

TWISTED ENDOSCOPY AND REDUCIBILITY OF INDUCED REPRESENTATIONS FOR p -ADIC GROUPS

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1. Introduction. This is the first in a series of papers in which we study the reducibility of representations induced from discrete series representations of the Levi factors of maximal parabolic subgroups of p -adic groups on the unitary axis. The problem is equivalent to determining certain local Langlands L -functions which are of arithmetic significance by themselves. One of the main aims of this paper is to interpret our results in the direction of the parametrization problem by means of the theory of twisted endoscopy for which our method seems to be very suitable. When the representation is supercuspidal, we also study the reducibility of the induced representations off the unitary axis.

Let F be a p -adic field of characteristic zero. Fix a positive integer $n > 1$ and let \mathbf{G} be either of the groups Sp_{2n} , SO_{2n} , or SO_{2n+1} . In all three cases there is a conjugacy class of maximal parabolic subgroups which has GL_n as its Levi factor. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be the standard parabolic subgroup of \mathbf{G} in this conjugacy class. Then $\mathbf{M} \cong GL_n$. Let σ be a discrete series representation of $M = GL_n(F)$ and, given $s \in \mathbb{C}$, let $I(s, \sigma)$ be the representation of $G = \mathbf{G}(F)$ induced from $\sigma \otimes |\det(\)|^s$. Let $I(\sigma) = I(0, \sigma)$.

Assume first that σ is supercuspidal. Let $A(s, \sigma, w_0)$ be the standard intertwining operator defined by equation (2.1) of Section 2, where w_0 is a representative for \tilde{w}_0 , the longest element in the Weyl group of \mathbf{G} . (See Section 2.) From the general theory of c -functions and R -groups (see [31, 23, 24, 16]), it is clear that $I(\sigma)$ is reducible if and only if $\tilde{w}_0(\sigma) \cong \sigma$ and $A(s, \sigma, w_0)$ is holomorphic at $s = 0$. By the properties of intertwining operators, there exists a unique polynomial $P_\sigma(q^{-s})$ satisfying $P_\sigma(0) = 1$, such that the operator

$$(1.1) \quad P_\sigma(q^{-2s})A(s, \sigma, w_0)$$

is holomorphic and nonzero. Thus if $\tilde{w}_0(\sigma) \cong \sigma$, then $I(\sigma)$ is irreducible if and only if $P_\sigma(1) = 0$.

On the other hand, let $\varphi: W_F \rightarrow GL_n(\mathbb{C})$ be the conjectural representation of the Weil group W_F of F/F , parametrizing σ . (See [4, 33].) Then from the general theory developed in [23] we must conjecturally have

$$L(s, \wedge^2 \rho_n \cdot \varphi) = P_\sigma(q^{-s})^{-1}$$

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where P_σ is the normalized polynomial which makes (1.1) holomorphic and nonzero for $\mathbf{G} = SO_{2n}$ or Sp_{2n} and where the L -function on the left is the corresponding Artin L -function [20, 33]. Here, $\wedge^2 \rho_n$ is the irreducible representation of $GL_n(\mathbb{C})$, the L -group of GL_n , on the space of alternating tensors of rank 2. (See Section 2.) The fact that the polynomial P_σ works for both SO_{2n} and Sp_{2n} follows from the general theory developed in [23].

Now if $P_\sigma(1) = 0$, then $L(s, \wedge^2 \rho_n \cdot \varphi)$ must have a pole at $s = 0$, and consequently the trivial representation of W_F must appear in $\wedge^2 \rho_n \cdot \varphi$. It then follows that $\text{Im}(\varphi)$ must fix a bilinear alternating form in n variables. Since φ is expected to be irreducible, the bilinear form must be nondegenerate. Thus, n must be even and φ must factor through $Sp_n(\mathbb{C})$. (See Conjecture 3.2.)

The theory of twisted endoscopy, which is now being developed by Kottwitz and Shelstad [18] (see Sections 3 and 7 here, and [2]) in general, is designed exactly to detect such homomorphisms. In fact, when n is even, the group $\mathbf{H} = SO_{n+1}$ is one of the twisted endoscopic groups attached to GL_n and self-dual representations σ for which $I(\sigma)$ are irreducible must then be exactly those for which φ factors through $Sp_n(\mathbb{C}) = {}^L H^0$, the connected component of the L -group of \mathbf{H} . In particular, for an irreducible *self-dual* supercuspidal representation σ of $GL_n(F)$, the induced representation $I(\sigma)$ of $Sp_{2n}(F)$ must always be reducible when n is odd. (See the remark after Proposition 3.5.)

If the theory of twisted endoscopy holds, those self-dual representations of $GL_n(F)$ which come from $SO_{n+1}(F)$ must be detected by means of the dual of a certain map $f \mapsto f^{\mathbf{H}}$, $f \in C_c^\infty(GL_n(F))$, $f^{\mathbf{H}} \in C_c^\infty(SO_{n+1}(F))$ defined through the matching of stable twisted orbital integrals (Section 7) on $GL_n(F)$ with stable ordinary orbital integrals on $SO_{n+1}(F)$ (Assumption 7.2). It is then natural to say that an irreducible supercuspidal self-dual representation σ of $GL_n(F)$ comes from $SO_{n+1}(F)$ if and only if $f^{\mathbf{H}}(e) \neq 0$ for some pseudocoefficient (by abuse of terminology any function $f \in C_c^\infty(GL_n(F))$ which defines a matrix coefficient of σ) f of σ (Definition 7.4 and the remark after it). One of the main results of this paper (Part b of Theorem 7.6) can be stated as the following theorem.

THEOREM 1.1. *Let $\mathbf{G} = Sp_{2n}$ and fix an irreducible (unitary) self-dual supercuspidal representation σ of $M = GL_n(F)$. Then $I(\sigma)$ is irreducible if and only if σ comes by twisted endoscopic transfer from a tempered L -packet of $SO_{n+1}(F)$. In particular, $I(\sigma)$ is reducible if n is odd.*

Parts (a) and (c) of Theorem 7.6 state similar results for $\mathbf{G} = SO_{2n+1}$ and $\mathbf{G} = SO_{2n}$, respectively.

Theorem 7.6 and in fact the complete reducibility criterion for $I(s, \sigma)$ are consequences of the general theory of R -groups, i.e., the determination of the polynomial $P_\sigma(q^{-s})$, which makes (1.1) nonzero and holomorphic. It turns out that it is best to compute P_σ , or equivalently the local Langlands L -function $L(s, \sigma, \wedge^2 \rho_n)$, using the group $SO_{2n}(F)$ (Proposition 5.1).

Reducibility criteria which are eventually interpreted as Theorem 7.6 are stated as Propositions 3.5 and 3.10 and Theorems 3.3, 5.3, and 6.3, in a way completely

independent of the theory of twisted endoscopy although the residue of $A(s, \sigma, w_0)$ for $SO_{2n}(F)$ at $s = 0$ is proportional to a twisted orbital integral on GL_n .

Our results generalize to discrete series and are stated as Theorem 9.1 and its corollaries. In particular, Corollary 9.3 states that the reducibility criteria for $SO_{2n+1}(F)$ and $Sp_{2n}(F)$ are dual to each other. More precisely, we have the following theorem.

THEOREM 1.2. *Let $I(\sigma)$ and $I(\sigma')$ be the representations of $Sp_{2n}(F)$ and $SO_{2n+1}(F)$ induced from a self-dual discrete series representation σ of their $GL_n(F)$ -Levi subgroups. Then $I(\sigma')$ is reducible if and only if $I(\sigma)$ is irreducible.*

Theorem 1.2 is a consequence of an important lemma (Lemma 3.6) whose use makes the treatment of the groups $SO_{2n+1}(F)$ and $Sp_{2n}(F)$, or equivalently the determination of local Langlands L -functions $L(s, \sigma, \text{Sym}^2(\rho_n))$ and $L(s, \sigma, \wedge^2 \rho_n)$, simultaneously possible, drastically reducing our original approach and reliance on unpublished results. (See Section 10.)

The results on L -functions are stated as Theorems 6.1 and 6.2 and Proposition 8.1. They completely determine local Langlands L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$ at every place, including the ramified ones. Although their product is the Rankin-Selberg L -function $L(s, \sigma \times \sigma)$ which is usually easy to compute (Lemma 3.6), it is how $L(s, \sigma \times \sigma)$ decomposes to their product which is delicate and requires the use of the theory of twisted endoscopy if one attempts to interpret them as Artin L -functions [20, 33].

In terms of representation theory, the simplicity of $L(s, \sigma \times \sigma)$ is equivalent to the simplicity of the irreducibility criterion for the representation of $GL_{2n}(F)$ induced from $\sigma \otimes \sigma$ on the Levi subgroup $GL_n(F) \times GL_n(F)$. It is always irreducible exactly because the L -function $L(s, \sigma \times \bar{\sigma})$ which gives the poles of the corresponding intertwining operator always has a pole at $s = 0$ if σ is in the discrete series. Consequently, the results of this paper may be looked at as a measure of how much deeper the reducibility criteria for other groups could be, as soon as one considers groups other than GL_n . Surprisingly enough, the reducibility criterion for Sp_{2n} turns out to be equivalent to nonvanishing of the integral of a certain matrix coefficient over the non-Riemannian symmetric space $Z_n(F)Sp_n(F) \backslash GL_n(F)$ when n is even (Criterion 5.2 and Remarks 2 and 3 of Section 5 for comparison with GL_n). This may have global implications.

We should remark that global integral representations for both of these L -functions are now available [6, 7, 13]. Whether their local definitions agree with those given here remains to be seen. The results in [7] follow the lead of Shimura, and Gelbart and Jacquet (for $n = 2$), and Patterson and Piatetski-Shapiro (for $n = 3$).

The last section of the paper is devoted to direct calculations of intertwining operators for $Sp_{2n}(F)$ and $SO_{2n+1}(F)$. Using the results of previous sections, they lead to certain nonvanishing statements about twisted orbital integrals, both semisimple and unipotent (Propositions 10.1 and 10.5 and their corollaries). The case of $SO_{2n+1}(F)$ is particularly interesting.

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2. Preliminaries. Let F be a nonarchimedean field of characteristic zero. If O and P are the ring of integers of F and its maximal ideal, respectively, we let q be the number of elements in the residue field of O/P . We use $|\cdot| = |\cdot|_F$ to denote the absolute value in F for which a prime element ϖ has $|\varpi|_F = q^{-1}$.

Fix a positive integer $n > 1$. In this paper \mathbf{G} will be one of the groups SO_{2n+1} , Sp_{2n} , or SO_{2n} . (For the purposes of this paper the cases of the groups SO_3 , Sp_2 , and SO_2 are well known). More precisely, we let

$$J_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \in GL_{2n+1},$$

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in GL_{2n},$$

or

$$J_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n},$$

according to whether $\mathbf{G} = SO_{2n+1}$, Sp_{2n} , or SO_{2n} , respectively. In each case we let \mathbf{G} be the connected component of

$$\{g \in GL_m \mid {}^t g J_0 g = J_0\}$$

where $m = 2n + 1$ in the first case and $m = 2n$, otherwise.

Once and for all, we shall fix a Borel subgroup \mathbf{B} for \mathbf{G} . For simplicity we do this by identifying the roots in the unipotent radical of the subgroup of upper triangular matrices in GL_m in the usual manner. Finally, we use \mathbf{T} to denote the Cartan subgroup of \mathbf{G} contained in the subgroup of diagonal matrices in GL_m . Clearly, $\mathbf{T} \subset \mathbf{B}$ and $\mathbf{B} = \mathbf{T}\mathbf{U}$. A parabolic subgroup \mathbf{P} of \mathbf{G} is standard if $\mathbf{P} \supset \mathbf{B}$. We shall frequently call $P = \mathbf{P}(F)$ as a parabolic subgroup of $G = \mathbf{G}(F)$, the F -rational points of \mathbf{G} .

In each case we let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be the standard parabolic subgroup of \mathbf{G} whose Levi subgroup is generated by the simple roots $\alpha_1, \dots, \alpha_{n-1}$; i.e., $\mathbf{M} \cong GL_n$.

In every case let σ be an irreducible admissible representation of $M = GL_n(F)$. Fix a complex number $s \in \mathbb{C}$ and let

$$I(s, \sigma) = \text{Ind}_{MN \uparrow G} (\sigma \otimes |\det(\)|^s) \otimes 1.$$

If $\alpha = \alpha_n$ denotes the remaining simple root of \mathbf{G} (i.e., not in \mathbf{M}) in each case, then in the notation of [23] this is $I(2s\tilde{\alpha}, \sigma)$, where $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$ in which ρ is half the sum of simple roots in \mathbf{N} (see [23]), unless $\mathbf{G} = Sp_{2n}$ for which $I(s, \sigma) = I(s\tilde{\alpha}, \sigma)$. It is useful to record that $\langle \rho, \alpha \rangle = (n+1)/2$, $n-1$, or n , according to whether $\mathbf{G} = Sp_{2n}$, SO_{2n} , or SO_{2n+1} , respectively. We use $V(s, \sigma)$ to denote the space of $I(s, \sigma)$. Finally, set $I(\sigma) = I(0, \sigma)$ and $V(\sigma) = V(0, \sigma)$.

Let \mathbf{T} be the standard Cartan subgroup of \mathbf{G} fixed before. Then \mathbf{T} is also a Cartan subgroup for \mathbf{M} . Denote by $W(\mathbf{T})$ the Weyl group of \mathbf{T} in \mathbf{G} . Let \tilde{w}_0 be the longest element in $W(\mathbf{T})$. We use w_0 to denote a fixed representative for \tilde{w}_0 in G . Throughout this paper we shall reserve the right of changing \tilde{w}_0 by an element in the Weyl group of \mathbf{T} in \mathbf{M} as we feel necessary. It will have no effect on our results.

Given $h \in V(s, \sigma)$, let

$$(2.1) \quad A(s, \sigma, w_0)h(g) = \int_N h(w_0^{-1}ng) dn,$$

$Re(s) \gg 0$, $g \in G$, be the standard intertwining operator, sending $I(s, \sigma) = I(ls\tilde{\alpha}, \sigma)$ to $I(\tilde{w}_0(ls\tilde{\alpha}), \tilde{w}_0(\sigma))$, $l = 1$ or 2 accordingly (notation as in [23]). Given $h \in V(s, \sigma)$ and $g \in G$, $A(s, \sigma, w_0)h(g)$ defines a rational function of q^{-s} .

Let ${}^L M = GL_n(\mathbb{C})$ be the L -group of M . Denote by r the adjoint action of ${}^L M$ on the Lie algebra ${}^L \mathfrak{n}$ of ${}^L N$, the L -group of \mathbf{N} . (See [4, 19, 23].) In what follows we shall order the irreducible constituents r_i of r according to the ordering in [23].

