mathbf{M} \in \mathcal{M}$ . If  $\lambda_{\mathbf{M}}(\Sigma)\mu(\sigma) - \lambda_{\mathbf{G}}(\Sigma) \neq 0$ , then  $\lambda_{\mathbf{M}}(\Sigma_\nu)\mu(\sigma_\nu) - \lambda_{\mathbf{G}}(\Sigma_\nu) \neq 0$  for  $\nu$  in some open subset of  $i\alpha^*$ . Choose  $h$  such that

$$\int_{\nu \in i\alpha^*} (\lambda_{\mathbf{M}}(\Sigma_\nu)\mu(\sigma_\nu) - \lambda_{\mathbf{G}}(\Sigma_\nu))h(\nu) \neq 0.$$

But this contradicts (9.3.2), which completes the proof of the proposition.

*Remark.* Even though there is a Paley-Wiener theorem for representations induced from supercuspidal representations of  $p$ -adic groups [4], none has yet been stated in any form for tempered representations. This is the reason for being so careful in proving the proposition. I would like to thank Clozel and Rogawski for useful communications on this.

To continue, let  $\Pi$  be a finite set of equivalence classes of irreducible admissible representations of  $G$ . We shall say  $\Pi$  is *generic* if it contains an

element which is generic with respect to some generic character of  $U$ . Our second conjecture is:

CONJECTURE 9.4. *Every tempered  $L$ -packet is generic.*

When  $G$  is a real group, this was proved by Vogan in [48].

Observe that if  $\Pi$  is a tempered  $L$ -packet and  $\pi \in \Pi$ , then  $\pi \cdot \mathbf{Ad} g$  is also expected to belong to  $\Pi$ , if  $\mathbf{Ad} g$  is defined over  $F$ ,  $g \in \mathbf{G}(\bar{F})$ ; and therefore if we assume Conjecture 9.4, every tempered  $L$ -packet of  $M$  contains a  $\chi_0$ -generic element (cf. Section 3).

Now let  $\sigma$  be an irreducible admissible representation of  $M$ . Let  $\sigma = J(P', \nu, \sigma')$  be the corresponding Langlands quotient. Then  $\mu(\sigma) = \mu(\sigma'_\nu) \mu_M(\sigma'_\nu)^{-1}$ , where  $\sigma'$  is tempered and  $\mu_M(\sigma'_\nu)$  is the Plancherel measure for  $\sigma'_\nu$  and  $P'$  as a parabolic subgroup of  $M$ . In fact if we compose the standard intertwining operator  $A(\sigma'_\nu, w_l)$  induced from the one attached to the longest element  $\tilde{w}_l$  of the Weyl group of  $\mathbf{A}'$ , the split torus in the center of  $\mathbf{M}'$ , in  $\mathbf{M}$  modulo that of  $\mathbf{M}'$ , which sends  $J(P', \nu, \sigma')$  to the unique subrepresentation of  $I_M(w_l(\nu), w_l(\sigma'))$  with the operator  $A(\sigma'_\nu, w_0)$ , we obtain the operator  $A(\sigma'_\nu, w_l w_0)$ . Observe that by inducing in stages

$$A(\sigma'_\nu, w_0) | A(\sigma'_\nu, w_l) \operatorname{Ind}_{P' \uparrow G} J(P', \nu, \sigma') = A(\sigma, w_0).$$

The equality  $\mu(\sigma) = \mu(\sigma'_\nu) \mu_M(\sigma'_\nu)^{-1}$  is now clear.

By ranging  $\sigma'$  in its  $L$ -packet  $\Sigma'$ ,  $\{J(P', \nu, \sigma'), \sigma' \in \Sigma'\}$  then becomes the  $L$ -packet of  $\sigma$ . Assuming Conjecture 9.4, we see there exists a  $\chi_0$ -generic representation  $\bar{\sigma}'$  in the  $L$ -packet  $\Sigma'$  of  $\sigma'$  and therefore by Proposition 9.3,  $\mu(\sigma) = \mu(\sigma'_\nu) \cdot \mu_M(\bar{\sigma}'_\nu)^{-1}$ . Let  $\bar{\sigma} = J(P', \nu, \bar{\sigma}')$ . We remark that  $\bar{\sigma}$  may not be generic. Define  $L(s, \bar{\sigma}, r_i)$ ,  $1 \leq i \leq m$ , by analytic continuation of the corresponding  $L$ -function for  $\bar{\sigma}'$  which was defined in Section 7. Similarly define the root number  $\varepsilon(s, \bar{\sigma}, r_i, \psi_F)$ . We then set

$$L(s, \sigma, r_i) = L(s, \bar{\sigma}, r_i),$$

and

$$\varepsilon(s, \sigma, r_i, \psi_F) = \varepsilon(s, \bar{\sigma}, r_i, \psi_F),$$

