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The notion of norm and the representation theory of orthogonal groups

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1. Introduction

The purpose of this paper is to lay the foundation for the study of nondiscrete tempered spectrum of classical groups over local fields by means of the theory of endoscopy [1, 19, 20, 21, 23, 24, 37, 46] which relates them to self-dual representations of $GL(n)$ [1, 32, 37]. We do this here by taking the modest but crucially instructive step of completing the problem for the case of split even orthogonal groups supported on their maximal parabolic subgroups. When combined with results of Goldberg [9], this solves the problem in this case completely (by means of the theory R-groups [17]). We expect the method and techniques of this paper to apply to other cases of classical groups as well.

To explain, let τ and τ' be two discrete series representations of $SO_{2m}(F)$ and $GL_n(F)$, respectively, and consider $\tau \otimes \tau'$ as a representation of the Levi factor $SO_{2m}(F) \times GL_n(F)$ of $SO_{2r}(F)$, $r = m + n$. Let $I(\tau \otimes \tau')$ be the representation of $SO_{2r}(F)$, unitarily induced from $\tau \otimes \tau'$. To determine the reducibility of $I(\tau \otimes \tau')$ we need to study the residue of the standard intertwining operator $A(s, \tau \otimes \tau', w_0)$ at $s = 0$ (see Section 3 for the definition of $A(s, \tau \otimes \tau', w_0)$).

Even to understand this residue, one needs to introduce a norm correspondence. More precisely, let w_n be the $n \times n$ matrix whose only nonzero entries are ones on its second diagonal. Define the automorphism θ of GL_n by $\theta(g) = w_n^{-1} \cdot {}^t g^{-1} \cdot w_n$. It is the subject matter of our Section 5 to define a norm correspondence, dictated to us by the theory of intertwining operators, from a certain subset of θ-conjugacy classes in $GL_n(F)$ into the set of conjugacy classes in $SO_{2m}(F)$. Then by Proposition 5.9 and Corollary 5.14, the norm correspondence is a surjection if $n \geqq 2m$ and always with finite fibers (cf. Lemma 8.1 for $n < 2m$). Moreover, by Corollary 5.21 and Lemma 5.20, our

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norm correspondence agrees with the one defined by Kottwitz and Shelstad in [19] whenever the θ-conjugacy class is θ-semisimple. More precisely, if $n = 2m$ and Y belongs to the F-points of a θ-stable Cartan subgroup of $\mathbf{G}'$, then, except on a set of measure zero, its norm is the set of all the $SO_n(F)$-conjugacy classes of elements of $SO_n(F)$ which are $GL_n(F)$-conjugate to $-Y\theta(Y) \in SO_n(F)$ (see Remark 5.22 for the minus sign). Finally for the sake of this introduction, assume $n \geqq 2m$ and let the image correspondence $\mathscr{A}$ from the set of conjugacy classes of $SO_{2m}(F)$ into the set of all θ-conjugacy classes in $GL_n(F)$ be the inverse of the norm correspondence. Then by the finiteness of fibers, the image of a conjugacy class always consists of a finite number of θ-conjugacy classes.

Now assume τ' is supercuspidal and let $f' \in C_c^\infty(GL_n(F))$ define a matrix coefficient of τ'. Denote by f an arbitrary matrix coefficient of τ. Note that we are NOT assuming τ is generic. Let

$$R_G(f,f') = \sum_{\{\mathbf{T}\}} |W(\mathbf{T})|^{-1} \int_{\sigma \in \mathbf{T}(F)} \boldsymbol{\Phi}(\{\sigma\},f)\boldsymbol{\Phi}_\theta(\mathscr{A}(\{\sigma\}),f')|D(\sigma)|\,d\sigma,$$

where the sum runs over conjugacy classes of all the elliptic tori of $\mathbf{G} = SO_{2m}$, and $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}_\theta$ denote the ordinary and θ-twisted orbital integrals on SO_{2m} and GL_n, respectively. Here $D(\sigma)$ is the usual discriminant (cf. [13]). Simply put, this is the Weyl integration formula applied to the class function

$$\{\gamma\} \mapsto \boldsymbol{\Phi}_\theta(\mathscr{A}(\{\gamma\}),f') = \sum_{\{\gamma'\} \in \mathscr{A}(\{\gamma\})} \boldsymbol{\Phi}_\theta(\{\gamma'\},f') \cdot \kappa(\{\gamma\},\{\gamma'\}),$$

where $\kappa(\{\gamma\},\{\gamma'\})$ is simply a ratio of discriminants defined in Section 7.

Next assume $\sigma \in G$ is singular and elliptic. Let $\mathscr{U}_{\mathbf{G}_\sigma^0}$ be the set of equivalence classes of elements of $\mathscr{U}_{G_\sigma^0}$, the set of unipotent conjugacy classes in G_σ^0, for which u is equivalent to u' if and only if $u' = h_0^{-1} u h_0$ for some $h_0 \in \mathbf{G}_\sigma^0$. When $\mathbf{G}_\sigma$ is connected this is the same as stable conjugacy defined in [18].

For each $u \in \mathscr{U}_{\mathbf{G}_\sigma^0}$, set

$$R^u_{\{\mathbf{G}_\sigma\}}(f,f') = \sum_{\{\mathbf{G}_{\sigma_1}\}} \sum_{\hat{\sigma}_1 \in Z_{\sigma_1}(F)'} \sum_{u_1} c_{\sigma_1 u_1} \boldsymbol{\Phi}(\{\hat{\sigma}_1 u_1\},f)\boldsymbol{\Phi}_\theta(\mathscr{A}(\{\hat{\sigma}_1 u_1\}),f'),$$

with κ identically 1. The outer sum runs over $\mathbf{G}(F)$-conjugacy classes of centralizers of elements of the form $\sigma_1 = h^{-1}\sigma h$, $h \in \mathbf{G}(\bar{F})$. The sum over u_1 is over unipotent conjugacy classes in $G_{\sigma_1}^0$ which are of the form $h^{-1}u'h$, where u' belongs to the class of $u \in \mathscr{U}_{\mathbf{G}_\sigma^0}$. The constant $c_{\sigma_1 u_1}$ is the inverse of an important index defined in Sections 2 and 7. Finally set

$$\mathbf{Z}_{\sigma_1}(F') = \{\hat{\sigma}_1 \in \mathbf{Z}_{\sigma_1}(F) | \mathbf{G}_{\hat{\sigma}_1} = \mathbf{G}_{\sigma_1}\}.$$

It may be called the subset of *regular* elements in $\mathbf{Z}_{\sigma_1}(F)$.

It is the sum $R_{\mathscr{A}}$ of $R_{\mathbf{G}}$ and different $R^u_{\{\mathbf{G}_\sigma\}}$'s which is proportional to the residue of $A(s, \tau \otimes \tau', w_0)$ at $s = 0$. We expect such sums to appear also as residues of intertwining operators for other classical groups. To generalize such sums, in Section 2 we develop an integral for a pair of reductive groups over a local field F by means of an *image* correspondence from the set of F-conjugacy classes of one into the set of equivalence classes of F-points of the other group defined by an equivalence relation. Our hope is that this more general setting may be useful in encountering new cases, optimistically put, of functoriality (cf. Remark 6.2 of this paper and the introduction of [32], Theorem 3 of [5], and [26] for $GL(n)$).

Besides our own calculations which led us to these definitions, results, and generalizations, our model and guide has been the work of Kottwitz, Langlands, and Shelstad on the theory of endoscopy and more generally twisted endoscopy [19, 20, 21, 23, 24]. In fact, the reader who is familar with the theory of twisted endoscopy [19, 20, 23] should immediately notice the special case when the equivalence relation on the second group is defined by means of a rational automorphism of finite order, i.e. twisted endoscopy, for which the image map is defined in [19, 23].

The main result of this paper when $n \geqq 2m$ (Theorem 7.8) is

Theorem 1.1 *Suppose $n \geqq 2m$ and $\tau' \cong \tilde{\tau}'$. Then the induced representation $I(\tau \otimes \tau')$ is irreducible if and only if either $R_G \neq 0$ or $R^u_{\{G_\sigma\}} \neq 0$ for some singular elliptic σ and $u \in \mathscr{U}_{G^0_\sigma}$. In this case the central character of τ' is trivial.*

Keeping in mind the results of [5, 12, 13], this is the first example (as far as we know) where a pairing between twisted characters of $GL_n(F)$ and those of one of its twisted endoscopic groups shows up. In fact, the nonvanishing of R_G points to τ' coming from the representation τ (or its L-packet, to be more precise) of $SO_{2m}(F)$ through twisted endoscopy (cf. Definition 9.1 and Theorem 9.10). We refer to Definition 9.7 and again Theorem 9.10 for an interpretation of nonvanishing of $R^u_{\{G_\sigma\}}$.

When $n < 2m$, the image of norm correspondence no longer intersects any regular elliptic conjugacy class of SO_{2m} and consequently the reducibility is controlled only by singular terms $R^u_{\{G_\sigma\}}$ as it is evident from our Theorem 8.2.

As experience has shown [2, 29, 44] we expect that by their analytic nature, such results, besides being of interest by themselves, to play a role in applications of Arthur's trace formula, especially when global counterparts of such results are to be proved.

In view of the theory developed in [33, 34] these results are equivalent to a determination of local Langlands L-functions [3] attached to the Rankin-Selberg products of representations of classical groups and $GL(n)$. In fact, following the method suggested in [33, 35], as used in Sections 6 and 8 of [32], as well as in [10], the Rankin-Selberg L-function $L(s, \tau \times \tau')$ can now be defined for any pair of irreducible admissible representations of $SO_{2m}(F)$ and $GL_n(F)$. Observe that as in Theorem 9.1 of [32] this will also lead to a direct determination of reducibility when τ' is any member of discrete series. We leave these to a future paper. We should mention that global analogues of these L-functions are now being studied by a number of mathematicians [8, 28, 39, 40] for a very interesting application of the converse theorem in obtaining similar liftings of automorphic forms from classical groups to $GL(n)$.

We have also left out the case of odd n which we expect to be easier since twisted endoscopy is not as rich in the odd case (cf. for example Proposition 3.5 of [32]).

There are many other aspects of this project such as, as it was pointed out to us by Assem, its possible connection with the theory of prehomogeneous spaces [19, 31], as well as its relation with algebraic methods (cf. [4, 14, 25, 30, 42]).

For the cases of quasisplit unitary groups the twisted endoscopy is that of base change as the work of Goldberg [10, 11] has shown. In this case the norm map is well understood through the work of Kottwitz [18]. These cases and more generally those of remaining classical groups are the subject matter of further joint work with Goldberg. As explained before, we expect our approach in this paper to play an important role in establishing the general case.

2. A pairing between groups via an image correspondence

The purpose of this section is to introduce certain sums of integrals for a pair of reductive groups which in certain special cases become residues of intertwining operators for parabolically induced representations of classical groups.

To explain we let F be a local field of characteristic zero. Let $\mathbf{G}$ and $\mathbf{G}'$ be two connected reductive algebraic groups over F. As usual we use $G = \mathbf{G}(F)$ and $G' = \mathbf{G}'(F)$ for their F-points. We shall assume that the center of G is compact which is to say that the connected component of the center of $\mathbf{G}$ is anisotropic.

Let $\sim$ be an equivalence relation on G'. We use $\mathscr{C}'$ to denote the set of equivalence classes in G'. We shall assume the existence of a map $\boldsymbol{\Phi}'$, sending each equivalence class $\{\gamma'\} \in \mathscr{C}'$, $\gamma' \in G'$, to a distribution

$$\boldsymbol{\Phi}'(\{\gamma'\}, \cdot) = \boldsymbol{\Phi}'(\gamma', \cdot): f' \mapsto \boldsymbol{\Phi}'(\gamma', f')$$

on $G', f' \in C_c^\infty(G')$. If the image consists of tempered distributions, we will then allow f' to be a Schwartz function [5,38].