Let ρ_n be the standard representation of $GL_n(\mathbb{C})$. Then for $\mathbf{G} = SO_{2n+1}$, $r = r_1 = \text{Sym}^2(\rho_n)$, the irreducible $\frac{1}{2}n(n+1)$ -dimensional representation of $GL_n(\mathbb{C})$ on the space of symmetric tensors of rank 2.

For $\mathbf{G} = Sp_{2n}$, $r = r_1 \oplus r_2$ with $r_1 = \rho_n$ and $r_2 = \wedge^2 \rho_n$, the irreducible $\frac{1}{2}n(n-1)$ -dimensional representation of $GL_n(\mathbb{C})$ on the space $\wedge^2 \mathbb{C}^n$ of alternating tensors of rank 2.

Finally, if $\mathbf{G} = SO_{2n}$, then $r = r_1 = \wedge^2 \rho_n$.

Observe that

$$(2.2) \quad \rho_n \otimes \rho_n = \wedge^2 \rho_n \oplus \text{Sym}^2(\rho_n).$$

Suppose σ has a vector fixed by $GL_n(O)$. Then there exists n unramified characters (not necessarily unitary) μ_1, \dots, μ_n of F^* such that $\sigma \subset \text{Ind}_{(B \cap M) \uparrow M} (\mu_1 \otimes \dots \otimes \mu_n) \otimes 1$.

Let A_σ be the (semisimple) conjugacy class of the matrix $\text{diag}(\mu_1(\varpi), \dots, \mu_n(\varpi))$ in $GL_n(\mathbb{C}) = {}^L M$. Then the Langlands L -function for each of the representations ρ_n ,

$\wedge^2 \rho_n$, and $\text{Sym}^2(\rho_n)$, and σ are

$$\begin{aligned} L(s, \sigma, \rho_n) &= \det(I - \rho_n(A_\sigma)q^{-s})^{-1} \\ &= \prod_{j=1}^n (1 - \mu_j(\varpi)q^{-s})^{-1}, \\ L(s, \sigma, \wedge^2 \rho_n) &= \det(I - \wedge^2 \rho_n(A_\sigma)q^{-s})^{-1} \\ &= \prod_{1 \leq i < j \leq n} (1 - \mu_i(\varpi)\mu_j(\varpi)q^{-s})^{-1}, \end{aligned}$$

and

$$\begin{aligned} L(s, \sigma, \text{Sym}^2(\rho_n)) &= \det(I - \text{Sym}^2(\rho_n)(A_\sigma)q^{-s})^{-1} \\ &= \prod_{1 \leq i \leq j \leq n} (1 - \mu_i(\varpi)\mu_j(\varpi)q^{-s})^{-1}. \end{aligned}$$

Let h_0 be a function in $V(s, \sigma)$ such that $h_0(k) = 1$ for all $k \in \mathbf{G}(O)$. Then by Langlands [19]

$$A(s, \sigma, w_0)h_0(e) = L(s, \sigma, \rho_n)L(2s, \sigma, \wedge^2 \rho_n)/L(1 + s, \sigma, \rho_n)L(1 + 2s, \sigma, \wedge^2 \rho_n)$$

if $\mathbf{G} = Sp_{2n}$ and

$$A(s, \sigma, w_0)h_0(e) = L(s, \sigma, r_1)/L(1 + s, \sigma, r_1)$$

where $r_1 = \text{Sym}^2(\rho_n)$ or $r_1 = \wedge^2 \rho_n$, according to whether $\mathbf{G} = SO_{2n+1}$ or $\mathbf{G} = SO_{2n}$, respectively.

We should remark that our parametrization A_σ of σ is the inverse of the one which results if one uses the definition in [19, 23]. This justifies the changes of \tilde{r}_i to r_i when using equation (2.7) of [25].

One of the aims of this paper is to determine the L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$ for any irreducible admissible representation of $GL_n(F)$.

3. A conjecture. From now until Section 8 we shall assume σ is irreducible and supercuspidal. Let ω be the central character of σ . Assume further that σ is unitary. Since σ is supercuspidal, this is equivalent to ω being unitary.

In Section 7 of [23] we defined local L -functions $L(s, \sigma, r)$ for many representations r of $GL_n(\mathbb{C})$ (each inverse of a polynomial in q^{-s}), particularly for $r = \rho_n, \wedge^2 \rho_n$, and $\text{Sym}^2(\rho_n)$. It then follows from Corollary 7.6 of [23] that for $\mathbf{G} = Sp_{2n}$

$$(3.1) \quad L(s, \sigma, \rho_n)^{-1}L(2s, \sigma, \wedge^2 \rho_n)^{-1}A(s, \sigma, w_0)$$

is a nonzero and holomorphic operator for all $s \in \mathbb{C}$, while for $\mathbf{G} = SO_{2n+1}$ or $\mathbf{G} = SO_{2n}$

$$(3.2) \quad L(2s, \sigma, r_1)^{-1} A(s, \sigma, w_0)$$

behaves as such if $r_1 = \text{Sym}^2(\rho_n)$ or $r_1 = \wedge^2 \rho_n$, respectively. The appearance of $2s$ in the L -function in (3.2) is due to the fact that $I(s, \sigma) = I(2s\tilde{\alpha}, \sigma)$ (in the notation of [23]) in these two cases.

It follows from Lemma 7.4 of [23], the uniqueness of $\gamma(s, \sigma, \rho_n, \psi_F)$, and case (ii) of [19] that for $n > 1$

$$L(s, \sigma, \rho_n) = 1.$$

Thus, for $\mathbf{G} = Sp_{2n}$, the poles of $A(s, \sigma, w_0)$ are determined completely by the zeros of the polynomial $L(2s, \sigma, \wedge^2 \rho_n)^{-1}$. The L -function $L(s, \sigma, \wedge^2 \rho_n)$ can also be obtained using the holomorphy and nonvanishing of operator (3.2). In fact, it turns out to be much easier and natural to use (3.2) rather than (3.1) to compute $L(s, \sigma, \wedge^2 \rho_n)$ as well as the poles of both intertwining operators. This we owe to the uniqueness of these L -functions.

Besides its arithmetic significance, knowledge of these L -functions also determines the reducibility of $I(s, \sigma)$ completely. For example, when σ is unitary (still assuming σ is irreducible and supercuspidal), Corollary 7.6 of [23] implies that $I(\sigma)$ is reducible if and only if $\sigma \cong \tilde{w}_0(\sigma)$ and $P_\sigma(1) \neq 0$, where

$$P_\sigma(q^{-s}) = L(s, \sigma, r)^{-1}$$

with $r = \text{Sym}^2(\rho_n)$ if $G = SO_{2n+1}(F)$ and $r = \wedge^2 \rho_n$ otherwise. We remark that in the notation of [23], $P_\sigma = P_{\sigma,2}$ if $\mathbf{G} = Sp_{2n}$ and $P_\sigma = P_{\sigma,1}$, otherwise.

On the other hand, σ is expected to be parametrized by a homomorphism

$$\varphi: W_F \rightarrow GL_n(\mathbb{C}) \times W_F,$$

$\varphi(w) = (\varphi_0(w), w)$, $w \in W_F$, where W_F is the Weil group of \bar{F}/F . Moreover, as a representation of W_F , φ_0 must be irreducible (σ is supercuspidal). The parametrization must be such that

$$L(s, \sigma, r) = L(s, r \cdot \varphi_0)$$

for any representation r of $GL_n(\mathbb{C})$, where the L -function on the right is the Artin L -function attached to the representation $r \cdot \varphi_0$ of W_F . (See [4, 20, 33].)

Suppose $\mathbf{G} = Sp_{2n}$ or SO_{2n} . If $P_\sigma(1) = 0$, then $L(s, \wedge^2 \rho_n \cdot \varphi_0)$ must have a pole at $s = 0$; i.e., the trivial representation of W_F must appear in $\wedge^2 \rho_n \cdot \varphi_0$. Consequently, under $\wedge^2 \rho_n$, $\text{Im}(\varphi_0) \subset GL_n(\mathbb{C})$ must fix a vector in the alternating space $\wedge^2 \mathbb{C}^n$. This then implies that $(\wedge^2 \mathbb{C}^n)^*$ has a vector fixed by every $g \in \text{Im}(\varphi_0)$. But elements of

$(\wedge^2 \mathbb{C}^n)^*$ can be realized as alternating bilinear forms on \mathbb{C}^n , and since φ_0 is an irreducible representation of $GL_n(\mathbb{C})$, this form must be nondegenerate. Consequently, n must be even and $\text{Im}(\varphi_0) \subset Sp_n(\mathbb{C})$. Thus, $I(\sigma)$ is reducible if and only if $\sigma \cong \tilde{w}_0(\sigma)$ and φ_0 does not factor through $Sp_n(\mathbb{C})$; in particular, if n is odd and $\sigma \cong \tilde{w}_0(\sigma)$, then $I(\sigma)$ must be reducible.

If $\mathbf{G} = SO_{2n+1}$ and $P_\sigma(1) = 0$, then $L(s, \text{Sym}^2(\rho_n) \cdot \varphi_0)$ must have a pole at $s = 0$. A similar argument applies, and φ_0 must factor through $SO_n(\mathbb{C})$. Therefore, $I(\sigma)$ is reducible if and only if $\sigma \cong \tilde{w}_0(\sigma)$ and φ_0 does not factor through $SO_n(\mathbb{C})$.

We shall now formulate these in terms of the important phenomenon of twisted endoscopy [2, 18]. (See Section 7 here.)

Let $w \in GL_n$ be

$$w = \begin{bmatrix} & & & 1 \\ 0 & & -1 & \\ & 1 & & \\ -1 & & & 0 \\ \cdot & & & \end{bmatrix}.$$

Let $\theta: GL_n \rightarrow GL_n$ be the outer automorphism of GL_n defined by

$$\theta(g) = w^{-1}g^{-1}w.$$

It is a F -automorphism and preserves the subgroups of diagonals and upper triangulars of GL_n , as well as its standard splitting. (See [21].) Let $\hat{\theta}$ be the outer automorphism of $GL_n(\mathbb{C})$ defined the same way as θ .

Let ${}^L H^\circ$ be the (connected component of the) centralizer of $1 \times \hat{\theta} \in GL_n(\mathbb{C}) \times \{1, \hat{\theta}\}$ inside $GL_n(\mathbb{C})$. (See Section 7 here and [2, 18].) It is easily checked that ${}^L H^\circ = Sp_n(\mathbb{C})$ or $Sp_{n-1}(\mathbb{C})$ according to whether n is even or odd. Suppose n is even. Then the group $\mathbf{H} = SO_{n+1}$ has $Sp_n(\mathbb{C}) \times W_F$ as its L -group and is one of the θ -twisted elliptic endoscopic groups of GL_n . (See [2, 18].)

Next, let ${}^L H^\circ$ be the connected component of the centralizer of

$$\begin{bmatrix} 1 & & & \\ & -1 & & 0 \\ & & 1 & \\ & 0 & & -1 \\ & & & \ddots \end{bmatrix} \times \hat{\theta}$$

inside $GL_n(\mathbb{C})$. Then ${}^L H^\circ = SO_n(\mathbb{C})$. Let $\mathbf{H} = SO_n^*$ be any of the quasi-split orthogonal groups which has $SO_n(\mathbb{C})$ as the connected component of its L -group if n is even, and $\mathbf{H} = Sp_{n-1}$ otherwise. Again, \mathbf{H} is one of the θ -twisted elliptic endoscopic groups of GL_n .

If σ is supercuspidal and $\varphi(w) = (\varphi_0(w), w)$ is attached to σ , then φ_0 can only factor through (a conjugate of) either $SO_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$ with n even since σ is self-dual.

Definition 3.1. Let σ be an irreducible supercuspidal representation of $GL_n(F)$ parametrized by $\varphi: W_F \rightarrow GL_n(\mathbb{C}) \times W_F$, $\varphi(w) = (\varphi_0(w), w)$. Then σ is said to come from $SO_{n+1}(F)$, n even, $SO_n^*(F)$, n even, or $Sp_{n-1}(F)$, n odd, according to whether φ factors through $Sp_n(\mathbb{C}) \times W_F$, $SO_n(\mathbb{C}) \rtimes W_F$, or $SO_n(\mathbb{C}) \times W_F$, respectively.

Remark. In Section 7 we shall give a conjecturally equivalent definition (Definition 7.4) of those defined in Definition 3.1 which will allow us to prove a theorem (Theorem 7.6), which is again conjecturally equivalent to a proof of Conjecture 3.2 below.

It is now clear that for $\mathbf{G} = Sp_{2n}$ or SO_{2n} , $P_\sigma(1) = 0$ if and only if n is even and σ comes from $SO_{n+1}(F)$. On the other hand, if $\mathbf{G} = SO_{2n+1}$, then $P_\sigma(1) = 0$ if and only if σ comes from $SO_n^*(F)$ when n is even, or $Sp_{n-1}(F)$ otherwise. The odd case must be automatic. In fact, if σ is self-dual, φ will have to factor through $SO_n(\mathbb{C}) \times W_F$. Using Theorem 8.1 of [23], we can now formulate the following conjecture. We continue to assume $n > 1$.

CONJECTURE 3.2. (a) *Let $\mathbf{G} = SO_{2n+1}$ and fix an irreducible (unitary) self-dual supercuspidal representation σ of $M = GL_n(F)$. Then $I(\sigma)$ is irreducible if and only if either n is odd or n is even and σ comes from $SO_n^*(F)$.*

(b) *Suppose $\mathbf{G} = Sp_{2n}$ and fix σ as in part (a). Then $I(\sigma)$ is irreducible if and only if n is even and σ comes from $SO_{n+1}(F)$.*

(c) *Suppose $\mathbf{G} = SO_{2n}$ and fix σ as in part (a). Assume n is even. Then $I(\sigma)$ is irreducible if and only if σ comes from $SO_{n+1}(F)$. If n is odd, then $I(s, \sigma)$ is always irreducible.*

The following theorem then complements Conjecture 3.2 and completely answers the reducibility questions. It is a consequence of Theorem 8.1 of [23] and our definition of $I(s, \sigma)$.

THEOREM 3.3. *In all three cases, let σ be an irreducible (unitary) supercuspidal self-dual representation of σ . Assume $I(\sigma)$ is irreducible. (Corresponding σ is characterized by Conjecture 3.2.) Then the only points of reducibility for $I(s, \sigma)$, $s \in \mathbb{R}$, are at $s = \pm 1/2$. If $I(\sigma)$ is reducible, then there is no more reducibility and no complementary series.*

There are two parts of the conjecture that we can answer immediately.

Recall [26] that a maximal parabolic subgroup \mathbf{P}_θ , $\theta \in \Delta$, of a connected reductive algebraic group \mathbf{G} is called self-conjugate if and only if $\tilde{w}_0(\theta) = \theta$. Here, \tilde{w}_0 is the longest element in the Weyl group of \mathbf{A}_0 in \mathbf{G} . The following lemma was first mentioned to me by David Goldberg. Here, we include his proof which is clearly equivalent to the fact that $\tilde{w}_0(\alpha_{n-1}) = -\alpha_n$ for the group SO_{2n} when n is odd.