$1 \leq i \leq m$ , for every  $\sigma \in \Sigma$ . If  $\sigma$  is unitary, they then satisfy Proposition 7.8. We shall now normalize the intertwining operator  $A(\sigma, w)$  as in (7.12) to get  $\mathcal{A}(\sigma, w)$ . The equality  $\mu(\sigma) = \mu(\sigma'_\nu) \mu_M(\bar{\sigma}'_\nu)^{-1}$  for each rank-one component of  $A(\sigma, w)$ , Corollary 3.6, and equations (7.10) and (7.11) now imply:

THEOREM 9.5. *Assume Conjectures 9.2 and 9.4 are valid and  $\sigma$  is unitary. Then the normalized operator  $\mathcal{A}(\sigma, w)$  satisfies conditions a and b of Theorem 7.9.*

The rest of this section is aimed at proving an inductive result (Proposition 9.6) in the direction of Conjecture 9.4. It reduces the conjecture to discrete series  $L$ -packets with only one element in them (these are expected to be the only ones which are not endoscopic transfers from a non-trivial endoscopic group).

Let  $\mathbf{H}$  be a quasi-split group, endoscopic to  $\mathbf{G}$ , both defined over a non-archimedean field  $F$  of characteristic zero. (The referee has informed me that the condition  $\text{char } \bar{F} \neq 2$ , where  $\bar{F}$  is the residue field of  $F$ , is not necessary for the purposes of this paper.)

**PROPOSITION 9.6.** *Suppose the map  $f \mapsto f^H$  exists. Let  $\Sigma$  be a tempered  $L$ -packet of  $H$ . Denote by  $\Pi$  the  $L$ -packet of  $G$  obtained from  $\Sigma$  by endoscopic transfer. Assume  $\Sigma$  is generic. Then so is  $\Pi$ .*

Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be the Lie algebras of  $G$  and  $H$ , respectively. Set  $n = \dim \mathfrak{g}$  and  $h = \dim \mathfrak{h}$ . Let  $f \in C_c^\infty(G)$  be a function supported around the origin. Moreover, assume that on the support of  $f$ , the inverse of  $\exp: \mathfrak{g} \rightarrow G$  is defined. Take  $\tilde{f} \in C_c^\infty(\mathfrak{g})$  such that  $\tilde{f}(X) = f(\exp X)$ ,  $X \in \mathfrak{g}$ . Given a function  $h \in C_c^\infty(\mathfrak{g})$  and  $t \in F^*$ , let

$$h_t(X) = h(t^{-1}X) \quad (X \in \mathfrak{g}).$$

Assume  $|t|$  is small and define  $f_t$  by  $\tilde{f}_t = \tilde{f} \cdot \exp$ . If  $f^H \in C_c^\infty(H)$  is defined by means of  $f$  and  $f$  is of sufficiently small support, we may assume the same is true for  $f^H$ . Let  $(f^H)_t$  be the corresponding function for  $f^H$ .

Now, let  $\Delta$  be the transfer factor defined by Langlands and Shelstad in [32]. We need:

**LEMMA 9.7.** *Suppose  $f$  has sufficiently small support around the origin. Then for  $|t|$  small,  $f_{t^2}$  and*

$$|t|^{n-h}(f^H)_{t^2}$$

*have  $\Delta$ -matching orbital integrals.*

*Proof.* Let  $\gamma_H$  and  $\gamma_G$  be strongly  $\mathbf{G}$ -regular and regular elements of  $H$  and  $G$ , respectively, which for the purpose of this lemma we may assume to lie in small neighborhoods of origins in  $H$  and  $G$ . The transfer factor  $\Delta(\gamma_H, \gamma_G)$  is a product of five factors which in the notation of [32] are denoted by  $\Delta_I, \Delta_{II}, \Delta_1, \Delta_2$ , and  $\Delta_{IV}$ . If  $\mathbf{T}_H$  is the centralizer of  $\gamma_H$  in  $\mathbf{H}$ , we fix an admissible embedding  $\mathbf{T}_H \rightarrow \mathbf{T} \subset \mathbf{G}$  (cf. [32], [22]) and let  $\gamma$  be the image of  $\gamma_H$  under this embedding. If  $\gamma_H = \exp(X)$ ,  $X \in \mathfrak{h}$ , we then use  $t^{-2}\gamma_H$  to denote  $\exp(t^{-2}X)$ , where throughout this proof we shall assume  $|t|$  is small. Clearly  $\mathbf{T}_H$  is also the centralizer of  $t^{-2}\gamma_H$  in  $\mathbf{H}$ . Moreover if  $\{\gamma'_H\}$  is a set of representatives for

conjugacy classes inside the stable conjugacy class of  $\gamma_H$ , then so is  $\{t^{-2}\gamma'_H\}$  for  $t^{-2}\gamma_H$ .

It is now clear that

$$\Delta_I(t^{-2}\gamma_H, t^{-2}\gamma_G) = \Delta_I(\gamma_H, \gamma_G).$$

If  $h \in \mathbf{G}(\bar{F})$  is such that  $h\gamma_G h^{-1} = \gamma$ , then the same  $h$  can be used to define  $\Delta_I(t^{-2}\gamma_H, t^{-2}\gamma_G)$  and therefore  $\Delta_I$  will remain unchanged.

Next consider  $\Delta_{II}(t^{-2}\gamma_H, t^{-2}\gamma_G)$ . Then

$$\Delta_{II}(t^{-2}\gamma_H, t^{-2}\gamma_G) = \prod_{\alpha} \chi_{\alpha}(t^{-2}) \Delta_{II}(\gamma_H, \gamma_G),$$

where  $\{\chi_{\alpha}\}$  is a  $\chi$ -data for  $\Gamma = \text{Gal}(\bar{F}/F)$  and  $R = R(\mathbf{T}, \mathbf{G})$  (notation as in §2.5 of [32]), and the product is over representatives  $\alpha$  for orbits of  $\Gamma$  in the roots of  $\mathbf{T}$  outside  $\mathbf{H}$ . We now choose our  $\chi$ -data such that  $\chi_{\alpha} \equiv 1$  if  $F_{+\alpha} = F_{\pm\alpha}$ . Otherwise, i.e., if  $[F_{+\alpha} : F_{\pm\alpha}] = 2$ , then  $t^{-2}$  is a norm and  $\chi_{\alpha}(t^{-2}) = 1$  by condition (iii) of subsection 2.5 of [32]. Then with this  $\chi$ -data,

$$\Delta_{II}(t^{-2}\gamma_H, t^{-2}\gamma_G) = \Delta_{II}(\gamma_H, \gamma_G).$$

Fix this  $\chi$ -data to define  $\Delta_2$  (cf. §3.5 of [32]). Reducing the supports of  $f$  and  $f^H$ , if necessary, we may assume  $\Delta_2(\gamma_H, \gamma_G) = 1$  (it is the value of a certain character of  $T$ , depending on the  $\chi$ -data, at  $\gamma$ , where  $\gamma$  is the image of  $\gamma_H$  as above). Choosing  $|t|$  small enough we may assume  $\Delta_2(t^{-2}\gamma_H, t^{-2}\gamma_G) = 1$ .

Observe that (§3.6 of [32])

$$\Delta_{IV}(t^{-2}\gamma_H, t^{-2}\gamma_G) = |t|^{-(n-h)} \Delta_{IV}(\gamma_H, \gamma_G).$$

Now it follows from the definition of  $\Delta(\gamma_H, \gamma_G)$  that for functions  $f$  and  $f^H$  with small supports around the origins of  $G$  and  $H$  respectively, and  $|t|$  small,

$$\Delta(t^{-2}\gamma_H, t^{-2}\gamma_G) = |t|^{-(n-h)} \Delta(\gamma_H, \gamma_G).$$

Finally observe that

$$\begin{aligned} \Phi^{\text{st}}(\gamma_H, (f^H)_{t^2}) &= \sum_{\gamma} \Delta(t^{-2}\gamma_H, t^{-2}\gamma) \Phi(\gamma, f_{t^2}) \\ &= |t|^{-(n-h)} \sum_{\gamma} \Delta(\gamma_H, \gamma) \Phi(\gamma, f_{t^2}), \end{aligned}$$

which completes the proof of the lemma.