One interesting and important example of this is obtained by fixing an automorphism of finite order θ of $\mathbf{G}'$ defined over F and letting G' act on itself by θ-twisted conjugation

$$h \mapsto g^{-1} h\theta(g),$$

$g, h \in G'$. We then let

$$\boldsymbol{\Phi}'(\gamma', f') = \boldsymbol{\Phi}_\theta(\gamma', f') = \int_{\mathbf{G}'_{\theta,h}(F)\backslash G'} f'(g^{-1} h\theta(g)) d\dot{g},$$

where the θ-twisted centralizer $\mathbf{G}'_{\theta,h}$ of h is defined by

$$\mathbf{G}'_{\theta,h} = \{g \in \mathbf{G}' | g^{-1} h\theta(g) = h\},$$

(cf. [1, 6, 19]).

Next let $\mathscr{C}$ be the set of conjugacy classes (i.e. $\theta = 1$) in G. Given a Schwartz function f on G and a conjugacy class $\{\gamma\} \in \mathscr{C}$, we use $\boldsymbol{\Phi}(\gamma, f)$ to denote the orbital integral of f at γ (cf. [5]).

Crucial to this paper and our formulation is the existence of a one to finite correspondence $\mathscr{A}$ from, possibly, a subset of $\mathscr{C}$ into $\mathscr{C}'$ and certain complex numbers

$$\Delta(\gamma, \gamma') = \Delta(\{\gamma\}, \{\gamma'\})$$

defined on $\mathscr{C} \times \mathscr{C}'$. The reader who is familiar with the theory of endoscopy should at least see the notational similarity of $\mathscr{A}$ and Δ with the image map and transfer factors of Langlands-Shelstad [23] and Kottwitz-Shelstad [19]. Our hope in this paper is to substantiate a much deeper connection.

We then set

$$\Phi'(\mathscr{A}(\{\gamma\}), \cdot) = \sum_{\{\gamma'\} \in \mathscr{A}(\{\gamma\})} \Delta(\gamma, \gamma')\Phi'(\gamma', \cdot). \tag{2.1}$$

The goal of this section is to introduce an integral $R_{\mathscr{A}}(\cdot, \cdot)$ of the product $\Phi(\gamma, \cdot)\Phi'(\mathscr{A}(\{\gamma\}), \cdot)$ over $\mathscr{C}$. Such integrals appear as residues of intertwining operators with far reaching consequences in representation theory of p-adic groups and L-functions (cf. Sections 7–9).

Fix $\gamma \in G$. Write $\gamma = \sigma u = u\sigma$ for its Jordan decomposition, where σ is semisimple and u is unipotent. Given $x \in G$, we use the standard notation $\mathbf{G}_x$ to denote its centralizer in $\mathbf{G}$. Then $u \in \mathbf{G}_\sigma^0$, the connected component of $\mathbf{G}_\sigma$, a reductive group. Moreover $\mathbf{G}_\gamma^0 = \mathbf{G}_{\sigma u}^0$ is equal to the connected component of the centralizer of u in $\mathbf{G}_\sigma$. Let $\mathbf{Z}_\sigma$ be the center of $\mathbf{G}_\sigma$. Set

$$\mathbf{Z}_\sigma(F)' = \{\hat{\sigma} \in \mathbf{Z}_\sigma(F) | \mathbf{G}_{\hat{\sigma}} = \mathbf{G}_\sigma\},$$

the set of *regular* elements in $\mathbf{Z}_\sigma(F)$. We want to integrate $\Phi(\{\hat{\sigma}u\}, \cdot)\Phi'(\mathscr{A}(\{\hat{\sigma}u\}), \cdot)$ over $\mathbf{Z}_\sigma(F)'$. We apply the usual techniques used by Harish-Chandra in the regular semisimple case [12, 13, 38]. We need:

Lemma 2.1 *Suppose $\mathbf{Z}_\sigma^0 = \mathbf{Z}(\mathbf{G}_\sigma)^0$ is a non-trivial torus. Fix a euclidean measure on $\mathbf{Z}_\sigma(F)$. Then $\mathbf{G}_{\hat{\sigma}}$, $\hat{\sigma} \in \mathbf{Z}_\sigma(F)$, is connected except for $\hat{\sigma}$ in a set of measure zero in $\mathbf{Z}_\sigma(F)$.*

Proof. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing $\mathbf{Z}_\sigma^0$. The group $\mathbf{G}_\sigma^0$ is generated by $\mathbf{T}$ and all those X_α for which $\alpha(\sigma) = 1$, where α is a nonrestricted root of $\mathbf{T}$ in $\mathbf{G}$. Moreover $\mathbf{Z}(\mathbf{G}_\sigma^0)$ consists of exactly those $\hat{\sigma} \in \mathbf{T}$ for which $\alpha(\hat{\sigma}) = 1$ with α's as above. Let W be the Weyl group of $\mathbf{T}$ in $\mathbf{G}$. If $\hat{\sigma} \in \mathbf{Z}(\mathbf{G}_\sigma^0)$ is such that $\mathbf{G}_{\hat{\sigma}}$ is not connected, then there exists a $w \in W$ which is not a product of w_α's but which fixes $\hat{\sigma}$(cf. [41]). But then $\hat{\sigma}$ satisfies a new equation different from the defining equations for $\mathbf{Z}(\mathbf{G}_\sigma^0)$. They then make a set of measure zero with respect to the euclidean measure on $\mathbf{Z}_\sigma(F)$.

Given $\hat{\sigma} \in \mathbf{Z}_\sigma(F)'$, $\mathbf{G}_{\sigma u}$ and $\mathbf{G}_{\hat{\sigma}u}$ are equal, both being centralizer of u in $\mathbf{G}_\sigma = \mathbf{G}_{\hat{\sigma}}$. Consider the map

$$(g, \hat{\sigma}u) \mapsto g^{-1}(\hat{\sigma}u)g \tag{2.2}$$

from $\mathbf{G}_{\sigma u}(F)\backslash G \times \mathbf{Z}_\sigma(F)'u$ onto $(\mathbf{Z}_\sigma(F)'u)^G$.

Let $N(\mathbf{Z}_\sigma(F)')$ be the normalizer of $\mathbf{Z}_\sigma(F)'$ in G. By uniqueness of Jordan decompositions g is in the normalizer of $\mathbf{Z}_\sigma(F)'u$ in G if and only if

$g \in N(\mathbf{Z}_\sigma(F)') \cap \mathbf{G}_u(F)$. If $\mathbf{Z}_\sigma^0$ is trivial, $[N(\mathbf{Z}_\sigma(F)') \cap \mathbf{G}_u(F){:}\ \mathbf{G}_{\sigma u}(F)]$ is trivially finite. Now suppose $\mathbf{Z}_\sigma^0$ is a non-trivial torus. The quotient

$$N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F)/\mathbf{G}_{\sigma u}(F)$$

is then a subgroup of $N(\mathbf{Z}_\sigma(F))/\mathbf{G}_\sigma(F)$ since $\mathbf{G}_\sigma \cap \mathbf{G}_u = \mathbf{G}_{\sigma u}$. It is easy to see that for σ outside a set of measure zero, $\mathbf{G}_\sigma(F)$ is equal to the centralizer of $\mathbf{Z}_\sigma(F)$ in G. Then $[N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F){:}\ \mathbf{G}_{\sigma u}(F)]$ is finite.

Suppose first that $\mathbf{Z}_\sigma^0$ is non-trivial. To introduce the contribution to the integral in question coming from σu, by Lemma 2.1, we may assume σ is such that $\mathbf{G}_\sigma$ is connected. Then for almost all $\hat{\sigma} \in \mathbf{Z}_\sigma(F)$, $[N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F){:}\ \mathbf{G}_{\sigma u}(F)]$ is equal to the index of the map (2.2) whose Jacobian is equal to the absolute value of

$$D(\hat{\sigma}u) = \det(I - Ad(\hat{\sigma}u))_{\mathfrak{g}/\mathfrak{g}_{\hat{\sigma}u}} = D(\hat{\sigma})$$

with obvious notation for Lie algebras. The contribution from all those $\hat{\sigma}u$, $\hat{\sigma} \in \mathbf{Z}_\sigma(F)$, for which $\mathbf{G}_{\hat{\sigma}}^0 = \mathbf{G}_\sigma$ is then equal to

$$[N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F){:}\ \mathbf{G}_{\sigma u}(F)]^{-1} \int_{\hat{\sigma} \in \mathbf{Z}(\mathbf{G}_\sigma)(F)} \Phi(\{\hat{\sigma}u\}, \cdot)\Phi'(\mathscr{A}(\{\hat{\sigma}u\}), \cdot)|D(\hat{\sigma})|\,d\hat{\sigma},$$

where $d\hat{\sigma}$ is the euclidean measure on $\mathbf{Z}(\mathbf{G}_\sigma)(F)$. Observe that we are now assuming that $\mathbf{G}_\sigma$ is connected.

If $\mathbf{Z}_\sigma^0$ is trivial, one can easily see that the contribution is just

$$[N(\mathbf{Z}_\sigma(F)') \cap \mathbf{G}_u(F){:}\ \mathbf{G}_{\sigma u}(F)]^{-1} \sum_{\hat{\sigma} \in \mathbf{Z}(\mathbf{G}_\sigma)(F)'} \Phi(\{\hat{\sigma}u\}, \cdot)\Phi'(\mathscr{A}(\{\hat{\sigma}u\}), \cdot).$$

Observe that we can no longer assume that $\mathbf{G}_\sigma$ is connected.

When $u = 1$, the constant $[N(\mathbf{Z}_\sigma(F)'){:}\ \mathbf{G}_\sigma(F)]^{-1}$ seems to be curiously close to the constants $i(\mathbf{G}, \mathbf{H})$, defined by Langlands [21] and evaluated by Kottwitz [46], which appear in the stabilization of the trace formula (cf. page 39 of [45]).

Finally, we consider the sum of these integrals over G-conjugacy classes of centralizers of semisimple elements in G which is a finite set. Given a subgroup $\mathbf{H} \subset \mathbf{G}$, let $\{\mathbf{H}\}$ denote the equivalence class of $\mathbf{H}$ under conjugation by elements of G. Then our integral can be written as the sum of

$$\sum_{\substack{\{\mathbf{G}_\sigma\} \\ \mathbf{G}_\sigma = \mathbf{G}_\sigma^0}} \sum_{\{u\} \in \mathscr{U}_{G_\sigma}} [N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F){:}\, \mathbf{G}_{\sigma u}(F)]^{-1} \cdot \int_{\hat{\sigma} \in \mathbf{Z}_\sigma(F)} \Phi(\{\hat{\sigma}u\}, \cdot)\Phi'(\mathscr{A}(\{\hat{\sigma}u\}), \cdot)\,|D(\hat{\sigma})|\,d\hat{\sigma}, \tag{2.3}$$

where the first sum is over the equivalence classes of *connected* centralizers of semisimple elements of G for which $\mathbf{Z}_\sigma(F)$ is infinite (σ is so that $\mathbf{G}_\sigma$ is connected) together with

$$\sum_{\{\mathbf{G}_\sigma\}} \sum_{\{u\} \in \mathscr{U}_{G_\sigma}} [N(\mathbf{Z}_\sigma(F)') \cap \mathbf{G}_u(F){:}\, \mathbf{G}_{\sigma u}(F)]^{-1} \sum_{\hat{\sigma} \in \mathbf{Z}_\sigma(F)'} \Phi(\{\hat{\sigma}u\}, \cdot)\Phi'(\mathscr{A}(\hat{\sigma}u\}), \cdot) \tag{2.4}$$

in which the first sum is over equivalence classes of centralizers of semisimple elements of G for which $\mathbf{Z}_\sigma(F)$ is finite. Observe that we may no longer consider only those which are connected.

In both sums the second sum is over unipotent conjugacy classes $\mathscr{U}_{G^0_\sigma}$ of G^0_σ. The choice of u in its class is irrelevant. We use $R_{\mathscr{A}}(\cdot,\cdot)$ to denote the sum of (2.3) and (2.4).