LEMMA 3.4. *Suppose $\mathbf{G} = SO_{2n}$ and n is odd. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be the standard parabolic subgroup of \mathbf{G} for which $\mathbf{M} \cong GL_n$. Then \mathbf{P} is not self-conjugate.*

Proof. Changing \tilde{w}_0 by an element in the Weyl group of \mathbf{T} in \mathbf{M} , we may assume it is only a product of sign changes. Since \mathbf{G} is SO_{2n} , this product must in fact have an even number of factors [5]. Clearly,

$$\mathbf{T} = \{\text{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}) \mid a_i \in \bar{F}^*\}.$$

If $\mathbf{P} = \mathbf{P}_\theta$ is self-conjugate, then $\tilde{w}_0(\theta) = \theta$ and \tilde{w}_0 must fix both \mathbf{M} and \mathbf{T} and sends every root of \mathbf{T} in \mathbf{N} to a negative one. But then up to a permutation of (a_1, \dots, a_n) , w_0 must send $(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1})$ to $(a_1^{-1}, \dots, a_n^{-1}, a_1, \dots, a_n)$. But since n is odd, this requires an odd number of sign changes, a contradiction.

We therefore have the following proposition.

PROPOSITION 3.5. (a) *Suppose $\mathbf{G} = Sp_{2n}$ and $n > 1$ is odd. Let σ be an irreducible (unitary) self-dual supercuspidal representation of $\mathbf{M} = GL_n(F)$. Then $I(\sigma)$ is reducible, and there are no complementary series.*

(b) *Suppose $\mathbf{G} = SO_{2n}$ and $n > 1$ is odd. Let σ be an irreducible supercuspidal representation of $\mathbf{M} = GL_n(F)$. Then $I(s, \sigma)$ is always irreducible.*

Proof. Suppose $\mathbf{G} = SO_{2n}$. By Lemma 3.4 the parabolic subgroup $\mathbf{P} = \mathbf{MN}$, $\mathbf{M} = GL_n$, is not self-conjugate. Therefore, Lemma 7.4 of [23] implies that P_σ is identically 1. But

$$P_\sigma(q^{-s}) = L(s, \sigma, \wedge^2 \rho_n)^{-1}$$

is the same for both Sp_{2n} and SO_{2n} and thus $P_\sigma \equiv 1$, where σ is a representation of $GL_n(F)$ now being considered as the corresponding Levi subgroup of $Sp_{2n}(F)$. Since $\tilde{w}_0(\sigma) \cong \sigma$ in $Sp_{2n}(F)$ is equivalent to self-duality of σ , part (a) is now a consequence of Theorem 8.1 of [23]. Part (b) follows from Lemma 3.4.

Remark. The fact that self-dual supercuspidal representations of $GL_n(F)$, when n is odd, exist (which is less clear than the case of n even as pointed out to us by Howe) must follow from the recent important work of Bushnell and Kutzko [36]. Private communications with Kutzko suggest that they exist when $\text{char}(\bar{F}) = 2$. Our own examples are in the (tame) case of $\text{char}(\bar{F}) = 2$ and $n = 3$. In fact, irreducible supercuspidal representations of $GL_3(F)$ which are images of extraordinary (supercuspidal) representations of $GL_2(F)$ under the Gelbart-Jacquet's adjoint square map are all self-dual.

Let σ be an irreducible unitary supercuspidal representation of $GL_n(F)$. Denote by $L(s, \sigma \times \sigma)$ the Rankin-Selberg L -function attached to (σ, σ) [14, 27]. Suppose $\sigma \cong \tilde{\sigma}$. Using the results of Olšanskii [22] (also see [27]), $L(s, \sigma \times \sigma)$ can easily be computed. In fact, let r be the order of the cyclic group of unramified characters η of F^* such that $\sigma \cong \sigma \otimes \eta \cdot \text{det}$. Then

$$(3.3) \quad L(s, \sigma \times \sigma) = (1 - q^{-rs})^{-1}.$$

(See Remark 8.2 of [23].) The following lemma is crucial.

LEMMA 3.6. *Let σ be an irreducible unitary supercuspidal representation of $GL_n(F)$. Then*

$$L(s, \sigma \times \sigma) = L(s, \sigma, \wedge^2 \rho_n) L(s, \sigma, \text{Sym}^2(\rho_n)).$$

Proof. By Proposition 5.1 of [23], choose a globally generic cusp form $\pi = \bigotimes_v \pi_v$ of $GL_n(\mathbb{A}_K)$, K a global field having F as its completion at v_0 , such that $\pi_{v_0} = \sigma$, while for every other finite place $v \neq v_0$, π_v is unramified. The lemma is now a consequence of comparing functional equation (7.9) of [23] for $L(s, \pi \times \pi)$ together with those satisfied by $L(s, \pi, \wedge^2 \rho_n)$ and $L(s, \pi, \text{Sym}^2(\rho_n))$ using (see equation (2.2))

$$L(s, \pi_v \times \pi_v) = L(s, \pi_v, \wedge^2 \rho_n) L(s, \pi_v, \text{Sym}^2(\rho_n))$$

for every finite place $v \neq v_0$ and Theorem 3.1 of [28] for $v = \infty$. Observe that Conjecture 7.1 of [23] holds since σ is supercuspidal and unitary.

COROLLARY 3.7. *Let σ be an irreducible (unitary) supercuspidal self-dual representation of $GL_n(F)$. Let s_0 be a pole of $L(s, \sigma \times \sigma)$. Then one and only one of the two L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$ will have a pole at $s = s_0$.*

Proof. This is a consequence of the simplicity of the poles of $L(s, \sigma \times \sigma)$ and nonvanishing of local L -functions. (See [14], for example.)

Remark 3.8. We refer to the introduction of [13] for the very deep global version of these results.

Remark 3.9. It is now clear that every irreducible supercuspidal self-dual representation σ of $GL_n(F)$ is such that always one and only one of the two L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$ has a pole at $s = 0$. Therefore, if such a representation does not come from $SO_{n+1}(F)$, then it will have to come either from $Sp_{n-1}(F)$ or $SO_n^*(F)$ and conversely. We shall return to this in Section 7, and we in fact have the following proposition.

PROPOSITION 3.10. *Suppose $G = SO_{2n+1}$ and $n > 1$ is odd. Let σ be an irreducible (unitary) self-dual supercuspidal representation of $M = GL_n(F)$. Then $I(\sigma)$ is irreducible, and the points of reducibility for $I(s, \sigma)$, $s \in \mathbb{R}$, are exactly at $s = \pm 1/2$.*

Proof. This is a consequence of Proposition 3.5 and Corollary 3.7.

4. A useful general lemma. In this section we shall prove a lemma on intertwining operators for an arbitrary connected reductive algebraic group over F . The basic idea of the proof is due to Rallis. In fact, our original statement was more restricted with a more delicate proof. The lemma will be used in the next section.

Let G be a connected reductive algebraic group over F . Fix a parabolic subgroup P of G , containing a minimal parabolic subgroup P_0 , and let $P = MN$ be a Levi decomposition of P . Fix a maximal split torus A_0 of G lying in M .

Let $\mathbf{A} \subset \mathbf{A}_0$ be the maximal split torus in the center of \mathbf{M} , and if $X(\mathbf{A})$ denotes the group of F -rational characters of \mathbf{A} , let

$$\mathfrak{a}^* = X(\mathbf{A}) \otimes \mathbb{R}$$

and $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes \mathbb{C}$.

Given an irreducible unitary representation $(\sigma, \mathcal{H}(\sigma))$ of $M = \mathbf{M}(F)$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $I(\nu, \sigma)$ be the representation of G induced from

$$\sigma \otimes q^{\langle \nu, H_M(\cdot) \rangle}$$

on M , where

$$H_M: M \rightarrow \text{Hom}(X(\mathbf{M}), \mathbb{Z})$$

is defined by

$$|\chi(m)|_F = q^{\langle H_M(m), \chi \rangle}$$

for every $\chi \in X(\mathbf{M})$, the group of F -rational characters of \mathbf{M} .

Let \tilde{w}_0 be the longest element in the Weyl group of \mathbf{A}_0 in \mathbf{G} and set $\mathbf{A}' = \tilde{w}_0(\mathbf{A})$. Let \mathbf{M}' be the centralizer of \mathbf{A}' in \mathbf{G} and use \mathbf{P}' to denote the standard (in the sense that $\mathbf{P}' \supset \mathbf{P}_0$) parabolic subgroup of \mathbf{G} which has \mathbf{M}' as its Levi subgroup. Let \mathbf{N}' be the unipotent radical of \mathbf{P}' . Clearly, $w_0^{-1}\mathbf{N}'w_0 = \mathbf{N}^-$, the opposite of \mathbf{N} , where w_0 is any representation for \tilde{w}_0 .

Recall that (see [26]) given $f \in V(\nu, \sigma)$, the space of $I(\nu, \sigma)$, the integral (which converges for ν in an appropriate cone)

$$A(\nu, \sigma)f(g) = \int_{\mathbf{N}'} f(w_0^{-1}n'g) dn'$$

is the standard intertwining operator attached to ν, σ , and w_0 , where w_0 is a fixed representative of \tilde{w}_0 .

The set $\mathbf{P}\mathbf{N}^-$ is open and dense in G . Let $V(\nu, \sigma)_0$ be the subspace of all the functions in $V(\nu, \sigma)$ which are of compact support in $\mathbf{P}\mathbf{N}^-$ modulo \mathbf{P} . We shall now prove the following lemma.

LEMMA 4.1. (Rallis). *Every pole of $A(\nu, \sigma)$ is a pole of $A(\nu, \sigma)f_0(e)$ for some $f_0 \in V(\nu, \sigma)_0$.*

Proof. We shall freely use the notation and definitions from [26]. Suppose ν_0 is a pole of $A(\nu, \sigma)$. Let $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^N$ for some nonnegative integer N . Given $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, write $\nu = (\nu_1, \dots, \nu_N)$ for its components. Set $\nu_0 = (\nu_{0,1}, \dots, \nu_{0,N})$. By the multiplicative properties of intertwining operators (Theorem 2.1.1 of [26]), there exist nonnegative

integers m_1, \dots, m_N such that

$$(4.1.1) \quad \lim_{v \rightarrow v_0} (v_1 - v_{0,1})^{m_1} \dots (v_N - v_{0,N})^{m_N} A(v, \sigma)$$

is a nonzero and holomorphic operator at v_0 . Let

$$\beta(v) = \prod_{i=1}^N (v_i - v_{0,i})^{m_i}.$$

Suppose

$$(4.1.2) \quad \lim_{v \rightarrow v_0} \beta(v) A(v, \sigma) f(e) = 0$$

for every $f \in V(v, \sigma)_0$. Since f is supported in PN^- , we may assume (4.1.2) holds with e replaced by pn^- , $p \in P$, and $n^- \in N^-$, and finally by density for any $g \in G$. Therefore,

$$\lim_{v \rightarrow v_0} \beta(v) A(v, \sigma) f = 0$$

for every $f \in V(v, \sigma)_0$, or more generally, for every f in the G -span of $V(v, \sigma)_0$. Therefore, we need the following lemma.

LEMMA 4.2. *The G -span of $V(v, \sigma)_0$ is $V(v, \sigma)$.*

Proof. Let (\cdot, \cdot) be the inner product on $\mathcal{H}(\sigma)$. Then by [26]

$$\langle f, f' \rangle = \oint_G (f(g), f'(g)) d\mu(g)$$

defines a duality between $V(v, \sigma)$ and $V(-\bar{v}, \sigma)$ (with notation as in [26]).

Let V_0 be the G -span of $V(v, \sigma)_0$ and denote by $V_0^\perp \subset V(-\bar{v}, \sigma)$ the subspace of all the functions $h \in V(-\bar{v}, \sigma)$ for which $\langle f, h \rangle = 0$ for all $f \in V_0$. Using the discussion in Section 1.2.2, page 24 of [31], it is easy to see that $\langle f, h \rangle$, $f \in V(v, \sigma)$, $h \in V(-\bar{v}, \sigma)$, is basically integration of $(f(n^-), h(n^-))$ on N^- . Consequently, if $\langle f, h \rangle = 0$ for all $f \in V_0$, then h must vanish on PN^- . By density, $h = 0$ on all of G , thus completing the lemma.

COROLLARY (OF THE PROOF) 4.3. *Lemma 4.1 is still true if σ is any irreducible admissible representation of M .*

Remark 4.4. Due to its general nature, Lemma 4.1 can also be used if σ is not tempered. This must have interesting consequences when the inducing representation is nontempered.

Remark 4.5. It is now clear that to calculate the poles of intertwining operators one only needs to consider functions in $V(v, \sigma)_0$, an observation which would have made some of the calculations in [22] and [32] avoidable.

5. Poles of intertwining operators for $SO_{2n}(F)$. In this section we compute the polynomial $P_\sigma(q^{-s})$ for $G = SO_{2n}(F)$. By Proposition 3.5 we may assume n is even since, for odd n , $P_\sigma \equiv 1$.

Again, we shall assume $(\sigma, \mathcal{H}(\sigma))$ is an irreducible unitary supercuspidal representation of $M = GL_n(F)$ with central character ω . Let ψ be a matrix coefficient of σ . Then there exists a function $f \in C_c^\infty(GL_n(F))$ such that

$$(5.1) \quad \psi(g) = \int_{Z_n(F)} f(zg)\omega^{-1}(z) dz$$

when $Z_n(F) \cong F^*$ is the center of $GL_n(F)$. Recall from Section 3 that

$$w = \begin{bmatrix} & & & 1 \\ & 0 & -1 & \\ & & 1 & \\ -1 & & & 0 \end{bmatrix}.$$

To compute $P_\sigma(q^{-s})$, we need to prove the following result.

PROPOSITION 5.1. *Assume n is even. Let σ be an irreducible unitary supercuspidal representation of $GL_n(F)$. Consider σ as a representation of the Levi factor M of the parabolic subgroup P fixed before. Then the intertwining operator $A(s, \sigma, w_0)$ has a pole at $s = 0$ if and only if $\sigma \cong \tilde{\sigma}$, and there exists a function $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient ψ of σ by means of equation (5.1), for which*

$$(5.2) \quad \int_{S_{P_n(F)} \backslash GL_n(F)} f({}^t g w^{-1} g w) dg \neq 0.$$

Moreover, the nonvanishing condition (5.2) implies $\omega = 1$.

Proof. For the intertwining operator (2.1) to have a pole at $s = 0$, one may assume that $\tilde{w}_0(\sigma) \cong \sigma$. This then immediately implies that σ is self-dual. Consequently, from now on we shall assume $\sigma \cong \tilde{\sigma}$; in particular, $\omega^2 = 1$.

By Lemma 4.1, to compute the poles of $A(s, \sigma, w_0)$ we may only study the poles of $A(s, \sigma, w_0)h(e)$, where h is a smooth $\mathcal{H}(\sigma)$ -valued function which is of compact support in PN^- modulo P . Here, $N^- = \mathbf{N}^-(F)$, where \mathbf{N}^- denotes the unipotent subgroup opposed to \mathbf{N} .