To prove the proposition we shall need the following two results.

If  $\pi$  is an irreducible admissible representation of any  $p$ -adic group  $G$ , then by Harish-Chandra [13], the character  $\chi_{\pi}$  of  $\pi$  can be expanded in a sufficiently small neighborhood  $U$  of the origin as follows:

$$(9.6.1) \quad \chi_{\pi}(f) = \sum_{\mathfrak{o}} c_{\mathfrak{o}}(\pi) \hat{\mu}_{\mathfrak{o}}(\tilde{f}),$$

where  $f \in C_c^\infty(U)$  and  $f = \tilde{f} \cdot \exp$ ,  $\tilde{f} \in C_c^\infty(\mathfrak{g})$ . The sum is over all the nilpotent  $G$ -orbits (conjugacy classes)  $\mathcal{O}$  of  $\mathfrak{g}$  and  $\hat{\mu}_\mathcal{O}$  is the Fourier transform of  $\mu_\mathcal{O}$ , where

$$\mu_\mathcal{O}(\tilde{f}) = \int_\mathcal{O} \tilde{f}(g) dg.$$

The second result is an important result of Rodier [34], generalized by Mœglin and Waldspurger [33], that  $\pi$  is generic (which requires  $G$  to be  $F$ -points of a quasi-split  $F$ -group) if and only if  $c_\mathcal{O}(\pi) \neq 0$  for some maximal (regular) nilpotent  $G$ -orbit  $\mathcal{O}$  of  $\mathfrak{g}$ . This is a remarkable characterization of generic representations in terms of their characters which gives them a new meaning.

We now proceed to prove Proposition 9.6.

Given  $\pi \in \Pi$ , let  $c(\pi) \in \mathbf{C}^*$  be such that

$$\begin{aligned} (9.6.2) \quad \chi_\Sigma(f^H) &= \sum_{\sigma \in \Sigma} m(\sigma) \chi_\sigma(f^H) \\ &= \sum_{\pi \in \Pi} c(\pi) \chi_\pi(f). \end{aligned}$$

Let

$$c_\mathcal{O}(\Pi) = \sum_{\pi \in \Pi} c(\pi) c_\mathcal{O}(\pi)$$

and

$$c_{\mathcal{O}_H}(\Sigma) = \sum_{\sigma \in \Sigma} m(\sigma) c_{\mathcal{O}_H}(\sigma),$$

where for nilpotent orbits  $\mathcal{O}$  and  $\mathcal{O}_H$  of  $G$  and  $H$ ,  $c_\mathcal{O}(\pi)$  and  $c_{\mathcal{O}_H}(\sigma)$  denote the coefficients in the expansion (9.6.1) of the characters of  $\pi$  and  $\sigma$ , respectively. Now suppose  $f \mapsto f^H$  and they both have small supports near the origins of  $G$  and  $H$ , respectively. Take  $t \in F^*$  with  $|t|$  small. Then by Lemma 9.7,  $f_{t^2} \mapsto |t|^{n-h}(f^H)_{t^2}$ ; applying (9.6.1) and (9.6.2) to this pair in small neighborhoods of origins in  $G$  and  $H$ , respectively, we have

$$\sum_\mathcal{O} c_\mathcal{O}(\Pi) \hat{\mu}_\mathcal{O}(f_{t^2}) = \sum_{\mathcal{O}_H} c_{\mathcal{O}_H}(\Sigma) |t|^{n-h} \hat{\mu}_{\mathcal{O}_H}((f^H)_{t^2}).$$

But it is easily seen that if  $d(\mathcal{O})$  and  $d(\mathcal{O}_H)$  denote the dimensions of  $\mathcal{O}$  and  $\mathcal{O}_H$ , respectively, then

$$\hat{\mu}_\mathcal{O}(f_{t^2}) = |t|^{2n-d(\mathcal{O})} \hat{\mu}_\mathcal{O}(f)$$

and

$$\hat{\mu}_{\mathcal{O}_H}((f^H)_{t^2}) = |t|^{2h-d(\mathcal{O}_H)} \hat{\mu}_{\mathcal{O}_H}(f^H).$$