We now turn to the question of convergence of $R_{\mathscr{A}}(f,f')$. It will be enough to assume that

(2.5) f is a cuspidal Schwartz function on G. This means that the integral of f over the unipotent radical of every parabolic subgroup different from G is zero. Then the Selberg principle is valid for $\Phi(\gamma,f)$ whenever γ is regular semisimple but not elliptic (Proposition 5, part i, of [5]). By Proposition 2 of [5], $\Phi(\gamma,f)$ satisfies a germ expansion for γ in a neighborhood of a singular semisimple element σ. By linear independence of germs it then follows that $\Phi(\sigma u,f) = 0$ if σ is semisimple but not elliptic and u commutes with σ.

(2.6) the function

$$\hat\sigma \mapsto |D(\hat\sigma)|^{1/2}\,\Phi'(\mathscr{A}(\{\hat\sigma u\}),f')$$

is locally integrable on the F-points of the center of the connected centralizer $\mathbf{G}^0_\sigma$ of every semisimple element σ for every $\{u\} \in \mathscr{U}_{G_\sigma}$.

We therefore have

Proposition 2.2. *Suppose f and f' satisfy* (2.5) *and* (2.6). *Then $R_{\mathscr{A}}(f,f')$ converges.*

Proof. Since the center of G is compact and f is cuspidal we will only need to integrate over compact $\mathbf{Z}_\sigma(F)$. It is well known that $|D(\hat\sigma)|^{1/2}\,\Phi(\hat\sigma u,f)$ is locally bounded and therefore the convergence is a consequence of (2.6).

For further reference we repeat the formula for $R_{\mathscr{A}}(f,f')$ as follows

$$\begin{aligned} R_{\mathscr{A}}(f,f') = {} & \sum_{\substack{\{\mathbf{G}_\sigma\} \\ \mathbf{G}_\sigma = \mathbf{G}^0_\sigma \\ \mathrm{Card}\ \mathbf{Z}_\sigma(F) = \infty}} \ \sum_{\{u\} \in \mathscr{U}_{G_\sigma}} [N(\mathbf{Z}_\sigma(F)) \cap \mathbf{G}_u(F) : \mathbf{G}_{\sigma u}(F)]^{-1} \\ & \cdot \int_{\hat\sigma \in \mathbf{Z}_\sigma(F)'} \Phi(\hat\sigma u,f)\,\Phi'(\mathscr{A}(\{\hat\sigma u\}),f')\,|D(\hat\sigma)|\,d\hat\sigma + \sum_{\substack{\{\mathbf{G}_\sigma\} \\ \mathrm{Card}\ \mathbf{Z}_\sigma(F) < \infty}} \ \sum_{\{u\} \in \mathscr{U}_{G^0_\sigma}} \\ & [N(\mathbf{Z}_\sigma(F)') \cap \mathbf{G}_u(F) : \mathbf{G}_{\sigma u}(F)]^{-1} \sum_{\hat\sigma \in \mathbf{Z}_\sigma(F)'} \Phi(\hat\sigma u,f)\,\Phi'(\mathscr{A}(\{\hat\sigma u\}),f'). \end{aligned} \qquad (2.7)$$

Remark 2.3. When σ is strongly regular [23] so that $\mathbf{G}_\sigma$ is a torus, the contribution becomes a sum over $\mathbf{G}$-conjugacy classes of maximal tori in $\mathbf{G}$ and is in fact the Weyl integration formula applied to the class function

$$\hat\sigma \mapsto \Phi'(\mathscr{A}(\{\hat\sigma\}),f').$$

The sum reduces to those over elliptic tori since f is cuspidal.

The following lemma is a consequence of Weyl integration formula.

Lemma 2.4. *Suppose domain of $\mathscr{A}$ is all of $\mathscr{C}$ and the class function $\gamma \mapsto \Phi'(\mathscr{A}(\{\gamma\}), f')$ is locally integrable on G. Then* (2.6) *is valid on Cartan subgroups of G.*

3. The orthogonal group

Let F be a nonarchimedean field of characteristic zero. If O and R are the ring of integers of F and its maximal ideal, respectively, we let q be the number of elements in the residue field O/R. We use $|\ | = |\ |_F$ to denote the absolute value in F for which a prime element ϖ has $|\varpi|_F = q^{-1}$.

For the rest of this paper we shall fix a pair of non-negative integers m and n and we shall assume $\mathbf{G} = SO_{2m}$ and $\mathbf{G}' = GL_n$, both as groups over F. In this paper we shall only treat the case of n even. Letting $r = m + n$, the group $\mathbf{G} \times \mathbf{G}'$ will then be a Levi subgroup for the group SO_{2r}. We are interested in representations unitarily induced from discrete series representations of the Levi subgroup $SO_{2m}(F) \times GL_n(F)$ of $SO_{2r}(F)$. In fact reducibility of such representations, besides being interesting by itself, is equivalent to determination of the local L-functions attached to representations of the group $SO_{2m}(F) \times GL_n(F)$ (cf. [33, 34]). When $m = 0$, the problem was answered, via the theory of twisted endoscopy [19], in [32]. What is remarkable is that with some work and new ideas the general case can also be answered by the same theory. In fact, the aim of this paper is to prove that the residues of intertwining operators at the origin which determine the reducibility of induced representations, are proportional to $R_{\mathscr{A}}(f, f')$, where $\mathscr{A}$ is the image map [19] between SO_{2m} and GL_n and f and f' are matrix coefficients for the inducing discrete series representations of $SO_{2m}(F)$ and $GL_n(F)$, respectively (Proposition 7.3 and Theorems 7.8 and 8.2).

Although it is no longer possible to study all the split classical groups at once as in [32] when $m = 0$, the fact that residues are of type $R_{\mathscr{A}}(f, f')$ seems to be quite general. This is exactly why we treated Section 2 in full generality.

We recall that by SO_{2r} we mean the connected component of

$$O_{2r} = \{g \in GL_{2r} \mid {}^t g w_{2r} g = w_{2r}\},$$

where

$$w_{2r} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

Let $\sigma = \tau \otimes \tau'$ be a discrete series representation of $M = \mathbf{M}(F)$, where $\mathbf{M} = SO_{2m} \times GL_n$, considered as a Levi subgroup $\mathbf{M}$ of the standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\tilde{G} = SO_{2r}$, containing the Borel subgroup $\tilde{\mathbf{B}} = \tilde{\mathbf{T}}\tilde{\mathbf{N}}$ of upper triangular matrices in $\tilde{\mathbf{G}}$. Here $\tilde{\mathbf{T}}$ denotes the subgroup of diagonal elements of SO_{2r}. Given a complex number s we set

$$I(s, \sigma) = I(s, \tau \otimes \tau') = \mathrm{Ind}_{\mathbf{MN} \uparrow \tilde{\mathbf{G}}}\, \tau \otimes (\tau' \otimes |\det(\ \ )|^s) \otimes \mathbf{1},$$

where $\tilde{G} = \tilde{\mathbf{G}}(F) = SO_{2r}(F)$.

Given a positive integer a, let

$$w_a = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \in GL_a.$$

Then if m is also even, the matrix

$$w_0 = \mathrm{diag}(w_n, w_{2m}, w_n) \cdot w_{2r} \in SO_{2r}(F)$$

is a representative for the longest element of the Weyl group of $\tilde{\mathbf{T}}$ in $\tilde{\mathbf{G}}$ modulo that of the Weyl group of $\tilde{\mathbf{T}}$ in $\mathbf{M}$. If ℓ is odd, then the permutation matrix $w'_{2\ell} \in SO_{2\ell}(F)$, sending $1 \mapsto 2\ell, \ldots, \ell - 1 \to \ell + 2, \ell \to \ell, \ell + 2 \to \ell - 1, \ldots, 2\ell \to 1$, while fixing e_ℓ, sends $e_j \to -e_j$, $1 \leqq j < \ell$. Consequently for m odd (thus r odd), $\mathrm{diag}(w_n, w'_{2m}, w_n)$ (respectively w'_{2r}) represents the longest element of the Weyl group of $\tilde{\mathbf{T}}$ in $\mathbf{M}$ (respectively in $\tilde{\mathbf{G}}$). Observe that $\mathrm{diag}(w_n, w'_{2m}, w_n) \cdot w'_{2r}$ equals w_0 of the case where m is even. Thus in both cases w_0 represents the longest elements of the Weyl group of $\tilde{\mathbf{T}}$ in $\tilde{\mathbf{G}}$ modulo that of the Weyl group of $\tilde{\mathbf{T}}$ in $\mathbf{M}$.

Let α be the unique simple root in $\mathbf{N}$. Denote by ρ the usual half the sum of roots in $\mathbf{N}$. Then it is easy to see that $\langle 2\rho, \alpha \rangle = 2m + (n-1)$ if $m \neq 0$, while $\langle 2\rho, \alpha \rangle = 2(n-1)$ for $m = 0$. Now, as in [33], let $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$. Then again in the notation of [33], $I(s, \sigma) = I(s\tilde{\alpha}, \sigma)$ if $m \neq 0$, while $I(s, \sigma) = I(2s\tilde{\alpha}, \sigma)$ for $m = 0$, the case considered in [32].

Given a function $h \in V(s, \sigma)$, the space of $I(s, \sigma)$, let

$$A(s, \sigma, w_0)h(g) = \int_N h(w_0^{-1} ng)dn, \tag{3.1}$$

$Re(s) > 0$, $g \in \tilde{G}$, be the standard intertwining operator attached to $I(s, \sigma)$ (cf. [33]). To study the reducibility of $I(\sigma) = I(0, \sigma)$, it is enough to determine whether (3.1) has a pole at $s = 0$ (cf. [33, 38]). By the theory developed in [33, 34, 36] this is equivalent to determining certain local L-functions. In fact, when $m = 0$ we get the L-function $L(s, \tau', \Lambda^2 \rho_n)$, where $\Lambda^2 \rho_n$ is the second exterior power of the standard representation ρ_n of $GL_n(\mathbf{C})$. This was done in [32]. On the other hand, when $m \neq 0$, this determines $L(2s, \tau', \Lambda^2 \rho_n)$. $L(s, \tau \times \tau')$, if we assume τ is generic (cf. [34, 36]) and σ is supercuspidal. The L-function $L(s, \tau \times \tau')$ can then be defined inductively, using $L(2s, \tau', \Lambda^2 \rho_n)$. Here $L(s, \tau \times \tau')$ is the Rankin-Selberg L-function attached to the pair (τ, τ') which is also being studied using a very different method by a number of other authors [8, 28, 39, 40]. The significance of our method is that it relates to the parameterization problem by means of the theory of endoscopy [19, 23, 32].

Given $Z \in M_n$, let $\tilde{\theta}(Z) = w_n^{-1} {}^t Z w_n$ be the standard anti-involution of M_n. If $Z \in GL_n$, we set $\theta(Z) = \tilde{\theta}(Z^{-1}) = w_n^{-1} {}^t Z^{-1} w_n$.

To proceed we need some notation. Suppose

$$n = \begin{pmatrix} I_n & X & Y \\ 0 & I_{2m} & X' \\ 0 & 0 & I_n \end{pmatrix} \in SO_{2r} \tag{3.2}$$

with $X \in M_{n\times 2m}$, $X' \in M_{2m\times n}$, and $Y \in M_n$. Then X, X', and Y must satisfy:

$$w_n X + {}^tX' w_{2m} = 0 \tag{3.3}$$

and

$$Y + \tilde{\theta}(Y) = XX'. \tag{3.4}$$

Occasionally we will use $n(X, Y)$ to denote n as in (3.2).

By Rallis' Lemma (Lemma 4.1 of [32]), to compute the poles of $A(s, \sigma, w_0)$, we only need to study the poles of $A(s, \sigma, w_0)h(e)$, where h is a smooth $\mathscr{H}(\sigma)$-valued function which is of compact support in PN^- modulo P. Here $N^- = \mathbf{N}^-(F)$, where $\mathbf{N}^-$ denotes the unipotent subgroup opposed to $\mathbf{N}$.

According to the discussion of the previous paragraph we may assume that h is supported in PN^-. We therefore need the following lemma.