Thus, we must study

$$(5.1.1) \quad A(s, \sigma, w_0)h(e) = \int_{\iota_{a=-a}} h\left(w_0 \begin{pmatrix} I_n & a \\ 0 & I_n \end{pmatrix}\right) da.$$

Since n is even, we may take

$$w_0 = J_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Incidentally, when n is odd, one can take

$$w_0 = \begin{pmatrix} & 0 & & & & 0 \\ & 0_{n-1} & \vdots & & I_{n-1} & \vdots \\ & & 0 & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ & & & 0 & & & \vdots \\ & I_{n-1} & \vdots & & 0_{n-1} & & 0 \\ 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}.$$

It can be easily checked that for

$$w_0 \begin{pmatrix} I_n & a \\ 0 & I_n \end{pmatrix}$$

to belong to PN^- , a must be invertible and then

$$w_0 \begin{pmatrix} I_n & a \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} -a^{-1} & I_n \\ 0 & a \end{pmatrix} \begin{pmatrix} I_n & 0 \\ a^{-1} & I_n \end{pmatrix}.$$

Consequently, (5.1.1) reduces, at first formally, to

$$(5.1.2) \quad \int_{\substack{\iota_{a=-a} \\ \det a \neq 0}} \omega(-1)\sigma(a^{-1})h\left(\begin{pmatrix} I_n & 0 \\ a^{-1} & I_n \end{pmatrix}\right) |\det a|^{-s-(n-1)/2} da.$$

By smoothness of h and the compactness of its support in N^- , we may assume that h has the property that there exists an open compact subgroup L of $M_n(F)$ such that

$$h\left(\begin{pmatrix} I_n & 0 \\ a^{-1} & I_n \end{pmatrix}\right) = 0$$

unless $a^{-1} \in L$ in which case

$$h\left(\begin{pmatrix} I_n & 0 \\ a^{-1} & I_n \end{pmatrix}\right) = h(e)$$

where $h(e) = v$ is a nonzero vector in $\mathcal{H}(\sigma)$. Let \tilde{v} be an arbitrary vector in $\mathcal{H}(\tilde{\sigma})$. It is therefore enough to study

$$(5.1.3) \quad \omega(-1) \int_{\substack{a=-a \\ \det a \neq 0 \\ a^{-1} \in L}} \langle \sigma(a^{-1})v, \tilde{v} \rangle |\det a|^{-s-(n-1)/2} da.$$

Let $d^*a = |\det a|^{-(n-1)/2} da$. Consider the adjoint action of

$$M = \left\{ m = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \middle| a \in GL_n(F) \right\}$$

on N , sending $X \in N$ to $aX^t a$. Then

$$d(aX^t a)/dX = |\det a|^{n-1},$$

implying $q^{\langle \rho, H_M(m) \rangle} = |\det a|^{(n-1)/2}$ used in equation (5.1.2) as well (§1.2.1 of [31]). Let $X = {}^t a^{-1}$. Then ${}^t X = -X$ implies $dX = d(a^{-1})$. Consequently,

$$da/d(a^{-1}) = |\det a|^{n-1}.$$

Thus, $d^*(a^{-1}) = d^*a$, and (5.1.3) is then equal to

$$(5.1.4) \quad \omega(-1) \int_{\substack{a=-a \\ a \in GL_n(F) \cap L}} \langle \sigma(a)v, \tilde{v} \rangle |\det a|^s d^*a.$$

Set $\psi(a) = \langle \sigma(a)v, \tilde{v} \rangle$ and take J to be any matrix in $GL_n(F)$ satisfying ${}^t J = -J$. Then (5.1.4) equals

$$(5.1.5) \quad \omega(-1) \cdot \int_{Sp_n(F) \backslash GL_n(F)} \Phi({}^t g J g) \psi({}^t g J g) |\det({}^t g J g)|^s d\dot{g}$$

where Φ is the characteristic function of L . Here, we choose the measure $d\dot{g}$ on $Sp_n(F) \backslash GL_n(F)$ to agree with d^*a through the isomorphism of $Sp_n(F) \backslash GL_n(F)$ with the space of nonsingular skew-symmetric matrices in $M_n(F)$.

Next, choose $f \in C_c^\infty(GL_n(F))$ which defines ψ by means of equation (5.1). Then (5.1.5) is equal to

$$(5.1.6) \quad \omega(-1) \cdot \int_{g \in Sp_n(F) \backslash GL_n(F)} \int_{z \in Z_n(F)} \sum_{\varepsilon \in (F^*)^2 \backslash F^*} \Phi_z({}^t(zg)J(zg)) \\ \times f({}^t(zg)\varepsilon J(zg)) |\det({}^t(zg)J(zg))|^s \omega(\varepsilon) \omega(z^2) |\det z|^{-2s} dz d\dot{g}$$

where Φ_z is the characteristic function of z^2L . Changing zg to g in (5.1.6) implies

$$(5.1.7) \quad \omega(-1) \cdot \int_{Sp_n(F) \backslash GL_n(F)} \sum_{(F^*)^2 \backslash F^*} f({}^t g \varepsilon J g) |\det({}^t g J g)|^s \omega(\varepsilon) d\dot{g} \\ \times \int_{Z_n(F)} \Phi_z({}^t g J g) |\det z|^{-2s} dz.$$

Since f is of compact support in $GL_n(F)$, we may assume $\Phi_z({}^t g J g) = 0$ unless ${}^t g J g \in z^2L \cap \text{supp}(f)$. Thus, $\det(z^2l)$ must be bounded both from above and below for some nonzero $l \in L$. On the other hand, $\det(l)$ is always bounded above. Consequently, $|\det(z)|$ must be bounded from below. Thus, we may assume that, if

$$(5.1.8) \quad \int_{Z_n(F)} \Phi_z({}^t g J g) |\det z|^{-2s} dz$$

has any poles (as a function of s), they must come from those of

$$(5.1.9) \quad \int_{|\det z| < \kappa} |\det z|^{2s} dz$$

for some positive number κ . But (5.1.9) converges absolutely if and only if $\text{Re}(s) > 0$ and its poles are the same as

$$(5.1.10) \quad L(2ns, \mathbf{1}) = (1 - q^{-2ns})^{-1}.$$

Letting s go to zero in (5.1.7), we must now consider

$$(5.1.11) \quad \sum_{\varepsilon \in (F^*)^2 \backslash F^*} \int_{Sp_n(F) \backslash GL_n(F)} \omega(\varepsilon) f({}^t g \varepsilon J g) d\dot{g}.$$

The convergence of the integral

$$(5.1.12) \quad \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g \varepsilon J g) d\dot{g}$$

follows from the compactness of the support of f . In fact, as it will be discussed in Section 7, the integral in (5.1.12) is over the conjugacy class of $(\varepsilon J, \theta) \in GL_n \rtimes \{\theta\}$ inside $GL_n(F)$ to which we can apply Lemma 2.1 of [1] for the group $GL_n \rtimes \{1, \theta\}$. Consequently, all our calculations are justified as long as $Re(s) > 0$.

Since ${}^t(\varepsilon J) = -\varepsilon J$, one has that (5.1.12) is equal to

$$(5.1.13) \quad \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g J g) d\dot{g},$$

from which one concludes that (5.1.11) equals

$$(5.1.14) \quad \sum_{(F^*)^2 \backslash F^*} \omega(\varepsilon) \cdot \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g J g) d\dot{g}.$$

This is clearly zero unless $\omega \equiv 1$ in which case the residue of $A(s, \sigma, w_0)$ at $s = 0$ is proportional to (5.1.13). Thus, $A(s, \sigma, w_0)$ has a pole at $s = 0$ if and only if $\sigma \cong \tilde{\sigma}$, $\omega \equiv 1$, and there exists a function $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of σ such that

$$(5.1.15) \quad \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g J g) d\dot{g} \neq 0.$$

Since n is even, w is skew-symmetric. Thus, changing v to $\sigma(w)v$, (5.1.15) can be replaced with

$$\int_{Sp_n(F) \backslash GL_n(F)} f({}^t g w^{-1} g w) d\dot{g} \neq 0$$

for some f .

We only need to show that the nonvanishing condition (5.2) implies $\omega = 1$. For simplicity and reasons to be explained in Section 7, for every $z \in Z_n(F)$ set

$$\Phi_\theta(z, f) = \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g z w^{-1} g w) d\dot{g}.$$

Given $a \in F^*$, let \tilde{a} be the diagonal element in $GL_n(F)$ whose first $n/2$ diagonal entries are a while the remaining ones are equal to 1. It normalizes $Sp_n(F)$. Changing g to $\tilde{a}g$ in (5.2) now implies that

$$\Phi_\theta(1, f) = \Phi_\theta(1, R_a f)$$

where a denotes the central element defined by $a \in F^*$.

Given $f \in C_c^\infty(GL_n(F))$, let

$$\tilde{f}(g) = \int_{Z_n(F)} f(z^2 g) dz$$

be the projection onto $C_c^\infty(Z_n(F)^2 \backslash GL_n(F))$ in which every self-dual discrete series representation of $GL_n(F)$ appears discretely. Moreover,

$$\Phi_\theta(1, \tilde{f}) = \Phi_\theta(1, f)$$

where the left-hand side is defined by

$$\Phi_\theta(1, \tilde{f}) = \int_{Z_n(F)S_{P_n(F)} \backslash GL_n(F)} \tilde{f}({}^t g w^{-1} g w) d\dot{g}.$$

Thus, for every $a \in F^*$,

$$\Phi_\theta(1, \tilde{f}) = \Phi_\theta(1, R_a \tilde{f}).$$

Choose f in such a way that \tilde{f} appears in a subspace equivalent to σ (in fact, such functions f provide all the matrix coefficients of σ), and consequently

$$\Phi_\theta(1, R_a \tilde{f}) = \omega(a) \Phi_\theta(1, \tilde{f}).$$

This now implies that, if $\Phi_\theta(1, f) \neq 0$ for some f , then $\omega = 1$, completing the proposition.

We shall now interpret (5.2) in terms of matrix coefficients of σ . Since $z w^{-1}$ is skew-symmetric,

$$\Phi_\theta(z, f) = \Phi_\theta(1, f).$$

Suppose $\varepsilon \in (F^*)^2 \backslash F^*$. Then up to the cardinality of $(F^*)^2 \backslash F^*$, $\Phi_\theta(1, f)$ is equal to

$$\begin{aligned} \sum_{\varepsilon \in (F^*)^2 \backslash F^*} \Phi_\theta(\varepsilon, f) &= \sum_{\varepsilon \in (F^*)^2 \backslash F^*} \int_{z \in Z_n(F)} \int_{Z_n(F)S_{P_n(F)} \backslash GL_n(F)} f(\varepsilon z^2 \cdot {}^t g w^{-1} g w) d\dot{g} dz \\ &= \int_{Z_n(F)S_{P_n(F)} \backslash GL_n(F)} \left(\int_{z \in Z_n(F)} f(z {}^t g w^{-1} g w) dz \right) d\dot{g} \\ &= \int_{Z_n(F)S_{P_n(F)} \backslash GL_n(F)} \langle \sigma({}^t g w^{-1} g w)v, \tilde{v} \rangle d\dot{g} \end{aligned}$$

using $\omega = 1$. Conversely, the nonvanishing of the last integral implies $\omega = 1$.

We thus have the following criterion.

CRITERION 5.2. *The nonvanishing condition (5.2) for some f is equivalent to*

$$(5.3) \quad \int_{Z_n(F)Sp_n(F)\backslash GL_n(F)} \langle \sigma({}^t g w^{-1} g w) v, \tilde{v} \rangle d\dot{g} \neq 0$$

for some $v \in \mathcal{H}(\sigma)$ and $\tilde{v} \in \mathcal{H}(\tilde{\sigma})$, and therefore $A(s, \sigma, w_0)$ has a pole at $s = 0$ if and only if (5.3) holds for some v and \tilde{v} .

Remark 1. The nonvanishing condition (5.3) immediately implies the condition $\sigma \cong \tilde{\sigma}$.

Remark 2. Suppose $\mathbf{G} = GL_{2n}$ and $\mathbf{M} = GL_n \times GL_n$. Let σ_1 and σ_2 be two irreducible unitary supercuspidal representations of $GL_n(F)$. Then the work of Olšanskii [22] shows that $A(s, \sigma, w_0)$, with $w_0 = J_0$ as in the case $\mathbf{G} = SO_{2n}$ and $\sigma = \sigma_1 \otimes \sigma_2$, has a pole at $s = 0$ if and only if $\sigma_1 \cong \sigma_2$. In fact, the L -function that appears is $L(s, \sigma_1 \times \tilde{\sigma}_2)$ discussed in Section 3, and the residue at $s = 0$ is proportional to

$$(5.4) \quad \int_{Z_n(F)\backslash GL_n(F)} \langle \sigma_1(g)v_1, \tilde{v}_1 \rangle \langle \sigma_2(g^{-1})v_2, \tilde{v}_2 \rangle d\dot{g}$$

where we are assuming $\omega_1 = \omega_2$ to get the pole at $s = 0$. Here, $v_i \in \mathcal{H}(\sigma_i)$ and $\tilde{v}_i \in \mathcal{H}(\tilde{\sigma}_i)$, $i = 1, 2$. Clearly, (5.4) is nonzero if and only if $\sigma_1 \cong \sigma_2$. The beauty and depth of (5.3) is that it provides a condition analogue to (5.4) for giving the poles for $\mathbf{G} = SO_{2n}$ as well as $\mathbf{G} = Sp_{2n}$ which are much deeper than the case of GL_{2n} , as we shall discuss later in Section 7.

Remark 3. The integral in (5.3) is over the (non-Riemannian) symmetric space $Z_n(F)Sp_n(F)\backslash GL_n(F)$. ($Sp_n(F)$ is the stabilizer of an involution of $GL_n(F)$.) When $F = \mathbb{R}$, these spaces were studied by Flensted-Jensen. (See [10].) It is interesting to study these spaces for nonarchimedean fields as well. We would like to call (5.3) a *local period integral* although it is not over a subgroup of $GL_n(F)$ but rather over a symmetric space attached to it.

We shall now state one of the main results of this paper whose proof is a consequence of the fact that the polynomial P_σ works for both SO_{2n} and Sp_{2n} . (See [23].) It complements Proposition 3.5.

THEOREM 5.3. *Let σ be an irreducible (unitary) self-dual supercuspidal representation of $M = GL_n(F)$, where M is being considered as the Levi factor of the maximal parabolic subgroup of either $SO_{2n}(F)$ or $Sp_{2n}(F)$ generated by $\{\alpha_1, \dots, \alpha_n\}$. Assume n is even. Then $I(\sigma)$ is irreducible if and only if (5.2), or equivalently (5.3), holds for some function $f \in C_c^\infty(GL_n(F))$ which defines a matrix coefficient of σ . The points of reducibility for $I(s, \sigma)$, $s \in \mathbb{R}$, are then only at $s = \pm 1/2$ in both cases. Otherwise, $I(\sigma)$ is reducible and there are no complementary series.*

We shall discuss the case of $\mathbf{G} = SO_{2n+1}$ in the next section.