Consequently

$$(9.6.3) \quad \sum_{\mathcal{O}} c_{\mathcal{O}}(\Pi) |t|^{2n-d(\mathcal{O})} \hat{\mu}_{\mathcal{O}}(\tilde{f}) = \sum_{\mathcal{O}_H} c_{\mathcal{O}_H}(\Sigma) |t|^{n+h-d(\mathcal{O}_H)} \hat{\mu}_{\mathcal{O}_H}(\tilde{f}^H).$$

Let  $l = \text{rank } \mathbf{T}_H = \text{rank } \mathbf{T}_G$ . Then  $d(\mathcal{O}_{\text{Max}}) = n - l$ , while  $d(\mathcal{O}_{H, \text{Max}}) = h - l$ , for maximal nilpotent  $G$  and  $H$ -orbits, respectively. Thus (9.6.3) implies

$$\sum_{\mathcal{O} \text{ maximal}} c_{\mathcal{O}}(\Pi) \hat{\mu}_{\mathcal{O}}(\tilde{f}) = \sum_{\mathcal{O}_H \text{ maximal}} c_{\mathcal{O}_H}(\Sigma) \hat{\mu}_{\mathcal{O}_H}(\tilde{f}^H).$$

Now assume  $c_{\mathcal{O}}(\pi) = 0$  for every maximal nilpotent  $G$ -orbit  $\mathcal{O}$  and every  $\pi \in \Pi$ . Then

$$(9.6.4) \quad \sum_{\mathcal{O}_H \text{ maximal}} c_{\mathcal{O}_H}(\Sigma) \hat{\mu}_{\mathcal{O}_H}(\tilde{f}^H) = 0.$$

Observe that  $m(\sigma) > 0$ ,  $c_{\mathcal{O}_H}(\sigma) \geq 0$ , and  $c_{\mathcal{O}_H}(\sigma) > 0$  for at least some maximal  $\mathcal{O}_H$  [33]. Thus we have  $c_{\mathcal{O}_H}(\Sigma) \neq 0$  for some maximal  $\mathcal{O}_H$ . To conclude we need the following lemma.

LEMMA 9.8. *Let  $C_H$  be a fixed regular nilpotent  $H$ -orbit of  $\mathfrak{h}$ . Then there exists a function  $f^H \in C_c^\infty(H)$  with arbitrarily small support which lies in the image of the map  $f \mapsto f^H$  such that*

$$\hat{\mu}_{C_H}(f^H) = 1,$$

while

$$\hat{\mu}_{\mathcal{O}_H}(f^H) = 0$$

for any other nilpotent  $H$ -orbit  $\mathcal{O}_H$  of  $\mathfrak{h}$ .

*Proof.* We need to choose a function  $h \in C_c^\infty(H)$  with small support such that  $\mu_{\mathcal{O}_H}(h) = 0$  for every nilpotent orbit  $\mathcal{O}_H$ ,  $\hat{\mu}_{\mathcal{O}_H}(\tilde{h}) = 0$  for all nilpotent orbits  $\mathcal{O}_H \neq C_H$ , and finally  $\hat{\mu}_{C_H}(\tilde{h}) = 1$ . Such an  $h$  is then in the image of  $f \mapsto f^H$ . This results from the following lemma.

LEMMA 9.9. *Let  $V$  be an open subset of  $\mathfrak{h}$  containing the origin. Then the distributions  $\{\mu_{\mathcal{O}_H}\}$  and  $\{\hat{\mu}_{\mathcal{O}_H}\}$ , where  $\mathcal{O}_H$  ranges over nilpotent  $H$ -orbits of  $\mathfrak{h}$ , are linearly independent on  $C_c^\infty(V)$ .*

*Proof.* For simplicity we drop the index  $H$  and use  $\{\mathcal{O}_i\}$  to denote these orbits. Assume

$$(9.9.1) \quad \sum_{\mathcal{O}_j} d_{\mathcal{O}_j} \hat{\mu}_{\mathcal{O}_j} = \sum_{\mathcal{O}_i} c_{\mathcal{O}_i} \mu_{\mathcal{O}_i}$$

on  $C_c^\infty(V)$ , where  $c_{\mathcal{O}_i}$  and  $d_{\mathcal{O}_j}$  are complex numbers. Take  $f \in C_c^\infty(V)$  and

choose  $t \in F^*$  with  $|t|$  small enough so that  $f_{t^2} \in C_c^\infty(V)$ . Then applying (9.9.1) to  $f_{t^2}$  we have:

$$\sum_{\mathcal{O}_j} d_{\mathcal{O}_j} |t|^{2h-d(\mathcal{O}_j)} \hat{\mu}_{\mathcal{O}_j}(f) = \sum_{\mathcal{O}_i} c_{\mathcal{O}_i} |t|^{d(\mathcal{O}_i)} \mu_{\mathcal{O}_i}(f).$$