Lemma 3.1 *Suppose n as in* (3.2) *is such that $w_0 n = pn^-$, $p \in P$, $n^- \in N^-$, where*

$$p = \begin{pmatrix} a & \delta & \eta \\ 0 & b & \delta' \\ 0 & 0 & \theta(a) \end{pmatrix}.$$

and

$$n^- = \begin{pmatrix} I_n & 0 & 0 \\ \alpha & I_{2m} & 0 \\ \beta & \alpha' & I_n \end{pmatrix}.$$

Then $Y \in GL_n(F)$, $a = \theta(Y)$, $b = I_{2m} - X'Y^{-1}X$, $\delta = -\alpha' = -Y^{-1}X$, $\delta' = X'$, $\eta = I_n$, $\alpha = X'\theta(Y)$, *and* $\beta = Y^{-1}$.

Proof. Straightforward.

Corollary 3.2 *With notation as in Lemma* 3.1

$$(I_{2m} - X'Y^{-1}X)X' = -X'Y^{-1}\theta(Y^{-1}) \tag{3.5}$$

and

$$X(I_{2m} - X'Y^{-1}X) = -\theta(Y^{-1})Y^{-1}X. \tag{3.6}$$

Proof. To get the first one use $b\alpha = -X'Y^{-1}$. For the second one use (3.3) in (3.5) to get

$$w_{2m}(I_{2m} - X'Y^{-1}X)w_{2m}^{-1}\,{}^tX = -{}^tX\theta({}^tY)^tY.$$

Then use the fact that $I_{2m} - X'Y^{-1}X \in SO_{2m}(F)$.

4. Some useful lemmas

To compute poles of $A(s, \sigma, w_0)$ at $s = 0$ we need some preparation. First we prove a number of elementary but general lemmas which we expect to be useful for other classical groups as well. Although they may not be the exact lemmas needed for other cases, we expect them to be useful in formulating and proving the necessary lemmas.

Fix two positive integers m and n. Given $X \in M_{n \times 2m}(F)$, define $X' \in M_{2m \times n}(F)$ by means of (3.3). Moreover for $\mathbf{x} = (x_1, \dots, x_{2m}) \in F^{2m}$, define $\mathbf{x}' = {}^t(x_{2m}, x_{2m-1}, \dots, x_1)$. Then SO_{2m} is the connected component of O_{2m} defined by

$$Q(\mathbf{x}) = \mathbf{x}\mathbf{x}'.$$

The following lemma is crucial.

Lemma 4.1 *Let $g \in GL_{2m}(F)$ be such that $(Xg)(Xg)' = XX'$. Then there exists a $h \in O_{2m}(F)$ such that $Xg = Xh$.*

Proof. Let U be the image of F^n in F^{2m} under X, i.e. $U = F^n X$. Given $\mathbf{y} = (y_1, \dots, y_n) \in F^n$, (3.3) implies

$$\mathbf{y}Xw_{2m} = -{}^t(X'y'),$$

where $\mathbf{y}' = {}^t(y_n, \dots, y_1)$. Thus

$$-{}^t(\mathbf{y}X)' = {}^t(X'\mathbf{y}')$$

implying

$$(\mathbf{y}X)' = -X'\mathbf{y}'. \tag{4.1.1}$$

Now, using (4.1.1), $(Xg)(Xg)' = XX'$ implies $(\mathbf{x}g)(\mathbf{x}g)' = \mathbf{x}\mathbf{x}'$ for every $\mathbf{x} \in U$. Thus $Q(\mathbf{x}g) = Q(\mathbf{x})$ for $\forall \mathbf{x} \in U$. Consequently g defines an isometry of U into F^{2m} as a quadratic space. Applying Witt's Theorem (e.g. Theorem 42:17 of [27]) to g then implies the existence of $h \in O_{2m}(F)$ for which $\mathbf{x}g = \mathbf{x}h$ for all $\mathbf{x} \in U$. Then $\mathbf{y}Xg = \mathbf{y}Xh$ for all $\mathbf{y} \in F^n$, completing the lemma.

Lemma 4.2 *Fix a positive integer r. Let $U \subset V = F^r$ be a subspace of $V = F^r$ defined by a projection $X \in M_r(F)$. Let $h \in GL_r(F)$. Then there exists a $g_h \in GL_r(F)$ such that $g_h X = Xh$ if and only if $Uh = U$. The element g_h is unique up to an element in the left stabilizer of X.*

Proof. Choose $\mathbf{u} \in U$. Then $\mathbf{u}g_h X = \mathbf{u}h$. On the other hand $\mathbf{u}g_h X = \mathbf{u}g_h X^2 = \mathbf{u}XhX = \mathbf{u}hX$. Thus $\mathbf{u}h = \mathbf{u}hX$ implying $Uh = U$. Conversely, suppose $Uh = U$. Write $V = U \oplus W$, where W is defined by $I - X$. If g_h exists, then $wg_h X = \mathbf{w}Xh = 0$ for $\mathbf{w} \in W$. Consequently, we may assume $g_h | W$ is any element of $GL(W)$. For $\mathbf{u} \in U$, we define $\mathbf{u}gh = \mathbf{u}h$. Then $(\mathbf{u} + \mathbf{w})g_h X = \mathbf{u}h = (\mathbf{u} + \mathbf{w})Xh$, completing the lemma.

Given $X \in M_{n \times 2m}(F)$, let $\tilde{X}$ be the matrix obtained from X by adding rows, if $n \leqq 2m$ (resp. columns if $n \geqq 2m$), of zeros to the bottom (resp. right if $n \geqq 2m$) of X. We shall call X *a projection if* $\tilde{X}$ is, i.e. $\tilde{X}^2 = \tilde{X}$. Observe that if $n \geqq 2m$, then in every class in $GL_n(F)\backslash M_{n \times 2m}(F)$ there is always a unique projection. This may not be the case if $n < 2m$.

Corollary 4.3 *Fix positive integers n and m. Let $X \in M_{n\times 2m}$ be a projection. Let $U = F^n X$. Fix $h \in GL_{2m}(F)$. Then there exists a $g_h \in GL_n(F)$ with $g_h X = Xh$ if and only if $Uh = U$.*

Proof. Suppose $n \leqq 2m$. Let $\tilde{X} \in M_{2m}(F)$ be defined as before by adding rows of zeros to the bottom of X. Then $\tilde{X}$ is a projection and $F^{2m}\tilde{X} = F^n X = U$ if we identify $F^n \subset F^{2m}$ as its first n coordinates. Suppose $g_h X = Xh$. Extend g_h to $\tilde{g}_h \in GL_{2m}(F)$ such that $\tilde{g}_h \tilde{X}$ has zero rows from row $n+1$ on, which is always possible. Then $\tilde{g}_h \tilde{X} = \tilde{X}h$ which implies $Uh = U$. Conversely, suppose $Uh = U$. Then $h \in \mathrm{Aut}(U)$. On the other hand, given $h \in \mathrm{Aut}(U)$, define $g \in \mathrm{Aut}(\ker(X)\backslash F^n)$ by $gX = Xh$ to complete the lemma.

If $n \geqq 2m$, define $\tilde{X} \in M_n(F)$ by adding columns of zeros to X at the right. Enlarge h to an element of $GL_n(F)$ in the standard manner. Again apply Lemma 4.2 and $F^n X = F^n \tilde{X}$.

Corollary 4.4 *Fix X and U as in Corollary 4.3. Let H_X be the subgroup of $SO_{2m}(F)$ consisting of all those h for which there exists a $g_h \in GL_n(F)$ such that $g_h X = Xh$. Suppose $0 \not\subseteq U \not\subseteq F^{2m}$. Then H_X is a proper subgroup of $SO_{2m}(F)$.*

Proof. By Corollary 4.3, $h \in H_x$ if and only if $Uh = U$. The corollary is now a consequence of irreducibility of $SO_{2m}(F)$.

Lemma 4.5 *Fix $X \in M_{n\times 2m}(F)$ and let $U = F^n X$. Denote by H'_X the right stabilizer of U in $SO_{2m}(F)$. If U is a regular subspace of F^{2m} with respect to the quadratic form Q, then H'_X is the centralizer of an involution of $O_{2m}(F)$ in $SO_{2m}(F)$. Otherwise, i.e. if U is not regular, then H'_X is contained in a proper parabolic subgroup of $SO_{2m}(F)$. Moreover, if $n \geqq 2m$, then centralizer of every singular elliptic element of $SO_{2m}(F)$ is a H'_X for some X. If X is a projection, then $H_X = H'_X$ and therefore all the statements are true about H_X.*

Proof. Consider U as a subspace of the quadratic space $V = F^{2m}$, defined by Q. First suppose U is regular. Write $V = U \perp W$ (notation as in [27]) with respect to a polarization of Q. Suppose $h \in H_X$. Then h stabilizes both U and W and therefore centralizes $-1_U \perp 1_W$, an element of $O_{2m}(F)$. Conversely fix h in the centralizer of $-1_U \perp 1_W$ in $SO_{2m}(F)$. Take $u \in U$ and write $uh = u_1 + w_1$, $u_1 \in U$, $w_1 \in W$. Then

$$\begin{aligned} u(-1_u \perp 1_W)h &= -uh \\ &= -u_1 - w_1 \\ &= (u_1 + w_1)(-1_u \perp 1_W) \\ &= -u_1 + w_1. \end{aligned}$$

Thus $h \in H_X$.

Now suppose U is not regular. Write $U = W \perp \mathrm{Rad}U$. Clearly H'_X stabilizes $\mathrm{Rad}U \neq \{0\}$. Since $\mathrm{Rad}U$ is totally isotropic, H'_X is contained in the proper parabolic subgroup $P_X = M_X N_X$ of $SO_{2m}(F)$ which stabilizes exactly $\mathrm{Rad}U$. Write $M_X = GL(RadU) \times SO(T)$, $T \subset V$. Then $V = H \perp T$, where H is a union of hyperbolic planes $H = H_1 \perp \ldots \perp H_r$, stabilized by P as well as H'_X, with $r = \dim(\mathrm{Rad}U)$ and $x_i \in H_i$, $1 \leqq i \leqq r$, for an appropriate basis $\{x_1, \ldots, x_r\}$ of $\mathrm{Rad}U$. Moreover $W \subset T$. Write $T = W \perp S$. Then $V = H \perp W \perp S$. Since H'_X stabilizes H it must fix $W \perp S$. But U is stabilized by H'_X and therefore H'_X must send W to U. Now $S \cap H = \{0\}$ implies that W must be sent to itself by H'_X. The same is true of S. Conversely, if $h \in SO_{2m}(F)$ fixes $H \perp W \perp S$, then $h \in H'_X$. It is therefore clear that $H'_X = GL(RadU) \times SO(W \perp S) \rtimes N_X$, where $SO(W \perp S)$ is the subgroup of $SO(T)$ preserving both W and S, regular subspaces of T.

Suppose $n \geqq 2m$. If δ is singular and elliptic, then δ is an involution. Let U and W denote its negative and positive eigenspaces. Let $\bar{X}$ be the projection of F^{2m} onto U and denote by X the extension of $\bar{X}$ to an element in $M_{n \times 2m}(F)$ as in the proof of Corollary 4.3. Then $U = F^n X$, $V = U \perp W$, and $\delta = -1_U \perp 1_W$. Clearly H'_X is the centralizer of δ.

Remark 4.6. The subgroups H_X are in fact F-points of endoscopic groups of SO_{2m}. As we shall discuss in the next section, it is through the theory of endoscopy that we shall interpret the residue of $A(s, \sigma, w_0)$ at $s = 0$ which otherwise seems to be a completely unmanageable object.

5. Twisted endoscopy and the norm correspondence

In this section we discuss how the theory of twisted endoscopy of Kottwitz and Shelstad [19] relates to our problem. In particular, we define a norm correspondence which agrees with their norm map on strongly θ-regular θ-semisimple conjugacy classes in GL_n. We need some preparation.

Given $Y \in GL_n(F) = G'$, the set

$$\{g^{-1} Y \theta(g) | g \in GL_n(F)\}$$

is called the θ-conjugacy class of Y in $GL_n(F)$. Moreover the group

$$\mathbf{G}'_{\theta,Y} = \{g \in \mathbf{G}' | g^{-1} Y \theta(g) = Y\}$$

is called the θ-twisted centralizer of Y in $\mathbf{G}'$ as in Section 2.