Remark 4. In Section 7 we shall return to the interpretation of this result in the direction of Conjecture 3.2.

6. L -functions and Plancherel measures. In this section we shall compute the local L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$, where σ is an irreducible supercuspidal representation of $GL_n(F)$. We shall treat the general case in Section 8. We start with $L(s, \sigma, \wedge^2 \rho_n)$.

Clearly, $P_\sigma(q^{-s}) \equiv 1$ unless some unramified twist of σ , say $\sigma_1 = \sigma \otimes |\det(\)|^{s_1}$ is self-dual. Then by equation (3.12) of [23]

$$P_{\sigma_1}(q^{-s}) = P_\sigma(q^{-s-2s_1}),$$

and therefore

$$L(s, \sigma, \wedge^2 \rho_n) = L(s - 2s_1, \sigma_1, \wedge^2 \rho_n).$$

Thus, from now on we may assume σ is self-dual, and therefore $\omega^2 = 1$. We need to determine the normalized polynomial $P_\sigma(q^{-s})$ for which

$$P_\sigma(q^{-2s})A(s, \sigma, w_0)$$

is holomorphic and nonzero. The operator $A(s, \sigma, w_0)$ is the one considered in Section 5.

The poles of $A(s, \sigma, w_0)$ are among those of $L(2ns, \mathbf{1})$ defined by (5.1.10). Let s_0 be a root of $1 - q^{-2ns} = 0$. The representation $\sigma \otimes |\det(\)|^{s_0}$ will have $f \otimes |\det(\)|^{s_0} \in C_c^\infty(GL_n(F))$ as a function defining one of its matrix coefficients. The central character of $\sigma \otimes |\det(\)|^{s_0}$ is $|\omega|^{ns_0}$ and is well defined on $(F^*)^2 \backslash F^*$ since $q^{-2ns_0} = 1$. The residue of $A(s, \sigma, w_0)h(e)$ at s_0 is proportional to

$$(6.1) \quad \sum_{(F^*)^2 \backslash F^*} \omega(\varepsilon) |\varepsilon|^{ns_0} \cdot \int_{Sp_n(F) \backslash GL_n(F)} f({}^t g J g) |\det({}^t g J g)|^{s_0} dg.$$

The nonvanishing of (6.1) determines whether $A(s, \sigma, w_0)$ has a pole at s_0 . The polynomial $P_\sigma(q^{-2s})$ will then be a product over all (with no repetition of factors)

$$(1 - q^{2s_0} q^{-2s})$$

for which (6.1) does not vanish. In fact, the condition $q^{2ns_0} = 1$ will be automatically satisfied since the nonvanishing of (5.2) for $f \otimes |\det(\)|^{s_0}$ implies the self-duality of $\sigma \otimes |\det(\)|^{s_0}$ which in turn points to $q^{2ns_0} = 1$. Finally, observe that if $\eta = |\ \ |^{s_0}$, then self-duality of $\sigma \otimes \eta \cdot \det$ implies that $\sigma \otimes \eta \cdot \det \cong \sigma \otimes \eta^{-1} \cdot \det$. We have therefore proved the following theorem.

THEOREM 6.1. *Let σ be an irreducible supercuspidal representation of $GL_n(F)$. Then $L(s, \sigma, \wedge^2 \rho_n) \equiv 1$ unless n is even and some unramified twist of σ is self-dual. Suppose n is even and σ is self-dual. Let S be the set of all the unramified characters $\eta \in \widehat{F}^*$, no two of which have equal squares, such that*

$$(6.2) \quad \int_{S_{P_n(F)} \backslash GL_n(F)} f({}^t g w^{-1} g w) d\dot{g} \neq 0$$

for some $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of $\sigma \otimes \eta \cdot \det$. Then

$$L(s, \sigma, \wedge^2 \rho_n) = \prod_{\eta \in S} (1 - \eta^2(\varpi) q^{-s})^{-1}.$$

Remark. Clearly, we can replace (6.2) with condition (5.3) in Criterion 5.2.

Next, we consider the L -function $L(s, \sigma, \text{Sym}^2(\rho_n))$. Our tools are Lemma 3.6 and its Corollary 3.7. We shall return to its direct calculation using $SO_{2n+1}(F)$ in Section 10.

Using $SO_{2n+1}(F)$ and the fact that operator (3.2) must be nonzero and holomorphic, we see that $L(s, \sigma, \text{Sym}^2(\rho_n)) \equiv 1$ unless some unramified twist of σ is self-dual. Thus, from now on we shall assume σ is self-dual and therefore $\omega^2 = 1$.

Suppose n is odd. Then $L(s, \sigma, \wedge^2 \rho_n) \equiv 1$, and therefore by Lemma 3.6

$$(6.3) \quad L(s, \sigma, \text{Sym}^2(\rho_n)) = (1 - q^{-rs})^{-1}.$$

It clearly has a pole at $s = 0$.

Next, assume n is even and s_0 is a pole of $L(s, \sigma, \text{Sym}^2(\rho_n))$. Let $\eta = | \cdot |^{s_0}$. Then $L(s, \sigma \otimes \eta \cdot \det, \text{Sym}^2(\rho_n))$ will have a pole at $s = 0$. By Lemma 3.6, Corollary 3.7, and Theorem 6.1, $\sigma \otimes \eta \cdot \det$ must be self-dual and (6.2) must vanish for all of its matrix coefficients. Clearly, $(1 - \eta^2(\varpi) q^{-2s})$ must divide $L(2s, \sigma, \text{Sym}^2(\rho_n))^{-1}$.

Let S' denote the set of all the unramified characters $\eta \in \widehat{F}^*$, no two of which have equal squares, for which $\sigma \otimes \eta \cdot \det$ is self-dual and the nonvanishing condition (6.2) never holds for any $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of $\sigma \otimes \eta \cdot \det$ (in particular, if $\omega \neq 1$). Then by the self-duality of σ and $\sigma \otimes \eta \cdot \det$, we have $\sigma \otimes \eta \cdot \det \cong \sigma \otimes \eta^{-1} \cdot \det$, and therefore if $\eta \in S'$, then so does η^{-1} . Moreover, $\eta^2(\varpi)$ must be an r th root of unity, where r is as in Section 3. In particular,

$$(1 - \eta^2(\varpi) q^{-s}) | L(s, \sigma \times \sigma)^{-1}$$

for every $\eta \in S'$, as expected. We therefore have the following theorem.

THEOREM 6.2. *Let σ be an irreducible supercuspidal representation of $GL_n(F)$. Then $L(s, \sigma, \text{Sym}^2(\rho_n)) \equiv 1$ unless some unramified twist of σ is self-dual. Suppose σ is self-dual.*

(a) If $n > 1$ is odd, then

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = (1 - q^{rs})^{-1}$$

where r is the order of the cyclic group of unramified characters η of F^* for which $\sigma \cong \sigma \otimes \eta \cdot \det$.

(b) Suppose n is even. Let S' be as above. Then

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = \prod_{\eta \in S'} (1 - \eta^2(\varpi)q^{-s})^{-1}.$$

(c) In both cases

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = L(s, \sigma \times \sigma) / L(s, \sigma, \wedge^2 \rho_n).$$

Part (c) is in fact the statement of Lemma 3.6 which we have included here for completeness. We refer to Corollary 8.2 for the general case.

Finally, using Corollary 3.7, we formulate our reducibility criterion for $SO_{2n+1}(F)$ as follows.

THEOREM 6.3. *Let σ be an irreducible (unitary) self-dual supercuspidal representation of $M = GL_n(F)$, where M is considered as the Levi factor of the maximal parabolic subgroup of either $SO_{2n+1}(F)$ or $Sp_{2n}(F)$ generated by $\{\alpha_1, \dots, \alpha_{n-1}\}$. Let $I(\sigma)$ and $I(\sigma')$ denote the representations of $SO_{2n+1}(F)$ and $Sp_{2n}(F)$ induced from σ on $M = GL_n(F)$, respectively. Then $I(\sigma)$ is irreducible if and only if $I(\sigma')$ is reducible.*

Plancherel measure $\mu(s, \sigma)$ for $I(s, \sigma)$ can now be easily computed by means of Corollary 3.6 of [23]. For example, consider $G = Sp_{2n}(F)$. In this case $\mu(s, \sigma) = \mu(s\tilde{\alpha}, \sigma)$ in the notation of [23]. Let $\varepsilon(s, \sigma, \wedge^2 \rho_n, \psi_F)$ denote the root number attached to $\wedge^2 \rho_n$ and σ in Section 7 of [23]. Then

$$\varepsilon(s, \sigma, \wedge^2 \rho_n, \psi_F) = cq^{-n(\wedge^2 \sigma)s}$$

where c is a nonzero complex number and the symbol $n(\wedge^2 \sigma)$ is an integer. Let $n(\sigma)$ be the conductor of σ . (See [15, 27].) Fix $\gamma(G/P) > 0$ as in [23, 31]. The following corollary is easy to prove.

COROLLARY 6.4. *Let σ be an irreducible unitary supercuspidal representation of $M = GL_n(F)$. Then*

$$\begin{aligned} \mu(s, \sigma) &= \gamma(G/P)^2 q^{n(\sigma) + n(\wedge^2 \sigma)} L(1 + 2s, \sigma, \wedge^2 \rho_n) L(1 - 2s, \tilde{\sigma}, \wedge^2 \rho_n) / \\ &\quad \times L(2s, \sigma, \wedge^2 \rho_n) L(-2s, \tilde{\sigma}, \wedge^2 \rho_n) \end{aligned}$$

where L -functions are defined as in Theorem 6.1.

The cases of $SO_{2n+1}(F)$ and $SO_{2n}(F)$ are similar.

7. Twisted endoscopy. The theory of twisted endoscopy, which is now being developed by Kottwitz and Shelstad [18], can be used to interpret our results (Theorems 5.3, 6.1–6.3) as a proof of Conjecture 3.2. In fact, the major reason for studying twisted endoscopy is to detect representations coming from the corresponding twisted endoscopic groups. In what follows we shall freely use notation and results from Arthur [1, 2].

Let us first assume $\tilde{\mathbf{G}}$ is an algebraic group over F . Let \mathbf{G} denote one of its components. Let \mathbf{G}^+ be the subgroup of $\tilde{\mathbf{G}}$ generated by \mathbf{G} . Denote by \mathbf{G}° the connected component of identity in \mathbf{G}^+ . We shall assume \mathbf{G}^+ is reductive. This is to say that \mathbf{G}° is a connected reductive algebraic group. We shall assume $G = \mathbf{G}(F)$ is not empty.

Given $x \in \mathbf{G}$, define a polynomial in a variable t as

$$(7.1) \quad \sum_k D_k(x)t^k = \det((t+1) - \text{Ad}(x)).$$

The smallest integer r for which $D_r(x)$ does not vanish is called the rank of \mathbf{G} . An element $\gamma \in \mathbf{G}$ is called \mathbf{G} -regular if $D_r(\gamma) \neq 0$. Then by Lemma 1 of [8] it is semisimple. In particular, $\text{Ad}(\gamma)$ is a semisimple automorphism of \mathbf{G}° . Moreover, the same lemma of [8] implies that the connected component of the centralizer of γ in \mathbf{G}° is a torus.

Let γ be any element in G . Given $f \in C_c^\infty(G)$, define

$$J(\gamma, f) = \int_{\mathbf{G}_\gamma^\circ(F) \backslash \mathbf{G}^\circ(F)} f(g^{-1}\gamma g) d\dot{g}$$

where $d\dot{g}$ is a fixed measure on $\mathbf{G}_\gamma^\circ(F) \backslash \mathbf{G}^\circ(F)$. (See [1].) Here, \mathbf{G}_γ° is the connected component of centralizer of γ in \mathbf{G}° which is reductive if γ is semisimple. We refer to Lemma 2.1 of [1] for the convergence of $J(\gamma, f)$. We need the following proposition (germ expansion for twisted orbital integrals).

PROPOSITION 7.1. *Fix a semisimple element $\sigma \in G$. Let $\gamma_0 = \sigma v_0$ be an element of $\sigma \mathbf{G}_\sigma^\circ(F)$ with $v_0 \in \mathbf{G}_\sigma^\circ(F)$ regular and semisimple. For every unipotent orbit \mathcal{O} of $\mathbf{G}_\sigma^\circ(F)$, let $\Gamma_\mathcal{O}^\sigma$ be the Shalika germ attached to \mathcal{O} and the F -points of the connected centralizer $\mathbf{G}_{v_0}^\circ$ of v_0 in \mathbf{G}_σ° . Then*

$$J(\sigma v, f) = \sum_{\mathcal{O}} \Gamma_\mathcal{O}^\sigma(v) J(\sigma u, f)$$

for v in the intersection of a neighbourhood of e in $\mathbf{G}_\sigma^\circ(F)$ with regular elements of $\mathbf{G}_{v_0}^\circ(F)$, depending on f . Here, for each \mathcal{O} , u represents an arbitrary element of \mathcal{O} .

Proof. Suppose first that $v \in \mathbf{G}_{v_0}^\circ(F)$ is strongly regular as an element in $\mathbf{G}_\sigma^\circ(F)$. (See [21].) Then $\mathbf{G}_{\sigma v}^\circ \cap \mathbf{G}_\sigma^\circ$ is the (connected) centralizer of v in \mathbf{G}_σ° . By connectedness

of this centralizer, $\mathbf{G}_{\sigma v} \cap \mathbf{G}_\sigma^\circ = \mathbf{G}_{\sigma v}^\circ \cap \mathbf{G}_\sigma^\circ$. Clearly, $\mathbf{G}_{v_0}^\circ = \mathbf{G}_{\sigma v}^\circ \cap \mathbf{G}_\sigma^\circ$. Then

$$(7.1.1) \quad J(\sigma v, f) = \int_{y \in \mathbf{G}_\sigma^\circ(F) \backslash \mathbf{G}^\circ(F)} \int_{x \in \mathbf{G}_{v_0}^\circ(F) \backslash \mathbf{G}_\sigma^\circ(F)} f(y^{-1}x^{-1}\sigma v x y) d\dot{x} d\dot{y}.$$

For every function $h \in C_c^\infty(\mathbf{G}_\sigma^\circ(F))$ and every $\delta \in \mathbf{G}_\sigma^\circ(F)$, let

$$J(\delta, h) = \int_{\mathbf{G}_\sigma^\circ(F) \backslash \mathbf{G}_\sigma^\circ(F)} h(x^{-1}\delta x) d\dot{x}$$

where $d\dot{x}$ is as in (7.1.1) if $\delta = v$. Given $y \in \mathbf{G}^\circ(F)$, $\delta \in \mathbf{G}_\sigma^\circ(F)$, and $f \in C_c^\infty(G)$, set

$$f_\sigma^y(\delta) = f(y^{-1}\sigma\delta y).$$

Then $f_\sigma^y \in C_c^\infty(\mathbf{G}_\sigma^\circ(F))$ and

$$J(\sigma v, f) = \int_{\mathbf{G}_\sigma^\circ(F) \backslash \mathbf{G}^\circ(F)} J(v, f_\sigma^y) d\dot{y}.$$

Using Shalika germ expansion [30] for orbital integrals attached to regular elements of $\mathbf{G}_{v_0}^\circ(F)$, a maximal torus of $\mathbf{G}_\sigma^\circ(F)$, one has

$$(7.1.2) \quad J(v, f_\sigma^y) = \sum_{\mathcal{O}} \Gamma_{\mathcal{O}}^\sigma(v) \mu_{\mathcal{O}}(f_\sigma^y)$$

for v in a neighbourhood N of identity in $\mathbf{G}_\sigma^\circ(F)$, depending on f , σ , and y . Here, for every $h \in C_c^\infty(\mathbf{G}_\sigma^\circ(F))$,

$$\mu_{\mathcal{O}}(h) = \int_{\mathbf{G}_u(F) \backslash \mathbf{G}_\sigma^\circ(F)} h(z^{-1}uz) dz$$

where $u \in \mathcal{O}$.