Fix a non-negative integer  $m$ . Let

$$\tilde{\mathcal{O}}_m = \{\mathcal{O} \in \{\mathcal{O}_i\} | d(\mathcal{O}) = m\}.$$

Assume  $\tilde{\mathcal{O}}_m$  is not empty. Then

$$\begin{aligned} \sum_{\mathcal{O}_i} c_{\mathcal{O}_i} |t|^{d(\mathcal{O}_i)-(2h-d(\mathcal{O}))} \mu_{\mathcal{O}_i}(f) - \sum_{\mathcal{O}_j \notin \tilde{\mathcal{O}}_m} d_{\mathcal{O}_j} |t|^{m-d(\mathcal{O}_j)} \hat{\mu}_{\mathcal{O}_j}(f) \\ - \sum_{\mathcal{O} \in \tilde{\mathcal{O}}_m} d_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f) = 0. \end{aligned}$$

Observe that  $d(\mathcal{O}_i) - (2h - d(\mathcal{O})) < 0$ . Starting with the largest possible  $m$  so that  $m - d(\mathcal{O}_j) > 0$  for  $\mathcal{O}_j \notin \tilde{\mathcal{O}}_m$ , we see the linear independence of characters of  $F^*$  now implies inductively that

$$\sum_{\mathcal{O} \in \tilde{\mathcal{O}}_m} d_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f) = 0$$

for every  $f \in C_c^\infty(V)$  and every  $m$ . Thus  $d_{\mathcal{O}} = 0$  for every  $\mathcal{O} \in \tilde{\mathcal{O}}_m$  by Theorem 4 of [13] and consequently  $d_{\mathcal{O}_j} = 0$  for all  $j$ . Finally since the origin is in the closure of every nilpotent orbit, the observations before Lemma 4 of [13] imply  $c_{\mathcal{O}_i} = 0$  for all  $i$ , which completes the proof of the lemma.

When we choose  $f^H$  as in Lemma 9.8, it is clear that (9.6.4) cannot happen, which implies the existence of a  $\pi \in \Pi$  with  $c_{\mathcal{O}}(\pi) \neq 0$  for some regular unipotent orbit  $\mathcal{O}$ . This  $\pi$  is then generic by [33], [34].

**COROLLARY 9.10.** *Assume  $\Pi$  is a discrete series L-packet and  $h < n$ . For every  $\pi \in \Pi$ , let  $d(\pi)$  be its formal degree. Then*

$$\sum_{\pi \in \Pi} c(\pi) d(\pi) = 0.$$

*In particular, assuming Conjecture 9.2,*

$$\sum_{\pi \in \Pi} c(\pi) m(\pi) = 0.$$

*Proof.* If  $d(\mathcal{O}) = 0$  in which case  $\mathcal{O}$  is unique, then the term

$$c_{\mathcal{O}}(\Pi) |t|^{2n} \hat{\mu}_{\mathcal{O}}(\tilde{f})$$

is the only term with  $|t|^{2n}$  appearing in (9.6.3). Thus  $c_{\mathcal{O}}(\Pi) = 0$ . But by Theorem 6 of [13], there exists an absolute constant  $c$  such that  $c_{\mathcal{O}}(\pi) = cd(\pi)$

from which the corollary follows. We should remark that the proof given in [13], even though only for supercuspidal representations, applies to any discrete series if one uses the corresponding pseudo-coefficient to determine its character on the elliptic set (Part b of Proposition 5.3 of [16]).

*Remark.* This result can be used to determine  $c(\pi)$  (cf. [35]).

**COROLLARY 9.11.** *Assume  $\Pi$  is a discrete series  $L$ -packet which consists of only one element. Then  $\Pi$  cannot be an endoscopic transfer from any endoscopic group attached to  $\mathbf{G}$  other than  $\mathbf{G}$  itself.*

*Remark.* This is expected to hold in the opposite direction as well; i.e., if a discrete series  $L$ -packet is not an endoscopic transfer, then it must be a singleton.

PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA

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