The θ-twisfted orbital itegral of a function $f' \in C_c^\infty(G')$ at Y is

$$\Phi_\theta(Y, f') = \int_{\mathbf{G}'_{\theta,Y}(F)\backslash G'} f'(g^{-1} Y \theta(g)) d\dot{g}.$$

Observe that we are not choosing the F-points of the connected component of $\mathbf{G}'_{\theta,Y}$, but rather all of it, to define our θ-twisted orbital integral.

Given $Y \in GL_n(F)$, there is always a $X \in M_{n \times 2m}(\bar{F})$ so that Y and X satisfy (3.4). Clearly we are only interested in those Y for which $X \in M_{n \times 2m}(F)$. Suppose a pair (Y, X) is a rational solution of (3.4). Then so is $(g^{-1} Y \theta(g), g^{-1} X)$ for every $g \in GL_n(F)$. Consequently (3.4) is also satisfied by every element in the θ-conjugacy class of Y.

Fix $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ for which (3.4) has F-rational solutions. It then parametrizes a set of θ-conjugacy classes in $GL_n(F)$. Let $\mathcal{N}$ be the set of all the θ-conjugacy classes in $GL_n(F)$ so parametrized as $\{X\}$ varies. Observe that $\{Y\} \in \mathcal{N}$ if and only if $\{Y^{-1}\} \in \mathcal{N}$.

We start with the following lemma which will be useful later.

Lemma 5.1 *a) Assume $n \leqq 2m$. For each even integer r_0, $0 \leqq r_0 \leqq n$, let E_0 and $E_0^\vee$, both in $M_n(F)$, be defined by*

$$E_0 = \begin{bmatrix} I_{r_0/2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{r_0/2} \end{bmatrix} \quad (0 \leqq r_0 \leqq n)$$

and

$$E_0^\vee = \begin{bmatrix} I_{(r_0/2)+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{r_0/2} \end{bmatrix} \quad (0 \leqq r_0 < n).$$

Set

$$X_0 = [0 \;\; E_0 \;\; 0]$$

and

$$E_0^\vee = [0 \;\; E_0^\vee \;\; 0]$$

both in $M_{n\times 2m}(F)$ with E_0 and $E_0^\vee$ occupying the middle n columns. Then for every r_0, $0 \leqq r_0 \leqq n$, X_0 and $X_0^\vee$ parametrize θ-conjugacy classes in $\mathcal{N}$ which for a given r_0, $0 \leqq r_0 < n$, are in fact the same.

b) If $n \geqq 2m$, a similar statement is valid for ${}^tX_0$ and ${}^tX_0^\vee$.

Proof. Clear.

Given $\{Y^{-1}\} \in \mathcal{N}$, define $N_\theta(\{Y^{-1}\})$ to be the $SO_{2m}(F)$-conjugacy class of $I - X'Y^{-1}X$ in $SO_{2m}(F)$, where (Y, X) is a F-rational solution of (3.4) with $Y^{-1} \in \{Y^{-1}\}$. We have to be more precise since neither X nor Y are unique.

Suppose $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ parametrizes $\{Y^{-1}\} \in \mathcal{N}$. Then $I - X'Y^{-1}X$ is unchanged as long as a choice $gX \in \{X\}$ is compensated with $\theta(g)Y^{-1}g^{-1} \in \{Y^{-1}\}$, another solution of (3.4). On the other hand, suppose $\{X\}$ and $\{X_1\}$ both parameterize same elements of $\mathcal{N}$. If they have same ranks, then by elementary operations we may assume $X_1 = Xh$, $h \in GL_{2m}(F)$. Using (3.4), we may assume

$$X_1X_1' = (Xh)(Xh)' = XX'.$$

An application of Lemma 4.1 now allows us to choose $h \in O_{2m}(F)$. Consequently $I - X_1'Y^{-1}X_1$ is in the same $O_{2m}(F)$-conjugacy class as $I - X'Y^{-1}X$. We formulate our conclusion as:

Lemma 5.2 *An element $X_1 \in M_{n\times 2m}(F)$ of the same rank as X parametrizes the same θ-conjugacy classes in $\mathcal{N}$ as X if and only if $X_1 \in GL_n(F)XO_{2m}(F)$.*

Finally, it is clear from Lemma 5.2 that given $\{Y^{-1}\} \in \mathcal{N}$, $N_\theta(\{Y^{-1}\})$ defines at most a finite number of conjugacy classes in $SO_{2m}(F)$ since $SO_{2m}(F)$ is a normal subgroup of $O_{2m}(F)$ of index two.

Definition 5.3 *The correspondence*

$$N_\theta : \mathcal{N} \to \mathscr{C}$$

is called the norm correspondence from $GL_n(F)$ to $SO_{2m}(F)$, where $\mathscr{C}$ is the collection of all the conjugacy classes in $SO_{2m}(F)$.

We refer to Remark 5.15 concerning the corresponding norm map.

We shall now start our study of N_θ. We start with:

Proposition 5.4 *a) Assume $n = 2m$ and $\{X\} = \{1\}$. Choose Y such that $(Y, 1)$ satisfies* (3.4). *Then $Y\theta(Y)$ belongs to $SO_n(F)$ and*

$$\begin{aligned} N_\theta(\{Y^{-1}\}) &= \{(I + Y^{-1})\} \\ &= \{-Y^{-1}\theta(Y^{-1})\} \\ &= \{-\theta(Y^{-1})Y^{-1}\}, \end{aligned}$$

where $\{(I + Y^{-1})\}$ denotes the conjugacy class of $I + Y^{-1}$ in $O_n(F)$.

b) With no assumption on n (n even), m, and $\{X\}$

$$X(I_{2m} - X'Y^{-1}X) = -\theta(Y^{-1})Y^{-1}X.$$

In particular, if X is a projection, then $I_{2m} - X'Y^{-1}X \in H_X$, where H_X is the right stabilizer of $F^nX \subset F^{2m}$ in $SO_{2m}(F)$.

Proof. For a) use Corollary 3.2. Part b) is a consequence of (3.6) and Corollary 4.3.

Proposition 5.5 *Suppose $n < 2m$ and take X in $M_{n\times 2m}(F)$. Fix Y, if any, with $\{Y^{-1}\} \in \mathcal{N}$, so that (Y, X) is a rational solution of* (3.4). *Then $I - X'Y^{-1}X$ belongs to either a proper parabolic subgroup of $SO_{2m}(F)$ or a proper centralizer of a singular elliptic element of $O_{2m}(F)$ in $SO_{2m}(F)$. The classes in $N_\theta(\{Y^{-1}\})$ are never regular elliptic.*

Proof. Since $n < 2m$, $U = F^nX$ is a proper subspace of F^{2m}. The proposition now follows immediately from Lemma 4.5.

For $n \geqq 2m$, we need:

Lemma 5.6 *Let V be a finite dimensional vector space over F. Let $T \to T^\vee$ be an anti-involution of End (V), i.e. $(T_1T_2)^\vee = T_2^\vee T_1^\vee$ and $(T^\vee)^\vee = T$. Fix $S \in End(V)$. Suppose $S^\vee = AS$ for some $A \in Aut(V)$. Then there exist a projection $E \in End(V)$ and a $B \in Aut(V)$ such that $S = E^\vee BE = E^\vee B$.*

Proof. Let H be the projection onto $\ker(S)$ and set $E = I - H$. Choose $B \in \mathrm{Aut}(V)$ such that $BE = S$. Let G be the projection onto $Im(S)$. We now require that $B(\ker(S)) = \ker(G)$ as well which is clearly possible. Then

$S = GBE = GB$. Now $S^\vee = E^\vee B^\vee G^\vee$, $G^\vee$ being a projection as well. Since $S^\vee = AS$, $\ker(S) = \ker(S^\vee) = \ker(E)$. On the other hand $S = GB = BE = GBE$ implies $B^\vee G^\vee = E^\vee B^\vee = E^\vee B^\vee G^\vee = S^\vee$. Thus $\ker(S^\vee) = \ker(G^\vee) = \ker(E)$, proving $G = E^\vee$.

Corollary 5.7 *Given $I + S \in SO_n(F)$, there exist $Y \in GL_n(F)$ and $X \in M_n(F)$ such that $S = -X'Y^{-1}X = -X'Y^{-1}$, where $X' = -\tilde{\theta}(X)$.*

Proof. Take $S^\vee = \theta(S)$. Since $I + S \in SO_n(F)$,

$$(I + S)\tilde{\theta}(I + S) = I$$

implies $S + \tilde{\theta}(S) + S\tilde{\theta}(S) = 0$ or $S^\vee = -(I + S)^{-1}S$. Set $A = -(I + S)^{-1}$ and apply Lemma 5.6.

Lemma 5.8 *Let $g \in SO_n(F)$, n even. Then the dimension of the subspace of vectors in F^n fixed by g is even.*

Proof. Let V be the subspace of F^n fixed by g. If $g = h^{-1}\sigma h$, with h and σ in $SO_n(\bar{F})$, then hV is a vector space over F contained in $\bar{F}^n$ and if $\overline{hV}$ is the $\bar{F}$-span of hV, then the $\bar{F}$-dimension of $\overline{hV} = hV \otimes_F \bar{F}$ is equal to the F-dimension of V. Moreover, $\overline{hV}$ is the fixed point subspace of σ in $\bar{F}^n$ if and only if $\bar{V} = V \otimes_F \bar{F}$ is the fixed point subspace of g in $\bar{F}^n$. Now suppose $\bar{v} \in \bar{F}^n$ satisfies $g\bar{v} = \bar{v}$. Write $\bar{v} = \sum c_i v_i$, $c_i \in \bar{F}$ and $v_i \in F^n$ with both $\{c_i\}$ and $\{v_i\}$ linearly independent over F. Then a standard argument, using the F-rationality of g, shows that $v_i \in V$. Thus the dimension of V is equal to $\bar{F}$-dimension of the fixed subspace of σ in $\bar{F}^n$.

If g is semisimple, then by the above observation σ may be assumed to be diagonal for which the lemma is immediate. The general case reduces to g being unipotent which we may assume to be upper triangular.

Assume g is upper triangular and unipotent. Write

$$g = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

and suppose $Y = \begin{bmatrix} Z \\ W \end{bmatrix}$ is in the kernel of $X = g - I$, where Z and W are same size column matrices. Then $C = \theta(A)$, $B\tilde{\theta}(A)$ is $\tilde{\theta}$-skew symmetric, $AZ + BW = Z$, and finally $\theta(A)W = W$. Using the last equation, we can consider $AZ + B\tilde{\theta}(A)W = Z$ and $\theta(A)W = W$ instead.

We now define a $n/2$ by $n/2$ matrix B' by replacing every row or column of $B\tilde{\theta}(A)$ which has a non-zero entry in $A - I$ or $\theta(A) - I$, respectively, with a row or column of zeros. Since $g \in SO_n(F)$, the replaced rows and columns are symmetric with respect to the second diagonal and consequently B' is also $\tilde{\theta}$-skew-symmetric. We then set $A' = [A - I | B\tilde{\theta}(A)] - [0 | B']$. We therefore need to determine the dimension of the solution space of $A'Y = [0 | B']Y = [0 | \theta(A) - I]Y = 0$.

The solution space of a $\tilde{\theta}$-skew-symmetric matrix is always of even dimension and therefore the number of independent equations coming from $[0|B']Y = 0$ is even. Moreover, they are independent from equations coming from the two other sets of equations. On the other hand A' and $\theta(A) - I$ have the same ranks and therefore the number of independent equations coming from $A'Y = 0$ is equal to ones coming from $[0|\theta(A) - I]Y = 0$. Moreover the two sets of equations are independent of each other. The lemma is now proved.

Remark. Due to the existence of reflections the lemma is clearly false if SO_n is replaced by O_n.