On the other hand, we need to consider

$$\int_{y \in \mathbf{G}_\sigma^\circ(F) \backslash \mathbf{G}^\circ(F)} \mu_{\mathcal{O}}(f_\sigma^y) d\dot{y} = \int_{\mathbf{G}_u(F) \backslash \mathbf{G}^\circ(F)} f(y^{-1}\sigma u y) d\dot{y} = J(\sigma u, f)$$

using $\mathbf{G}_u = \mathbf{G}_{\sigma u}^\circ$, which is implied by the properties of Jordan decomposition. Now by Lemma 2.1 of [1], one may assume that y belongs to a compact subset of $\mathbf{G}_\sigma^\circ(F) \backslash \mathbf{G}^\circ(F)$. Since f is smooth, this will allow us to choose the neighbourhood N of identity for which (7.1.2) holds, independent of y . This proves the proposition for the strongly regular v . The regular case is immediate.

We shall now specialize to the case of interest to us. Let w and θ be as in Section 3, i.e.,

$$w = \begin{pmatrix} & & & 1 \\ 0 & & -1 & \\ & & 1 & \\ & -1 & & \\ \cdot & & & 0 \end{pmatrix} \in GL_n$$

and, for every $g \in GL_n$, $\theta(g) = w^{-1}g^{-1}w$. It is an F -rational automorphism (in fact, an involution) of GL_n which fixes the (Borel) subgroup of upper triangulars and the subgroup of diagonals, as well as the standard splitting of GL_n . We let $\tilde{\mathbf{G}} = GL_n \rtimes \{1, \theta\}$, and we take $\mathbf{G} = GL_n \rtimes \{\theta\}$. Then $\mathbf{G}^+ = \tilde{\mathbf{G}}$ and $\mathbf{G}^\circ = GL_n$. We shall use θ -regular to call \mathbf{G} -regular elements of \mathbf{G} .

From now on we shall assume n is even. We shall only consider the twisted endoscopic group of GL_n coming from $1 \rtimes \hat{\theta}$, i.e., the group SO_{n+1} . (See Section 3 here and Section 9 of [2].)

A θ -regular (semisimple) element $\tilde{\gamma} \in \mathbf{G}(F)$ is of the form (γ, θ) , $\gamma \in GL_n(F)$, such that the connected component of its centralizer

$$\{g \in GL_n | (g, 1)^{-1}(\gamma, \theta)(g, 1) = (\gamma, \theta)\}$$

in GL_n , which is the connected component $\mathbf{G}_{\theta, \gamma}^\circ$ of the θ -twisted centralizer

$$\mathbf{G}_{\theta, \gamma} = \{g \in GL_n | g^{-1}\gamma\theta(g) = \gamma\}$$

of γ in GL_n , is a torus. The element γ (or $\tilde{\gamma}$ with abuse of notation) will be called strongly θ -regular if $\mathbf{G}_{\theta, \gamma}$ is connected.

Given a function $f \in C_c^\infty(GL_n(F))$ and a θ -regular element (or θ -conjugacy class) γ of $GL_n(F)$, we define

$$\Phi_\theta(\gamma, f) = \int_{\mathbf{G}_{\theta, \gamma}^\circ(F) \backslash GL_n(F)} f(g^{-1}\gamma\theta(g)) dg$$

where the measure dg is the ratio of two Haar measures on $GL_n(F)$ and $\mathbf{G}_{\theta, \gamma}^\circ(F)$ fixed as in [21]. It is clearly an orbital integral defined on the disconnected group $\tilde{\mathbf{G}} = GL_n \rtimes \{1, \theta\}$.

Suppose γ is changed to $\gamma' \in GL_n(F)$, stably θ -conjugate to γ , i.e., $\gamma' = g^{-1}\gamma\theta(g)$ for some $g \in GL_n(\bar{F})$ satisfying $g\sigma(g)^{-1} \in \mathbf{G}_{\theta, \gamma}^\circ$ for all $\sigma \in \Gamma(\bar{F}/F)$, where $\Gamma = \Gamma(\bar{F}/F)$ is the Galois group of \bar{F}/F . It is then easily checked that $\mathbf{G}_{\theta, \gamma}^\circ$ and $\mathbf{G}_{\theta, \gamma'}^\circ$ are inner forms, which being tori must be isomorphic. Transferring measures to $\Phi_\theta(\gamma', f)$ can now be done with no ambiguity. Assume now that γ is strongly θ -regular and set

$$\Phi_\theta^{\text{st}}(\gamma, f) = \sum_{\gamma' \sim \gamma} \Phi_\theta(\gamma', f)$$

where γ' runs over representatives for all the θ -regular (semisimple) conjugacy classes $\{\gamma'\}$ which lie inside the stable θ -conjugacy class of γ .

Given a regular semisimple element $\delta \in SO_{n+1}(F)$, we let \mathbf{H}_δ° be the connected component of its centralizer. Again as in [21], we set

$$\Phi(\delta, f) = \int_{\mathbf{H}_\delta^\circ(F) \backslash SO_{n+1}(F)} f(h^{-1}\delta h) d\dot{h}$$

for every $f \in C_c^\infty(SO_{n+1}(F))$, where $d\dot{h}$ is again defined as in [21]. We call δ strongly regular [21] if $\mathbf{H}_\delta = \mathbf{H}_\delta^\circ$. We can then define

$$\Phi^{\text{st}}(\delta, f) = \sum_{\delta' \sim \delta} \Phi(\delta', f)$$

where δ is strongly regular and δ' runs over representatives for all the regular semisimple conjugacy classes $\{\gamma'\}$ which lie inside the stable conjugacy class of δ .

Next, we need to define a norm map. We follow [18] and [21]. Let \mathbf{T} be a Cartan subgroup of GL_n fixed by θ . Use \mathbf{T}_θ to denote $\mathbf{T}/(1 - \theta)\mathbf{T}$. It can then be identified over F with a Cartan subgroup of \mathbf{H} and conversely. As in [21], this then establishes a Γ -map $\mathcal{A} = \mathcal{A}GL_n/\mathbf{H}$ between semisimple conjugacy classes of $\mathbf{H}(\bar{F}) = \mathbf{H}$ and θ -semisimple θ -conjugacy classes of $GL_n(\bar{F})$. An element δ in $H = \mathbf{H}(F)$ is called strongly \mathbf{G}° -regular if the image of its conjugacy class under \mathcal{A} consists of strongly θ -regular elements. We now define the norm as follows.

An element δ in $H = \mathbf{H}(F)$ is called norm of an element γ in $GL_n(F)$ and is denoted by $\mathcal{N}\gamma$ if the $GL_n(\bar{F})$ - θ -conjugacy class of γ is the image of $\mathbf{H}(\bar{F})$ -conjugacy class of δ under \mathcal{A} . If γ is strongly θ -regular, then $\mathcal{N}\gamma$ is strongly regular.

The following assumption is just a special case of a basic general assumption in the theory of twisted endoscopy [18]. (See [21] for ordinary endoscopy.) Without it the stabilization of the twisted trace formula [9] will not be possible.

ASSUMPTION 7.2. *Given a function $f \in C_c^\infty(GL_n(F))$, there exists a function $f^{\mathbf{H}} \in C_c^\infty(SO_{n+1}(F))$, $\mathbf{H} = SO_{n+1}$, such that*

$$(7.2) \quad \Phi_\theta^{\text{st}}(\gamma, f) = \Phi^{\text{st}}(\delta, f^{\mathbf{H}})$$

for every strongly θ -regular (semisimple) element γ in $GL_n(F)$ if $\delta = \mathcal{N}\gamma$, and $\Phi^{\text{st}}(\delta, f^{\mathbf{H}}) = 0$ otherwise.

The fact that transfer factors are simply the ratios of discriminants (which is a constant in this case and can be absorbed in the defining measures) and that there are no signs is a reflection of the fact that ${}^LH^\circ = Sp_n(\mathbb{C})$ is the centralizer of $1 \times \hat{\theta}$ in $GL_n(\mathbb{C})$ rather than $\varepsilon \times \hat{\theta}$, where $\varepsilon = \text{diag}(1, -1, 1, \dots)$.

We shall now deduce a useful corollary of this assumption.

PROPOSITION 7.3. *Given $f \in C_c^\infty(GL_n(F))$, let $f^{\mathbf{H}}$ be as in Assumption 7.2. Then up to a nonzero constant*

$$\int_{Sp_n(F) \backslash GL_n(F)} f({}^t g w^{-1} g w) d\dot{g}$$

is equal to $f^{\mathbf{H}}(e)$. Consequently, if $f \in C_c^\infty(GL_n(F))$ is such that $f^{\mathbf{H}}(e) \neq 0$, then (5.2) (or equivalently (5.3)) holds and conversely.

Proof. This is a consequence of Proposition 7.1 applied to the left-hand side of (7.2), together with the homogeneity of Shalika germs and the fact that $\Gamma_{\{e\}}^{(1, \theta)}$ (attached to $Sp_n(F)$), as well as the one attached to the trivial orbit of $SO_{n+1}(F)$, are nonzero constants, if (v_0, θ) , $v_0 \in Sp_n(F)$, is appropriately chosen. In fact, choose a strongly θ -regular element $v_0 \in Sp_n(F)$ so that \mathbf{G}_{θ, v_0} is anisotropic. Then as an element of $Sp_n(F)$, v_0 is elliptic in the usual sense. The same is true of \mathcal{N}_{v_0} as an element of $\mathbf{H}(F)$.

We can now define what it means for a supercuspidal representation of $GL_n(F)$ to come from $SO_{n+1}(F)$.

Definition 7.4. Let σ be an irreducible self-dual supercuspidal representation of $GL_n(F)$. Then σ is said to come from $SO_{n+1}(F)$ if and only if n is even and there exists a function $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of σ , for which $f^{\mathbf{H}}(e) \neq 0$, where $\mathbf{H} = SO_{n+1}$.

We justify our definition by observing that if $f^{\mathbf{H}}(e) \neq 0$, then by the Plancherel formula for $SO_{n+1}(F)$, there must exist a stable tempered (in fact discrete series) character of an L -packet on $SO_{n+1}(F)$ which does not vanish on $f^{\mathbf{H}}$. (See [12, 23].) It is this L -packet which is supposed to lift to σ by means of twisted endoscopic transfer [18]. This must be then the packet attached to φ_0 as a map into $Sp_n(\mathbb{C}) = {}^L H^\circ$.

Remark. The automatic condition $\omega \equiv 1$ (beside $\sigma \cong \bar{\sigma}$), if σ comes from $SO_{n+1}(F)$, is in complete agreement with the parametrization problem. In fact, if $\varphi_0: W_F \rightarrow GL_n(\mathbb{C})$, attached to σ , factors through $Sp_n(\mathbb{C})$, then $\det(\varphi_0) = 1$ implies $\omega = 1$.

Definition 7.5. Let σ be an irreducible self-dual supercuspidal representation of $GL_n(F)$. Then

- (a) σ is said to come from $SO_n^*(F)$ if and only if n is even and σ does not come from $SO_{n+1}(F)$, and
- (b) σ is said to come from $Sp_{n-1}(F)$ if and only if n is odd.

We can now restate Propositions 3.5 and 3.9 and Theorems 5.3 and 6.3 as the following theorem. Assuming the existence of twisted endoscopic transfer, it answers our Conjecture 3.2 positively.

THEOREM 7.6. (a) *Let $\mathbf{G} = SO_{2n+1}$ and fix an irreducible (unitary) self-dual supercuspidal representation σ of $M = GL_n(F)$. Then $I(\sigma)$ is irreducible if and only if σ*

comes by twisted endoscopic transfer from a tempered L -packet of either $SO_n^*(F)$ or $Sp_{n-1}(F)$.

(b) Suppose $\mathbf{G} = Sp_{2n}$ and fix σ as in part (a). Then $I(\sigma)$ is irreducible if and only if σ comes by twisted endoscopic transfer from $SO_{n+1}(F)$.

(c) Suppose $\mathbf{G} = SO_{2n}$ and fix σ as in part (a). Then $I(\sigma)$ is irreducible if and only if σ comes by twisted endoscopic transfer from either $SO_{n+1}(F)$ or $Sp_{n-1}(F)$.

Finally, we restate Theorems 6.1 and 6.2 on L -functions as follows.

THEOREM 7.7. *Let σ be an irreducible supercuspidal representation of $GL_n(F)$.*

(a) *The L -function $L(s, \sigma, \wedge^2 \rho_n) \equiv 1$ unless some unramified twist of σ is self-dual. Assume σ is also self-dual. Let S be the set (possibly empty) of all the unramified characters η , no two of which have equal squares, for which $\sigma \otimes \eta \cdot \det$ comes from $SO_{n+1}(F)$. Then*

$$L(s, \sigma, \wedge^2 \rho_n) = \prod_{\eta \in S} (1 - \eta^2(\varpi)q^{-s})^{-1}.$$

(b) *The L -function $L(s, \sigma, \text{Sym}^2(\rho_n)) \equiv 1$ unless some unramified twist of σ is self-dual. Assume σ is also self-dual. If σ comes from $Sp_{n-1}(F)$, then*

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = (1 - q^{-rs})^{-1}.$$

Otherwise, let S' be the set (possibly empty) of all the unramified characters η , no two of which have equal squares, for which $\sigma \otimes \eta \cdot \det$ comes from $SO_n^(F)$. Then*

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = \prod_{\eta \in S'} (1 - \eta^2(\varpi)q^{-s})^{-1}.$$

8. L -functions in general. We shall now assume σ is any irreducible admissible representation of $GL_n(F)$ and compute the L -functions $L(s, \sigma, \wedge^2 \rho_n)$ and $L(s, \sigma, \text{Sym}^2(\rho_n))$ by means of the theory developed in [23]. By the construction of these factors explained in Section 7 of [23], we may assume σ is in the discrete series.