Proposition 5.9 *Suppose $n = 2m$. Then the norm correspondence $N_\theta: \mathcal{N} \to \mathcal{C}$ is surjective with finite fibers.*

Proof. We must first show that every conjugacy class in $SO_n(F)$ lies in the image of N_θ. Take $I - S \in SO_n(F)$ and by Corollary 5.7, choose $Y \in GL_n(F)$ and $X \in M_n(F)$, a projection, such that $S = X'Y^{-1}X = X'Y^{-1}$. Since $I - S \in SO_n(F)$, one gets

$$Y^{-1}X + \tilde{\theta}(Y^{-1})X = \tilde{\theta}(Y^{-1})XX'Y^{-1}X. \tag{5.9.1}$$

We must choose Y so that (Y, X) satisfies (3.4). We thus need to show whether and how uniquely we can choose Y so that $Y^{-1} + \tilde{\theta}(Y^{-1})$ vanishes on the kernel of the projection X, since $\ker(X) \subset \ker(\tilde{\theta}(Y^{-1})XX'Y^{-1}) = \ker(\tilde{\theta}(Y^{-1})XX'Y^{-1}X)$.

By (5.9.1)

$$Y^{-1} + \tilde{\theta}(Y^{-1}) = \tilde{\theta}(Y^{-1})XX'Y^{-1} \tag{5.9.2}$$

on the image of the projection X. On the other hand if $\mathbf{v} \in \ker(X)$, then

$$(Y^{-1} + \tilde{\theta}(Y^{-1}))(\mathbf{v}) = 0. \tag{5.9.3}$$

Since $-X'Y^{-1} = Y^{-1}X$ and $-X'\tilde{\theta}(Y^{-1}) = \tilde{\theta}(Y^{-1})X$, both Y^{-1} and $\tilde{\theta}(Y^{-1})$ send $\ker(X)$ isomorphically onto $\ker(-X')$. (Incidentally, they also send $\ker(Z)$, $Z = \tilde{\theta}(Y^{-1}).XX'Y^{-1}$, isomorphically onto $\ker(XX')$.) Fix a pair of bases for $\ker(X)$ and $\ker(-X')$. Then the matrix of the restriction of $\tilde{\theta}(Y^{-1})$ to $\ker(X)$ with respect to these bases is equal to the matrix of $Y^{-1}|\ker(X)$ when written with respect to transfer of these bases by $\tilde{\theta}$, where $\tilde{\theta}$ is induced by $\tilde{\theta}(\mathbf{v}) = {}^t(w_n(\mathbf{v}))$ for every $\mathbf{v} \in F^n$. We only need to observe $\tilde{\theta}(\ker(X))\tilde{\theta}(Y^{-1}) = \tilde{\theta}(Y^{-1}(\ker(X)))$. Consequently by (5.9.3), the matrix of $Y^{-1}|\ker(X)$ is $\tilde{\theta}$-skew-symmetric. But by Lemma 5.8, such transformations exist with a unique θ-conjugacy class in $GL_n(F)$. Then with Y as such we may drop X from (5.9.1) to conclude

$$Y^{-1} + \tilde{\theta}(Y^{-1}) = \tilde{\theta}(Y^{-1})XX'Y^{-1}, \tag{5.9.4}$$

proving (3.4).

We must now show that the θ-conjugacy class of Y^{-1} depends only on the $SO_n(F)$-conjugacy class of $I - X'Y^{-1}X$. Suppose X is changed to Xh, $h \in SO_n(F)$. Choose $g \in GL_n(F)$ and another projection X_1 such that $gX_1 = Xh$.

Then X_1 parametrizes the same θ-conjugacy classes as X and Y^{-1} changes to $\theta(g)^{-1}Y^{-1}g$ which is in the θ-conjugacy class of Y^{-1}.

We must now prove the finiteness of the fibers. We start with the following lemma.

Lemma 5.10 *Fix (Y, X) satisfying (3.4). Then $N_\theta(\{Y^{-1}\})$ determines the semisimple part of the conjugacy class of $\theta(Y^{-1})Y^{-1}$ in $GL_n(F)$ uniquely.*

Proof. Changing Y in its θ-conjugacy class, we may assume X is a projection. The conjugacy class of $\theta(Y^{-1})Y^{-1}$ in $GL_n(F)$ will not change. Then by equation (3.6)

$$\mathbf{v}\theta(Y^{-1})Y^{-1}X = -\mathbf{v}(I - X'Y^{-1}X)$$

for every $\mathbf{v}$ in the left image of X. On the other hand by (3.4)

$$\mathbf{v}\theta(Y^{-1})Y^{-1} = -\mathbf{v}$$

for every $\mathbf{v}$ in the left kernel of X. Since F^n is the direct sum of the left image and the left kernel of X, this determines the semisimple part of $\theta(Y^{-1})Y^{-1}$ uniquely.

Corollary 5.11 *The norm correspondence has finite fibers.*

Proof. In view of Lemma 5.10 and finiteness of unipotent conjugacy classes for $GL_n(F)$, we need to prove that the map

$$\{Y^{-1}) \mapsto \{\theta(Y^{-1})Y^{-1}\}$$

has finite fibers.

Given $g \in GL_n(F)$, let $\tilde{g} = (g, \theta) \in GL_n(F) \rtimes \{1, \theta\}$. Let $\tilde{g} = su$ be the Jordan decomposition of $\tilde{g}$. Then $\tilde{g}^2 = s^2u^2 = g\theta(g)$ with s^2 semisimple and u^2 unipotent, both in $GL_n(F)$. Write $s = (\sigma, \theta)$. Then in the terminology of [18], σ is θ-semisimple and $s^2 = \sigma\theta(\sigma) \in GL_n(F)$. Since the number of unipotent conjugacy classes are finite, the problem reduces to the finiteness of fibers for θ-semisimple θ-conjugacy classes in $GL_n(F)$.

First assume F is algebraically closed. Then we may assume σ is diagonal. The map $\{\sigma\} \mapsto \{\sigma\theta(\sigma)\}$ from GL_n-θ-conjugacy classes of diagonal elements of GL_n into their conjugacy classes is then easily seen to be an injection. For arbitrary F this is now a consequence of finiteness of H^1 for a linear algebraic group over a local field of characteristic zero.

Remark 5.12. It is now clear that our norm correspondence N_θ always picks up the semisimple part of the concrete norm map $\{Y^{-1}\} \mapsto \{\theta(Y^{-1})Y^{-1}\}$ (cf. [7, 16, 19, 22] and discussions at the end of this section). Of course the converse is not true since N_θ will always pick up all of $\{\theta(Y^{-1})Y^{-1}\}$ if $\{X\} = \{1\}$which is always the case if Y^{-1} is unipotent.

Remark 5.13. Here is an example where certain fibers of N_θ are non-trivial even when F is algebraically closed.

Assume F is algebraically closed and $n = 2$. Let

$$Y = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Observe that Y is semisimple but not θ-semisimple. We may then take $X = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and therefore $I - X'Y^{-1}X = I$. Consequently both θ-conjugacy classes

$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right\} \neq \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

are in the fiber of I. Observe that

$$-\theta(Y^{-1})Y^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

will still single out Y. But it is deficient since it does not belong to SO_2.

Corollary 5.14. The statement of Proposition 5.9 is valid for $n \geq 2m$.

Proof. By embedding SO_{2m} as the subgroup $I_{(n/2)-m} \times SO_{2m}$ of the standard Levi subgroup $GL_{(n/2)-m} \times SO_{2m}$ and choosing X in its coset in $GL_n(F)\backslash M_{n\times 2m}(F)$ so that its non-zero rows are its middle $2m$ rows, the problem reduces to the case of $n = 2m$ proved in Proposition 5.9.

Remark 5.15. It is clear that the norm correspondence N_θ becomes a map if we restrict the image to a given $O_{2m}(F)$-conjugacy class $\{H_X\}$, where (Y, X) is a solution of (3.4). In fact every double coset $GL_n(F)XO_{2m}(F)$ which parametrizes $\{Y\} \in \mathcal{N}$ through (3.4) defines a unique $O_{2m}(F)$-conjugacy class in $N_\theta(\{Y^{-1}\})$. Specifying the double coset $\{X\}$ will then allow us to define a norm *map* from $GL_n(F)$ to $\{H_X\}$.

Next observe that the choice of the preimage of each conjugacy class in $\mathscr{C}$ made for in Corollary 5.7 and Proposition 5.9 defines a well defined section for the inverse of N_θ. Let N_θ^{-1} denote this section. We have:

Lemma 5.16 *The map $N_\theta^{-1}: \mathscr{C} \to \mathscr{C}'$ is continuous.*

Proof. Let $\{I - S_n\}_n \in \mathscr{C}$ approach to $\{I - S\} \in \mathscr{C}$ by $\{S_n\}_n$ approaching to S, where $-S$ is as in Corollary 5.7. Given S_n, let X'_n and $Y_{S_n}^{-1}$ be defined as in Corollary 5.7. Similarly define X' and Y_S^{-1} for S. Since $S_n(V)$ approaches $S(V)$, $\lim_n X'_n$ exists and equals X'. Then

$$X'Y_S^{-1} = S = \lim_n S_n = \lim_n X'_n Y_{S_n}^{-1} = X' \lim_n Y_{S_n}^{-1}.$$

Applying $\tilde{\theta}$ we get $\tilde{\theta}(Y_S^{-1})X = \lim_n \tilde{\theta}(Y_{S_n}^{-1})X$ and therefore if W is defined by $V = W \oplus \ker(S)$, then $\left(\lim_n \tilde{\theta}(Y_{S_n}^{-1})\right)| W$ exists and equals $\tilde{\theta}(Y_S^{-1})| W$. On the other hand

$$Y_{S_n}^{-1} + \tilde{\theta}(Y_{S_n}^{-1}) = \tilde{\theta}(S_n)S_n \tag{5.11.1}$$

on W_n, $V = W_n \oplus \ker(S_n)$, and

$$Y_{S_n}^{-1} + \tilde{\theta}(Y_{S_n}^{-1}) = 0$$

on $\ker(\tilde{\theta}(S_n)S_n)$, i.e. (5.11.1) holds on V. Thus

$$\left(\lim_n Y_{S_n}^{-1} + \tilde{\theta}\left(\lim_n Y_{S_n}^{-1}\right)\right)(\mathbf{v}) = \tilde{\theta}(S)S(\mathbf{v})$$

for $\mathbf{v} \in V$. Now suppose $\mathbf{v} \in \ker(\tilde{\theta}(S)S)$. Then

$$\left(\lim_n Y_{S_n}^{-1} + \tilde{\theta}\left(\lim_n Y_{S_n}^{-1}\right)\right)(\mathbf{v}) = 0.$$

On the other hand

$$Y_S^{-1} + \tilde{\theta}(Y_S^{-1}) = 0$$

on the same kernel. This implies $\{\tilde{\theta}(Y_S^{-1})\} = \lim_n \{\tilde{\theta}(Y_{S_n}^{-1})\}$, i.e. $\{Y_S^{-1}\} = \lim_n \{Y_{S_n}^{-1}\}$, proving the lemma.

We shall now study the relation of N_θ with the norm map defined by Kottwitz and Shelstad in [19].

Suppose first that F is algebraically closed and again assume $n = 2m$. Then $\mathcal{N} = \mathscr{C}'$, the set of all the θ-conjugacy classes in $GL_n = GL_n(F)$. The correspondence $N_\theta: \mathscr{C}' \to \mathscr{C}$ is again defined. Take $\mathbf{T}'_0$ to be the Cartan subgroup of diagonal matrices in GL_n. Clearly θ fixes the pair $(\mathbf{B}'_0, \mathbf{T}'_0)$, where $\mathbf{B}'_0$ is the Borel subgroup of upper triangular matrices. But it does not fix the standard splitting of GL_n defined by standard root vectors of $\mathbf{T}'_0$ in $\mathbf{B}'_0$. Here we use notation and definitions from §1.1 of [19]. Let $g_\theta = \mathrm{diag}(1, -1, 1, -1, \dots) \in GL_n$. Set $\theta^* = \mathrm{Int}(g_\theta)\cdot\theta$, another automorphism of GL_n which now also fixes the standard splitting of GL_n (notation as in [19]).

A simple calculation shows that the kernel of the map

$$\bar{N}_{\theta^*}: Y \to Y\theta(Y)$$

from $\mathbf{T}'_0$ into itself is exactly $(1 - \theta^*)\,\mathbf{T}'_0$ and therefore $\bar{N}_{\theta^*}$ can be identified with the projection of $\mathbf{T}'_0$ onto $\mathbf{T}'_0/(1 - \theta^*)\mathbf{T}'_0$, i.e. $\bar{N}_{\theta^*}$ is the abstract norm map N_{θ^*} of Kottwitz and Shelstad defined in §3.2 of [19].

We now remove the assumption that F is algebraically closed. Suppose $\mathbf{T}_H$ is a Cartan subgroup of SO_n defined over F. By Lemma 3.3.B of [19], there exists a θ^*-stable pair $(\mathbf{B}', \mathbf{T}')$ of GL_n with $\mathbf{T}'$ defined over F so that the attached isomorphism $\mathbf{T}_H \to \mathbf{T}'_{\theta^*}$ is defined over F.

Lemma 5.17 *The map $Y \mapsto Y\theta^*(Y)$ from* $\mathbf{T}'$ into $\mathbf{T}'$ *has* $(\mathbf{T}')^{\theta^*}$ *as its image and can be identified with the projection of* $\mathbf{T}'$ *onto* $\mathbf{T}'_{\theta^*}$. *It is defined over* F.

Proof. Since $\mathbf{T}'$ and $\mathbf{T}'_0$ are both θ^*-stable, there exists a g in the group of fixed points of θ^* in GL_n such that $\mathbf{T}' = g^{-1}\mathbf{T}'_0 g$. Let K and K_0 be the kernels of the map $Y \to Y\theta^*(Y)$ defined for $\mathbf{T}'$ and $\mathbf{T}'_0$, respectively. Then $K = g^{-1}K_0 g$ and the lemma is a consequence of $K_0 = (1 - \theta^*)\mathbf{T}'_0$. A same type of argument using the fact that the image of $\mathbf{T}'_0$ is $(\mathbf{T}'_0)^{\theta^*}$, fixed points of θ^* in $\mathbf{T}'_0$, implies the same fact for $\mathbf{T}'$.

Lemma 5.18 *Assume $n = 2m$. Suppose $I - X'Y^{-1}X$ acts semisimply on the direct sum of the right image and kernel of X, where X and Y are as in Corollary 5.7. Then $I - X'Y^{-1}X = -\theta(Y^{-1})Y^{-1}$.*

Proof. By (3.6)

$$X(I - X'Y^{-1}X) = -\theta(Y^{-1})Y^{-1}$$

on the right image of X and $I - X'Y^{-1}X = I$ on the right kernel of X.

Unless $(I - X)(I - X'Y^{-1}X) = 0$ on the right image of X, the matrix of $I - X'Y^{-1}X$ in a basis respecting the decomposition of F^n according to the right kernel and right image of X will have a unipotent part, contradicting the semisimplicity of the action of $I - X'Y^{-1}X$. This implies

$$-\theta(Y^{-1})Y^{-1} = X(I - X'Y^{-1}X) = I - X'Y^{-1}X$$

on the right image of X. But by (5.9.3), $-\theta(Y^{-1})Y^{-1} = I$ on the right kernel of X as well, proving the lemma.

It is curious to know exactly when the equality of Lemma 5.18 holds. The following proposition answers this question.

Proposition 5.19 *Assume $n = 2m$. Choose (Y, X) satisfying* (3.4) *according to Corollary* 5.7. *Then $I - X'Y^{-1}X = -Y^{-1}\theta(Y^{-1})$ if and only if $X = \tilde{\theta}(X) = -X'$. In this case $Y^{-1}\theta(Y^{-1}) = \theta(Y^{-1})Y^{-1}$.*

Proof. Assume first that $X' = -X$. Suppose $\mathbf{v}$ is in the left kernel of the projection $-X'$. Then it is in the left kernel of $Z = \tilde{\theta}(Y^{-1})XX'Y^{-1}$ and therefore by (5.9.4), $-Y^{-1}\theta(Y^{-1})$ acts on (the right) on the left kernel of $-X'$ as identity. On the other hand by (3.5)

$$(\mathbf{v})(I - X'Y^{-1}X)X' = (\mathbf{v})(Y^{-1}\theta(Y^{-1}))$$

whenever $\mathbf{v}$ is in the left image of $-X'$. Now using $X' = -X$ and $X^2 = X$, one gets

$$(\mathbf{v})(I - X'Y^{-1}X) = (\mathbf{v})(-Y^{-1}\theta(Y^{-1}))$$

for such a $\mathbf{v}$. Finally observe that $I - X'Y^{-1}X = I$ on the left kernel of $-X'$. This proves $I - X'Y^{-1}X = -Y^{-1}\theta(Y^{-1})$.

Conversely, suppose

$$I - X'Y^{-1}X = -Y^{-1}\theta(Y^{-1}).$$

Applying X' on the left to both sides of (3.5) implies

$$X'(I - X'Y^{-1}X)X' = X'Y^{-1}\theta(Y^{-1})$$

which by our assumptions equals $-X'(I - X'Y^{-1}X)$. This now implies $X'Y^{-1}XX' = -X'Y^{-1}X$. Using

$$X'Y^{-1}X = X'Y^{-1} = -Y^{-1}X$$

we now have $-Y^{-1}XX' = Y^{-1}X$ or $XX' = -X$. We now apply $\tilde{\theta}$ to complete the lemma.

Lemma 5.20 *a) Assume F is algebraically closed. Suppose $\{Y\} \in \mathcal{N} = \mathcal{C}'$ is θ-semisimple with Y belonging to a θ-stable Cartan subgroup of GL_n. Then there exists a $X \in M_n(F)$, satisfying* (3.4) *with Y, such that $I - X'Y^{-1}X$ is semisimple in $SO_n(F)$.* Moreover *$\theta(Y^{-1})Y^{-1}$ belongs to $SO_n(F)$ and $I - X'Y^{-1}X$ is $GL_n(F)$-conjugate to $-\theta(Y^{-1})Y^{-1}$. Consequently, there exists a $Y_0 \in \{Y\}$ and X_0, satisfying* (3.4) *with Y_0, such that $I - X_0'Y_0^{-1}X_0 = -\theta(Y_0^{-1})Y_0^{-1}$. Moreover, every $GL_n(F)$-conjugate of $-\theta(Y^{-1})Y^{-1}$ belongs to the image of $\{Y^{-1}\}$ under the norm correspondence N_θ of Definition* 5.3.

b) Suppose F is not necessarily algebrically closed. Then above statements remain all valid whenever there exists a $X \in M_n(F)$, satisfying (3.4) *with Y as above, such that $I - X'Y^{-1}X$ is semisimple in $SO_n(F)$. This is, in particular, the case, if $\{Y\} \in \mathcal{N}$ corresponds to $\{X\} = \{I\}$, i.e. for almost all Y. In this case, $GL_n(F)$-conjugates of $-\theta(Y^{-1})Y^{-1}$ exhaust the image of $\{Y^{-1}\}$ under the norm correspondence N_θ of Definition* 5.3.

c) In either case, the semisimple part of every conjugacy class in $N_\theta(\{Y^{-1}\})$ is $GL_n(F)$-conjugate to $\{-\theta(Y^{-1})Y^{-1}\}$.

Proof. Proofs of parts b) and c) are the same as the proof of part a) and therefore we shall assume F is algebrically closed. Then Y is conjugate by an element $h \in SO_n(F)$ to a diagonal matrix $Y_1 = hYh^{-1} = \mathrm{diag}(a_1, \ldots, a_n) \in \mathbf{T}_0'$. Then

$$X_1 = \mathrm{diag}(a_1 + a_n, a_2 + a_{n-1}, \ldots, 1, \ldots, 1)\cdot\sqrt{-1}$$

satisfies (3.4) with Y_1 and

$$I - X_1'Y_1^{-1}X_1 = -\mathrm{diag}(a_n a_1^{-1}, a_{n-1}a_2^{-1}, \ldots, a_n^{-1}a_1)$$

which is semisimple. Set $X = h^{-1}X_1$. Then $I - X'Y^{-1}X = I - X_1'Y^{-1}X_1$ is semisimple.

Since Y belongs to a θ-stable Cartsan subgroup, $\theta(Y^{-1})Y^{-1}$ is semisimple and lies in $SO_n(F)$. Choose Y_2^{-1}, θ-conjugate to Y^{-1}, and a projection X_2, satisfying (3.4) with Y_2, such that $I - X_2'Y_2^{-1}X_2 = I - X'Y^{-1}X$ which is semisimple. Then $-\theta(Y_2^{-1})Y_2^{-1}$ is $GL_n(F)$-conjugate to $-\theta(Y^{-1})Y^{-1}$. Moreover, by Lemma 5.10 its eigenvalues different from 1 are among those of

$I - X_2' Y_2^{-1} X_2$. ($I - X_2' Y_2^{-1} X_2$ preserves the left image of X_2). Since $\theta(Y^{-1})Y^{-1} = Y^{-1}\theta(Y^{-1})$, one sees that the eigenvalues of $-\theta(Y_2^{-1})Y_2^{-1}$ and $-Y_2^{-1}\theta(Y_2^{-1})$ are the same. Therefore, one can apply the argument of Lemma 5.10 to equation (3.5) to show that eigenvalues of $I - X_2' Y_2^{-1} X_2$ which are not 1 are also among those of $-\theta(Y_2^{-1})Y_2^{-1}$. This now implies that $I - X'Y^{-1}X$ and $-\theta(Y^{-1})Y^{-1}$ are $GL_n(F)$-conjugate. The rest of the statements are now clear.

Corollary 5.21 *a) Suppose F is algebraically closed. Then the norm correspondence of Definition* 5.3 *agrees with the norm map of Kottwitz and Shelstad on strongly θ-regular θ-conjugacy classes in GL_n.*

b) If F is not necessarily algebraically closed, then the norm correspondence of Defnition 5.3 *agrees with the norm map of Kottwitz and Shelstad on the intersection of $\mathcal{N}$ with strongly θ-regular θ-conjugacy classes in GL_n, except possibly on a set of measure zero.*

c) In either case, the semisimple part of every conjugacy class in the norm N_θ of a θ-semisimple element agrees with the one defined by Kottwitz and Shelstad.

Proof. Fix a Cartan subgroup $\mathbf{T}_H$ of SO_n defined over F. Let $\mathbf{T}'$ be a θ^*-stable Cartan subgroup of $\mathbf{G}'$ so that the attached isomorphism $\mathbf{T}'_H \to \mathbf{T}'_{\theta^*}$ is defined over F. Choose Y^{-1} in $T'\, g_\theta$. Then

$$-Y^{-1}\theta(Y^{-1}) = Y^{-1}g_\theta^{-1}\theta^*(Y^{-1}g_\theta^{-1}) \tag{5.21.1}$$

and $Y^{-1}g_\theta^{-1} = m(Y^{-1})$ in the notation of §3.1 of [19]. Moreover if Y^{-1} is strongly θ-regular, then $Y^{-1}g_\theta^{-1}$ is strongly θ^*-regular. The sign in (5.21.1) results from $\theta(g_\theta) = -g_\theta$. By (3.3.4) of [19] and Lemma 5.17, $N_{\theta^*}(m(Y^{-1})) = Y^{-1}g_\theta^{-1}\,\theta^*(Y^{-1}g_\theta^{-1})$.

The corollary is now a consequence of Lemma 5.20, identity (5.21.1), Definition 5.3, and the definition of norm in [19].

Remark 5.22. Equation (5.21.1) now justifies the minus sign in Corollary 3.2, Proposition 5.4, and elsewhere, which at first looked disturbing. It points to the harmonious agreement of our results with those of Kottwitz and Shelstad [19].

Remark 5.23. From Proposition 5.9 and Corollary 5.21, it is clear that N_θ is the natural generalization of the norm of Kottwitz and Shelstad [19] in this case. One must notice that N_θ is a norm map which belongs to the split group and is only defined on the subset $\mathcal{N}$ of all the θ-conjugacy classes GL_n.