Let us first treat $L(s, \sigma, \wedge^2 \rho_n)$. Fix a nontrivial additive character ψ_F of F and let $\gamma(s, \sigma, \wedge^2 \rho_n, \psi_F)$ be the factor attached to σ and $\wedge^2 \rho_n$ by Theorem 3.5 of [23]. Then the L -function $L(s, \sigma, \wedge^2 \rho_n)$ is exactly the inverse of the normalized polynomial which gives us the numerator of $\gamma(s, \sigma, \wedge^2 \rho_n, \psi_F)$. More precisely, there exists a monomial $\varepsilon(s, \sigma, \wedge^2 \rho_n, \psi_F)$ such that

$$(8.1) \quad \gamma(s, \sigma, \wedge^2 \rho_n, \psi_F) = \varepsilon(s, \sigma, \wedge^2 \rho_n, \psi_F) L(1 - s, \tilde{\sigma}, \wedge^2 \rho_n) / L(s, \sigma, \wedge^2 \rho_n).$$

Since σ is in the discrete series, there exists two positive integers a and b with $ab = n$ and an irreducible unitary supercuspidal representation π_0 of $GL_a(F)$ such that σ is the unique discrete series component of the representation of $GL_n(F)$

induced from $\pi_1 \otimes \cdots \otimes \pi_b$, where

$$\pi_i = \pi_0 \otimes \alpha^{(b+1)/2-i},$$

$1 \leq i \leq b$, and $\alpha(g) = |\det(g)|$. (See [3, 35] and Proposition 9.2 of [14].)

Applying part 3 of Theorem 3.5 of [23] to the above data on the group SO_{2n} immediately implies

$$(8.2) \quad \gamma(s, \sigma, \wedge^2 \rho_n, \psi_F) = \prod_{i=1}^b \gamma(s, \pi_i, \wedge^2 \rho_a, \psi_F) \prod_{1 \leq i < j \leq b} \gamma(s, \pi_i \times \pi_j, \psi_F)$$

where each $\gamma(s, \pi_i \times \pi_j, \psi_F)$ is the Rankin-Selberg factor attached to the pair (π_i, π_j) . (See [14, 27].)

Clearly,

$$\gamma(s, \pi_i, \wedge^2 \rho_a, \psi_F) = \gamma(s + (b + 1) - 2i, \pi_0, \wedge^2 \rho_a, \psi_F),$$

$1 \leq i \leq b$, while

$$\gamma(s, \pi_i \times \pi_j, \psi_F) = \gamma(s + (b + 1) - (i + j), \pi_0 \times \pi_0, \psi_F).$$

They all satisfy equation (7.4) of [23], where the L -functions on the right-hand side of equation (7.4) of [23], when applied to $\gamma(s, \pi_0, \wedge^2 \rho_a, \psi_F)$, are those defined by Theorem 7.7 of this paper, while the L -function $L(s, \pi_0 \times \pi_0)$ for $\gamma(s, \pi_0 \times \pi_0, \psi_F)$ is that of the Rankin-Selberg L -function discussed in Section 3. (See [14, 27].)

Using the discussion in this paper and those of [14], it is clear that $L(s, \pi_0, \wedge^2 \rho_a)$ and $L(s, \pi_0 \times \pi_0)$ are both identically one unless some unramified twist of π_0 is self-dual. It then implies $\sigma \cong \tilde{\sigma}$ and conversely, and therefore we conclude that $\gamma(s, \sigma, \wedge^2 \rho_n, \psi_F) \equiv 1$ unless some unramified twist of σ is self-dual.

Since π_0 is both unitary and supercuspidal, the argument in the proof of Theorem 5.5 of [29] (page 287) implies that, up to a monomial in q^{-s} , the L -function $L(1 - s, \tilde{\pi}_0, \wedge^2 \rho_a)$ is equal to $L(s - 1, \pi_0, \wedge^2 \rho_a)$, and similarly for $L(1 - s, \tilde{\pi}_0 \times \tilde{\pi}_0)$ and $L(s - 1, \pi_0 \times \pi_0)$. Therefore, up to a monomial in q^{-s} , the right-hand side of (8.2) equals

$$(8.3) \quad \prod_{v=-(b-1)/2}^{(b-1)/2} L(s + 2v - 1, \pi_0, \wedge^2 \rho_a) / L(s + 2v, \pi_0, \wedge^2 \rho_a)$$

times

$$(8.4) \quad \prod_{1 \leq i < j \leq b} L(s + b - (i + j), \pi_0 \times \pi_0) / L(s + b + 1 - (i + j), \pi_0 \times \pi_0).$$

It is a trivial matter to show that (8.4) is equal to

$$(8.5) \quad \prod_{i=1}^{b-1} L(s-i, \pi_0 \times \pi_0) / L(s+b-2i, \pi_0 \times \pi_0).$$

We shall now apply Lemma 3.6 to every L -function in (8.5). First, assume b is even. Then the numerator of (8.5) is equal to

$$\prod_{k=0}^{(b-2)/2} L(s-(2k+1), \pi_0, \text{Sym}^2(\rho_a)) L(s-(2k+1), \pi_0, \wedge^2 \rho_a)$$

while its denominator, when divided by

$$\prod_{k=0}^{(b-2)/2} L(s+2k, \pi_0, \text{Sym}^2(\rho_a)),$$

is equal to the numerator of (8.3) divided by $L(s-b, \pi_0, \wedge^2 \rho_a)$. From this it follows that the normalized numerator of (8.2) is equal to

$$(8.6) \quad \prod_{i=1}^{b/2} L(s, \pi_i, \wedge^2 \rho_a)^{-1} L(s, \pi_i \otimes \alpha^{-1/2}, \text{Sym}^2(\rho_a))^{-1}$$

while its normalized denominator is

$$\prod_{i=1}^{b/2} L(1-s, \tilde{\pi}_i, \wedge^2 \rho_a)^{-1} L(1-s, \tilde{\pi}_i \otimes \alpha^{-1/2}, \text{Sym}^2(\rho_a))^{-1}$$

where $\tilde{\pi}_i = \tilde{\pi}_0 \otimes \alpha^{(b+1)/2-i}$. They have no factors in common, and therefore by (8.1), $L(s, \sigma, \wedge^2 \rho_n)$ is equal to the inverse of (8.6). The case of odd n can be treated the same way. We record our results as the following proposition.

PROPOSITION 8.1. *Let σ be a discrete series representation of $GL_n(F)$. Choose an irreducible unitary supercuspidal representation π_0 of $GL_a(F)$, $n = ab$, such that σ is the unique discrete series component of the representation of $GL_n(F)$ induced from $\pi_1 \otimes \cdots \otimes \pi_b$, $\pi_i = \pi_0 \otimes \alpha^{(b+1)/2-i}$, $1 \leq i \leq b$. Let $\tilde{\pi}_i = \tilde{\pi}_0 \otimes \alpha^{(b+1)/2-i}$.*

(a) *Suppose n is even. Then*

$$L(s, \sigma, \wedge^2 \rho_n) = \prod_{i=1}^{b/2} L(s, \pi_i, \wedge^2 \rho_a) L(s, \pi_i \otimes \alpha^{-1/2}, \text{Sym}^2(\rho_a)),$$

$$L(s, \tilde{\sigma}, \wedge^2 \rho_n) = \prod_{i=1}^{b/2} L(s, \tilde{\pi}_i, \wedge^2 \rho_a) L(s, \tilde{\pi}_i \otimes \alpha^{-1/2}, \text{Sym}^2(\rho_a)),$$

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = \prod_{i=1}^{b/2} L(s, \pi_i, \text{Sym}^2(\rho_n)) L(s, \pi_i \otimes \alpha^{-1/2}, \wedge^2 \rho_a),$$

and

$$L(s, \tilde{\sigma}, \text{Sym}^2(\rho_n)) = \prod_{i=1}^{b/2} L(s, \tilde{\pi}_i, \text{Sym}^2(\rho_a)) L(s, \tilde{\pi}_i \otimes \alpha^{-1/2}, \wedge^2 \rho_a).$$

(b) Suppose n is odd. Then

$$L(s, \sigma, \wedge^2 \rho_n) = \prod_{i=1}^{(b+1)/2} L(s, \pi_i, \wedge^2 \rho_a) \prod_{i=1}^{(b-1)/2} L(s, \pi_i \otimes \alpha^{-1/2}, \text{Sym}^2(\rho_a))$$

and

$$L(s, \sigma, \text{Sym}^2(\rho_n)) = \prod_{i=1}^{(b+1)/2} L(s, \pi_i, \text{Sym}^2(\rho_a)) \prod_{i=1}^{(b-1)/2} L(s, \pi_i \otimes \alpha^{-1/2}, \wedge^2 \rho_a),$$

and similarly for $\tilde{\sigma}$.

COROLLARY 8.2. *Let σ be an irreducible admissible representation of $GL_n(F)$. Then*

$$L(s, \sigma \times \sigma) = L(s, \sigma, \wedge^2 \rho_n) L(s, \sigma, \text{Sym}^2(\rho_n)).$$

Proof. It is enough to assume σ is in the discrete series. The corollary is then a consequence of the formulas in Proposition 8.1, Lemma 3.6, and the formula

$$L(s, \sigma \times \sigma) = \prod_{i=1}^b L(s, \pi_1 \times \pi_i)$$

of Proposition 9.2 of [14].

9. R-groups for representations induced from discrete series. In this section we shall use the results of Section 8 to determine the reducibility of $I(\sigma)$ when σ is now any discrete series representation of $GL_n(F)$. In fact, our Theorem 9.1 generalizes Theorem 7.6 to any discrete series representation. We may assume σ is self-dual.

Choose positive integers a and b with $ab = n$ and let π_0 be an irreducible unitary supercuspidal representation of $GL_a(F)$ such that σ is the unique discrete series component of the representation of $GL_n(F)$ induced from $\pi_1 \otimes \cdots \otimes \pi_b$, where $\pi_i = \pi_0 \otimes \alpha^{(b+1)/2-i}$. The representation σ being self-dual is then equivalent to $\pi_0 \cong \tilde{\pi}_0$.

By Corollary 3.6 of [23] we need to check whether

$$L(1 + s, \sigma, r_1) L(s, \sigma, r_1)^{-1}$$

has a zero at $s = 0$, where either $\mathbf{G} = SO_{2n}$ or SO_{2n+1} with $r_1 = \wedge^2 \rho_n$ or $r_1 =$

$\text{Sym}^2(\rho_n)$, respectively, while if $\mathbf{G} = Sp_{2n}$, we need to check

$$L(1 + s, \sigma, \rho_n)L(1 + 2s, \sigma, \wedge^2 \rho_n)L(s, \sigma, \rho_n)^{-1}L(2s, \sigma, \wedge^2 \rho_n)^{-1}$$

at $s = 0$. In this case $L(s, \sigma, \rho_n) \equiv 1$ unless $a = 1$ in which case $L(s, \sigma, \rho_n) = L(s, \pi_1)$, the Hecke L -function attached to π_1 .

Thus, let us first assume $a > 1$ so that the situation of $SO_{2n}(F)$ and $Sp_{2n}(F)$ becomes similar. Assume b is even. Using the formulas in Proposition 8.1, it can be checked that

$$L(1, \sigma, \wedge^2 \rho_n)L(0, \sigma, \wedge^2 \rho_n)^{-1}$$

is zero if and only if $L(s, \pi_{b/2}, \text{Sym}^2(\rho_a))$ has a pole at $s = 0$. Thus, for $Sp_{2n}(F)$ with $a > 1$ and $\sigma \cong \tilde{\sigma}$, $I(\sigma)$ is irreducible if and only if π_0 comes from either $SO_a^*(F)$ or $Sp_{a-1}(F)$. For $SO_{2n}(F)$ we observe that, since b is even, then so is n . Assume $\sigma \cong \tilde{\sigma}$. Then a similar criterion applies.

Now suppose b is odd and again $\sigma \cong \tilde{\sigma}$. For $Sp_{2n}(F)$ assume $a > 1$. Then by Proposition 8.1

$$L(1, \sigma, \wedge^2 \rho_n)L(0, \sigma, \wedge^2 \rho_n)^{-1}$$

is zero if and only if $L(s, \pi_{(b+1)/2}, \wedge^2 \rho_n)$ has a pole at $s = 0$. Consequently, $I(\sigma)$ is irreducible if and only if π_0 comes from $SO_{a+1}(F)$. For $SO_{2n}(F)$ we may assume n is even. Then the same criterion applies.

Now, suppose $a = 1$ and $G = Sp_{2n}(F)$. Assume $n = b$ is even. Then $L(s, \sigma) = L(s, \pi_1)$ will have no pole at $s = 0$. The L -function $L(s, \pi_i, \wedge^2 \rho_a) \equiv 1$ while the corresponding symmetric square L -function, which is

$$\prod_{i=1}^{n/2} L(n - 2i, \pi_0^2),$$

will always have a pole since $\pi_0^2 = 1$. Therefore, $I(\sigma)$ is always irreducible. This is exactly what it means for π_0 to come from $Sp_0(F) = \{e\}$. If n is odd, then

$$L(s, \sigma) = L(s + (n - 1)/2, \pi_0)$$

will have a pole at $s = 0$ exactly when $n = 1$ and $\pi_0 = 1$. Then $\sigma = \pi_0$ is a character. No other L -function appears and $I(\sigma)$ is irreducible if and only if $\pi_0 = 1$. Observe that $\pi_0^2 = 1$. This is the classical case of $SL_2(F)$. Therefore, we may assume $n > 1$. Then there is no contribution from $L(s, \sigma)$. Although $\pi_0^2 = 1$, the symmetric square L -function in the denominator can have no poles at $s = 0$ since n is odd. We therefore conclude that, if n is odd and $\sigma \cong \tilde{\sigma}$, then $I(\sigma)$ is reducible.

Similar arguments apply to $SO_{2n+1}(F)$, and if b is even, one obtains that, for $\sigma \cong \tilde{\sigma}$, $I(\sigma)$ is irreducible if and only if π_0 comes from $SO_{a+1}(F)$. On the other hand, if b is

odd, then for $\sigma \cong \tilde{\sigma}$, $I(\sigma)$ is irreducible if and only if π_0 comes from either $SO_a^*(F)$ or $Sp_{a-1}(F)$.

Putting everything together and leaving out the trivial case of $Sp_2 = SL_2$, we have proved the following theorem.

THEOREM 9.1. *Let G be either of the groups SO_{2n} , Sp_{2n} , $n > 1$, or SO_{2n+1} . Let σ be a discrete series representation of $M = GL_n(F)$, the F -points of the GL_n -Levi subgroup of either of these groups. Choose positive integers a and b , $ab = n$, and an irreducible unitary supercuspidal representation π_0 of $GL_a(F)$, defining σ as before. Assume $\sigma \cong \tilde{\sigma}$ or equivalently $\pi_0 \cong \tilde{\pi}_0$.*

If b is even, then for $G = SO_{2n}(F)$ and $Sp_{2n}(F)$, $I(\sigma)$ is irreducible if and only if π_0 comes either from $SO_a^(F)$ or $Sp_{a-1}(F)$, while for $G = SO_{2n+1}(F)$ a similar statement about $I(\sigma)$ is true if and only if π_0 comes from $SO_{a+1}(F)$.*

Suppose b is odd. Then exactly the opposite situation happens; i.e., one should only change the role of $SO_{2n+1}(F)$ with that of $SO_{2n}(F)$ and $Sp_{2n}(F)$ in the above statements.

The following corollary generalizes Proposition 3.5.

COROLLARY 9.2. *Let $G = Sp_{2n}(F)$ and take σ in the discrete series of $GL_n(F)$. Suppose $n > 1$ is odd and $\sigma \cong \tilde{\sigma}$. Then $I(\sigma)$ is always reducible.*

Proof. a and b are both odd.

COROLLARY 9.3. *Theorem 6.3 is still valid if σ is any discrete series representation of $GL_n(F)$.*

10. Further nonvanishing results. In this section we shall calculate directly the residues of the intertwining operators for the GL_n -Levi subgroups of $Sp_{2n}(F)$ and $SO_{2n+1}(F)$. Using the results from the previous sections, this leads to some very interesting nonvanishing statements about twisted orbital integrals, both semi-simple and unipotent, which may be interpreted directly. At any rate, the fact that their nonvanishing can be understood by means of our results is quite fascinating. We refer to Lemma 2.1 of [1] for the convergence of all the twisted orbital integrals that appear in this section. We start with $G = Sp_{2n}(F)$.

Notation will be exactly as before, and we shall calculate the poles of the intertwining operator by using functions which are of compact support in PN^- modulo P (Lemma 4.1). Since all the steps and techniques are exactly the same as in Proposition 5.1, we shall only record the results.

PROPOSITION 10.1. *Let σ be an irreducible unitary supercuspidal representation of $GL_n(F)$, considered as the GL_n -Levi subgroup of $Sp_{2n}(F)$. Then the intertwining operator $A(s, \sigma, w_0)$ has a pole at $s = 0$ if and only if $\sigma \cong \tilde{\sigma}$, $\omega \equiv 1$, and there exists a function $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of σ for which*

$$(10.1) \quad \sum_{\alpha} \int_{\mathbf{G}_{\theta, w^{-1}\alpha}(F) \backslash GL_n(F)} f(\theta(g)w^{-1}\alpha g^{-1}) d\dot{g} \neq 0.$$

The sum runs over diagonal matrices α , each defining an inequivalent class of quadratic forms over F , and $\mathbf{G}_{\theta, w^{-1}\alpha}$ is the θ -twisted centralizer of $w^{-1}\alpha$ with w as in Section 3.

Remark 1. The sum in (10.1) is already indexed by $H^1(\Gamma, O(n))$ and therefore is over all the θ -twisted conjugacy classes within a $GL_n(\bar{F})$ - θ -conjugacy class which is in fact singular. It should be pointed out that, since $O(n)$ is not connected, the sum is in fact over several stable θ -conjugacy classes. This is a good example of when $GL_n(\bar{F})$ - θ -conjugacy and stable θ -conjugacy are not the same.

COROLLARY 10.2. *The nonvanishing condition (10.1) is valid if σ comes from $SO_{n+1}(F)$.*

Remark 2. It would be interesting to interpret (10.1) by means of twisted endoscopy [18] and obtain further nonvanishing results. But since the orbits are singular, one has to wait until results of Kottwitz (Section 3 of [17]) on singular orbital integrals are generalized to both the ordinary (when the endoscopic group is of strictly lower dimension) and the twisted endoscopy.

Next, we study the mysterious and fascinating case of $SO_{2n+1}(F)$. The residue at the pole $s = 0$ is now much more complicated than the two other cases. But we still manage to write them in terms of θ -twisted orbital integrals. Curiously enough, they are no longer semisimple!

Let $G = SO_{2n+1}(F)$ and let $P = MN$ be the standard maximal parabolic subgroup of G , having $M = GL_n(F)$ as its Levi factor. The permutation w_0 can be taken to be

$$w_0 = \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \in SO_{2n+1}(F).$$

Here, we shall only treat the case of even n . The odd case is similar. An element $n \in N$ is of the form

$$(10.2) \quad n = \begin{bmatrix} 1 & 0 & X \\ -{}^tX & I_n & Y \\ 0 & 0 & I_n \end{bmatrix}$$

with $X \in F^n$ and $Y \in M_n(F)$, satisfying

$$(10.3) \quad Y + {}^tY + {}^tXX = 0.$$

We start with the following lemma.

LEMMA 10.3. *Suppose $w_0n = pn^-$ with $n \in N$ as in (10.2),*

$$p = \begin{bmatrix} 1 & 0 & \alpha \\ -a \cdot {}^t\alpha & a & a\beta \\ 0 & 0 & {}^t a^{-1} \end{bmatrix} \in P,$$

and

$$n^- = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & I_n & 0 \\ -{}^t\gamma & \delta & I_n \end{bmatrix} \in N^-.$$

Then $Y \in GL_n(F)$, $\alpha = X$, $\beta = a^{-1} = {}^tY$, $\delta = {}^t a = Y^{-1}$, and $\gamma = X{}^tY^{-1}$. Moreover, X and Y must satisfy (10.3) and

$$(10.4) \quad XY^{-1} \cdot {}^tX = 0.$$

Write $Z = Y{}^tY^{-1}$. Then (10.3) and (10.4) imply

$$(10.5) \quad (Z + I)^2 = 0$$

where $I = I_n$. Conversely, if ${}^tY \neq -Y$, then (10.3) and (10.5) imply (10.4).

Proof. The only extra conditions which must be satisfied are $\alpha \cdot {}^t\gamma = 0$ which leads to (10.4),

$$(10.3.1) \quad \gamma + \alpha\delta = 0,$$

$$(10.3.2) \quad a{}^t\alpha + a\beta{}^t\gamma = 0,$$

and

$$(10.3.3) \quad -a{}^t\alpha\gamma + a + a\beta\delta = 0.$$

But (10.3) and (10.4) imply

$$XY^{-1}(Y + {}^tY) = 0$$

which leads to

$$(10.3.4) \quad X{}^tY^{-1} + XY^{-1} = 0.$$

This then implies (10.3.1) and (10.3.2) immediately. Finally, observe that (10.3.3) together with (10.3) imply

$$Y{}^tY^{-1} + 2I + {}^tYY^{-1} = 0$$

and therefore (10.5).

Condition (10.3) with (X, Y) replaced with (α, β) and (γ, δ) are automatically satisfied. Finally, observe that ${}^tY = -Y$ if and only if $X = 0$. Moreover, equation

(10.5) gives

$$({}^tY + Y)Y^{-1}({}^tY + Y) = 0$$

which together with (10.3) imply ${}^tXXY^{-1} \cdot {}^tXX = 0$. This leads to (10.4) if we assume $X \neq 0$, completing the lemma.

LEMMA 10.4. *Suppose $Z = Y{}^tY^{-1}$, $Y \in GL_n(F)$, satisfies (10.5). Then its Jordan canonical form is*

$$D_m = \begin{pmatrix} -I_m & 0 \\ 0 & -U_{n-m} \end{pmatrix}$$

for some nonnegative even integer m , where

$$U_{n-m} = \begin{pmatrix} u_0 & & & \\ & u_0 & & \\ & & \ddots & \\ 0 & & & u_0 \end{pmatrix}$$

with $u_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and therefore there exists a $g \in GL_n(\bar{F})$ such that

$$\mathcal{N}(g^{-1}Yw\theta(g)) = -D_m.$$

Here, w and θ are as in Sections 3 and 7, and the norm of a θ -twisted conjugacy class v is defined by the conjugacy class

$$\mathcal{N}(v) = \{v\theta(v)\}$$

inside $GL_n(\bar{F})$.

Proof. It is trivial.

Let σ be an irreducible (unitary) self-dual supercuspidal representation of $GL_n(F)$, where $GL_n(F)$ is considered as GL_n -Levi subgroup of $SO_{2n+1}(F)$. To study the poles of the corresponding intertwining operator, we again choose a function of compact support in PN^- modulo P and proceed as in the proof of Proposition 5.1. Using Lemmas 10.3 and 10.4, we see that we need to consider

$$(10.6) \quad \sum_{\substack{n \geq m \geq 0 \\ m \equiv 0(2)}} \sum_{\mathcal{N}(u) = -D_m} \int_{\mathbf{G}_{\theta, u}(F) \backslash GL_n(F)} \Phi(g^{-1}u\theta(g))\psi(g^{-1}u\theta(g))|\det(g^{-1}u\theta(g))|^s dg$$

where, given m , u runs over all the θ -twisted conjugacy classes in $GL_n(F)$ whose

norm is $-D_m$. Here, Φ is a Schwartz function on $M_n(F)$ and ψ is a matrix coefficient of σ . The fact that there are only a finite number of θ -conjugacy classes of a given norm can be verified directly.

Choose $f \in C_c^\infty(GL_n(F))$ which defines ψ by equation (5.1). Then (10.6) is basically equivalent to

$$(10.7) \quad \sum_{\substack{n \geq m \geq 0 \\ m \equiv 0(2)}} \sum_{\mathcal{N}(u) = -D_m} \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \int_{\mathbf{G}_{\theta, \varepsilon u}(F) \setminus GL_n(F)} f(g^{-1} \varepsilon u \theta(g)) |\det(g^{-1} u \theta(g))|^s dg \\ \cdot \int_{|\det z| > \kappa} |\det z|^{-2s} dz$$

for some $\kappa > 0$. Observe that the integral over $|\det z| > \kappa$ is basically $L(2ns, \mathbf{1})$. Therefore, the residue at $s = 0$ is proportional to

$$(10.8) \quad \sum_{\substack{n \geq m \geq 0 \\ m \equiv 0(2)}} \sum_{\mathcal{N}(u) = -D_m} \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \int_{\mathbf{G}_{\theta, \varepsilon u}(F) \setminus GL_n(F)} f(g^{-1} \varepsilon u \theta(g)) dg.$$

Using the notation of Section 7, we let

$$\Phi_\theta(\varepsilon u, f) = \int_{\mathbf{G}_{\theta, \varepsilon u}(F) \setminus GL_n(F)} f(g^{-1} \varepsilon u \theta(g)) dg.$$

We have proved the following proposition.

PROPOSITION 10.5. *Let σ be an irreducible unitary supercuspidal representation of $GL_n(F)$, where $GL_n(F)$ is considered as the GL_n -Levi subgroup of $SO_{2n+1}(F)$. Assume n is even. Then the intertwining operator $A(s, \sigma, w_0)$ has a pole at $s = 0$ if and only if $\sigma \cong \tilde{\sigma}$ and there exists a function $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of σ for which*

$$(10.9) \quad \sum_{\substack{n \geq m \geq 0 \\ m \equiv 0(2)}} \sum_{\mathcal{N}(u) = -D_m} \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \Phi_\theta(\varepsilon u, f) \neq 0.$$

COROLLARY 10.6. *The nonvanishing condition (10.9) is valid if and only if σ comes from $SO_n^*(F)$.*

Now suppose $n = 2$ as to study the case of $SO_5(F)$. In particular, we give a completely local proof of Proposition 8.4 of [23] which may also be considered as a new proof of Proposition 5.1 of [34]. (See the remark after Proposition 10.7.) Let ε and ε' be representatives for two distinct classes in $(F^*)^2 \setminus F^*$. Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ represent the regular unipotent conjugacy class in $GL_2(F)$. Then εu and $\varepsilon' u$ are θ -conjugate if and only if $\varepsilon = \varepsilon'$. Consequently, the left-hand side of (10.9) can be

written as

$$\sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \Phi_\theta(\varepsilon u, f) + \Phi_\theta(1, f).$$

Suppose σ comes from $SO_n^*(F)$. Then $\Phi_\theta(1, f) = 0$, which can be easily shown to be equivalent to $\omega \neq 1$ since $n = 2$. Consequently, (10.9) equals

$$(10.10) \quad \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \Phi_\theta(\varepsilon u, f).$$

Finally, observe that $\{\varepsilon u\}_\varepsilon$ gives a complete set of representatives for the unipotent θ -conjugacy classes of GL_2 of highest dimension.

Assume $\omega \neq 1$, $\omega^2 = 1$, and let $\mathbf{H} = SO_2^*$ be the anisotropic torus defined by ω in SL_2 . If the theory of twisted endoscopy also holds for GL_2 and SO_2^* , then (10.10) must equal $f^{\mathbf{H}}(e)$ if one makes a similar use of Shalika germs as in Section 7. One then clearly expects that $f^{\mathbf{H}}(e) \neq 0$ for some f , proving one direction of Proposition 8.4 of [23] (also [34]) using $\mathbf{H} = SO_2^*$ directly (rather than using SO_3).

Now suppose $\omega = 1$. Then σ must come from H , where $\mathbf{H} = SO_3 = PGL_2$. (In fact, $L(s, \sigma, \wedge^2 \rho_2)$ has a pole at $s = 0$ or equivalently $\Phi_\theta(1, f) \neq 0$.) Using equation (7.2) and the matching of top-dimensional unipotent θ -orbital integrals, we see that

$$(10.11) \quad \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \Phi_\theta(\varepsilon u, f)$$

is nonzero if and only if $\Phi(u', f^{\mathbf{H}}) \neq 0$, where u' is the regular unipotent orbit of PGL_2 . We have therefore proved the following proposition.

PROPOSITION 10.7. *Let σ be an irreducible self-dual supercuspidal representation of $GL_2(F)$, considered as a representation of the GL_2 -Levi subgroup of $SO_5(F)$. Then $I(\sigma)$ is irreducible if and only if $\omega \neq 1$. In particular,*

$$(10.12) \quad \sum_{\varepsilon \in (F^*)^2 \setminus F^*} \omega(\varepsilon) \Phi_\theta(\varepsilon u, f) + \Phi_\theta(1, f) = 0$$

for every $f \in C_c^\infty(GL_n(F))$ defining a matrix coefficient of σ if and only if $\omega = 1$ and therefore (10.10) is never zero no matter what ω is. Consequently, under assumption (7.3),

$$\Phi(u', f^{\mathbf{H}}) \neq 0$$

for every f defining a matrix coefficient of σ with $\omega = 1$.

Remark. This basically gives a new proof of Proposition 8.4 of [23] since $SO_5 \cong PSP_4$ and \mathbf{M} is the image of the $SL_2 \times GL_1$ -Levi subgroup of Sp_4 in PSP_4 . It is a completely local proof and makes no use of the global functional equation for symmetric square L -functions for GL_2 . The second and the third assertions of

Proposition 10.7 follow from the equivalence of nonvanishing of $\Phi_\theta(1, f)$ and $\omega = 1$ when $n = 2$.

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