6. Poles of intertwining operators

For the rest of the paper we shall assume τ' is supercuspidal but still allow τ to be any discrete series representation of $SO_{2m}(F)$. Observe that τ is quite general and in particular we are not assuming τ to be generic at all. In this section we shall calculate the residue of $A(s, \sigma, w_0)$ at $s = 0$.

From the general theory [33, 38] we may assume $\tau' \simeq \tilde{\tau}'$. If ω' is the central character of τ', this then implies $\omega'^2 = 1$. Given a matrix coefficient ψ of τ', there exists a function $f' \in C_c^\infty(G')$ such that

$$\psi(g) = \int f'(zg)\omega'(z)dz, \tag{6.1}$$

where the integral is over the center of $G' = GL_n(F)$. Let f be a matrix coefficient of τ.

As discussed before we may assume h has compact support in PN^- modulo P. Then by Lemma 3.1 it is enough to study poles of

$$\int \psi(\theta(Y))f(I_{2m} - X'Y^{-1}X)|\det Y|^{-s-m-(n-1)/2}d(X, Y), \tag{6.2}$$

where the integral is over all $Y \in GL_n(F)$ and $X \in M_{n\times 2m}(F)$ satisfying (3.3) and (3.4) with Y^{-1} and $Y^{-1}X$ belonging to compact subsets of $M_n(F)$ and $M_{n\times 2m}(F)$, respectively. Observe that in view of the result of [32], we have assumed $m \neq 0$. The measure $d(X, Y)$ denotes dn.

We first show that the condition on X can be dropped. In fact, we will soon break the integral over θ-conjugacy classes of elements in G'. Then the pairs $(gX, gY\theta(g)^{-1})$ as g ranges over G' will all have the same norm and for all of these $(gY\theta(g)^{-1})^{-1}gX$ must belong to the same open compact subset of $M_{n\times 2m}(F)$ which we may assume to be a basic neighborhood of zero. Then $\theta(g)Y^{-1}X$ belongs to the same set for all $g \in G'$. Now, by stretching the basic neighborhood by $Y\theta(g)^{-1}$ as g, for example, ranges over the center of G', X will have no restriction.

Let $\mathrm{diag}(a, I_{2m}, \theta(a))$, $a \in GL_n(F)$, act by adjoint action on N, sending $n(X, Y)$ to $n(a^{-1}X, a^{-1}Y\theta(a))$. Let $a = Y$. Then

$$d(Y^{-1}X, \theta(Y))/d(X, Y) = |\det Y|^{-2m-(n-1)}.$$

Set

$$d^*(X, Y) = |\det Y|^{-m-(n-1)/2}d(X, Y).$$

Then

$$d^*(Y^{-1}X, \theta(Y)) = d^*(X, Y).$$

Moreover (X, Y) and $(Y^{-1}X, \theta(Y))$ have same norms and therefore (6.2) can be written as

$$\int \psi(Y)f(I_{2m} - X'Y^{-1}X)\omega'(z)|\det Y|^s d^*(X, Y). \tag{6.3}$$

Finally, using (6.1), (6.3) equals

$$\int\int f'(zY)f(I_{2m} - X'Y^{-1}X)\omega'(z)|\det Y|^s dz d^*(X, Y), \tag{6.4}$$

where we have used $\omega'^2 = 1$. We need a technical lemma.

Lemma 6.1 *Choose X and Y satisfying* (3.3) *and* (3.4). *Assume $g \in \mathbf{G}'_{\theta,Y}(F)$. Define, if possible, $h \in G$ by $gX = Xh$. Then h whose class modulo the right stabilizer of X is unique belongs to the centralizer of $I - X'Y^{-1}X$ in G. Conversely, given an element h in the centralizer of $Z = I - X'Y^{-1}X$ in G,* *satisfying* $gX = Xh$, $g \in G'$, *one may choose g such that* $g \in \mathbf{G}'_{\theta,Y}(F)$, *provided that X and Y are as in Corollary* 5.7.

Proof. We shall prove the lemma for the case $n = 2m$ and leave the general case to the reader. We only need to prove the second assertion. Then X is a projection. Let $U = F^n X$ be the left image. Then $Uh = U$ and if $H' \subset G'$ is the subgroup of all $g \in G'$ for which $gX = Xh$, $h \in H_X$, then the quotient of H' by the left stabilizer of X is isomorphic to the quotient of H_X by the right stabilizer of X (Lemma 4.2). If $g \in \mathbf{G}'_{\theta,Y}(F) \cap H'$, then h will centralize $X'Y^{-1}X$. On the other hand, if $h \in H_X$ centralizes $X'Y^{-1}X$ which is the truncation of Y^{-1} by the projection X and its twist X', then, using an appropriate element in the intersection of $\mathbf{G}'_{\theta,Y}(F)$ with the left stabilizer of X, h can be extended to an element of $g \in \mathbf{G}'_{\theta,Y} \cap H'$, since $Uh = F^n Xh = U$ and Y is as in Corollary 5.7. This completes the lemma.

The integration requires Y to be in G' and therefore we can break the integration over different θ-conjugacy classes in $\mathcal{N}$. This requires a measure $d\dot{g}$ on $\mathbf{G}'_{\theta,Y}(F)\backslash GL_n(F)$ for every $\{Y\} \in \mathcal{N}$. We fix (X, Y) with X a projection. As h ranges in $O_{2m}(F)$, every pair (Y, Xh) satisfies (3.4) and by Lemma 5.2 these are the only possibilities if X parametrizes the θ-twisted conjugacy class of Y. Suppose $Z = I - X'Y^{-1}X$ is so that for every $h \in \mathbf{G}_Z(F)$ there exists a $g \in \mathbf{G}'(F)$ such that $gX = Xh$ and that through this, the quotient of $\mathbf{G}_Z(F)$ by the right stabilizer of X in G embeds into the quotient of $\mathbf{G}'_{\theta,Y}(F)$ by the left stabilizer of X in G'. Since g varies in $\mathbf{G}'_{\theta,Y}(F)\backslash G'$, h must change in $\mathbf{G}_Z(F)\backslash G$. Consequently we can choose a measure $d\dot{h}$ on $\mathbf{G}_Z(F)\backslash G$ which together with $d\dot{g}$ accounts for the contribution from the θ-twisted conjugacy classes of Y and the conjugacy classes of Z.

When $n = 2m$ and given Z, X and Y are chosen as in Corollary 5.7, then for $h \in \mathbf{G}_Z(F)$, $h^{-1}X'Y^{-1}Xh = -h^{-1}Y^{-1}Xh = -Y^{-1}X$ and therefore $F^n Xh = F^n X$. Hence by Lemma 4.2, every $h \in \mathbf{G}_Z(F)$ satisfies $gX = Xh$, $g \in G'$, and therefore by Lemma 6.1 we are in the above situation. This happens in particular for an important and large subset of $\mathcal{N}$ if $2m = n$. In fact, if $n = 2m$, then for almost all the θ-regular (and therefore θ-semisimple) θ-twisted conjugacy classes in $\mathcal{N}$, X may be taken to be I. Clearly if $\{X\} = \{I\}$, then by (3.4), $\mathbf{G}'_{\theta,Y} \cong \tilde{\mathbf{G}}_Z$, where $\tilde{\mathbf{G}}_Z$ is the centralizer of Z in O_{2m}.

On the other hand the example of Remark 5.13 shows that this is not the case in general. In fact, since the integration is over different θ-twisted conjugacy classes in $\mathcal{N}$, the corresponding orbital integrals in $SO_{2m}(F)$ may not converge as the integration will include points in $\mathbf{G}_Z(F)$ non-trivially. But such terms cannot contribute to the residue since their divergence is independent of s and it is well known that the intertwining integrals converge absolutely for $Re(s)$ large. In these cases the corresponding stable θ-twisted orbital integrals must vanish and a proof of that along the lines of the proof of Proposition 3.12, pg. 31 of [2] must not be difficult. But we leave this to a future paper and at present content ourselves with the above argument. Now, let dn be a measure which is supported in every θ-twisted conjugacy class in $\mathcal{N}$ (up to a set of measure zero if the class is θ-regular). Then, using

Definition 5.3, equation (6.4) can be written as

$$\int_{z\in F^*}\int_{\{Y\}\in\mathcal{N}} \tilde{\Phi}_\theta(zY,f')\Phi(N_\theta(\{Y^{-1}\}),f)\omega'(z)dzd\mathrm{n}, \tag{6.5}$$

where

$$\tilde{\Phi}_\theta(zY,f') = \int_{G'_{\theta,Y}(F)\backslash G'} f'(g^{-1}zY\theta(g))|\det(g^{-1}Y\theta(g))|^s d\dot{g}.$$

Finally (6.5) can be written as

$$\int_{\{Y\}\in\mathcal{N}}\sum_{\varepsilon\in(F^*)^2\backslash F^*} \tilde{\Phi}_\theta(\varepsilon Y,f')\Phi(N_\theta(\{Y^{-1}\}),f)\omega'(\varepsilon)d\mathrm{n} \int_{|\det z|<\kappa} |\det z|^{2s}dz, \tag{6.6}$$

where as usual the bound κ for $|\det z|$ is a consequence of Y belonging to a compact subset of $M_n(F)$ in (6.3) and $f' \in C_c^\infty(G')$.

We shall soon show that for $s = 0$ the first integral is always convergent. Therefore the poles of $A(s, \sigma, w_0)$ at $s = 0$, if any, always come from the second one. But

$$\int_{|\det z|<\kappa} |\det z|^{2s}dz$$

converges absolutely if and only if $Re(s) > 0$ and its poles are the same as

$$L(2ns, 1) = (1 - q^{-2ns})^{-1}.$$

Letting s go to zero in (6.6) we must therefore consider, at least formally,

$$\sum_{\varepsilon\in(F^*)^2\backslash F^*}\int_{\mathcal{N}} \Phi_\theta(\varepsilon Y,f')\Phi(N_\theta(\{Y^{-1}\}),f)\omega'(\varepsilon)d\mathrm{n}. \tag{6.7}$$

To analyze (6.7) or more precisely to understand what the measure $d\mathrm{n}$ is, we need to consider two cases. We do this in two separate sections.

Remark 6.2. Using the decomposition $w_0n = pn^-$ of Lemma 3.1, it is clear that the integration in hand is in fact over orbits of N (with $Y \in GL_n$) under the adjoint action of M. Therefore, studying these orbits must play a role in studying the general case (i.e. non-classical). Whether they can lead us to new cases of functoriality remains to be seen.

7. The case $n \geqq 2m$

We shall first assume $n \geqq 2m$. With notation as in Section 2, let $\mathscr{C}$ and $\mathscr{C}'$ be the sets of conjugacy classes and θ-conjugacy classes in $SO_{2m}(F)$ and $GL_n(F)$, respectively. By Proposition 5.9 and Corollary 5.14, we can define a finite set

$$\mathscr{A}(\{\gamma\}) = \{\{\varepsilon\gamma'\} \mid \varepsilon \in (F^*)^2\backslash F^*, \{\gamma\} \in N_\theta(\{(\gamma')^{-1}\})\},$$

for every $\{\gamma\} \in \mathscr{C}$. Moreover, we have

Lemma 7.1 *(no assumption on n and m except n even). Fix a representative $\bar{\varepsilon}$ for a class in $(F^*)^2\backslash F^*$. Let $\varepsilon_0 = diag(\bar{\varepsilon}, \ldots, \bar{\varepsilon}, 1, \ldots, 1) \in GO_{2m}(F)$. Let $\varepsilon = diag(\bar{\varepsilon}, \ldots, \bar{\varepsilon}) \in GL_n(F)$. Then*

$$N_\theta(\{\varepsilon\gamma'\}) = \varepsilon_0^{-1}N_\theta(\{\gamma'\})\varepsilon_0,$$

for every $\{\gamma'\} \in \mathcal{N}$. *Moreover, if f is a matrix coefficient for* τ, *then* f^{ε_0}, *defined by* $f^{\varepsilon_0}(g) = f(\varepsilon_0^{-1} g \varepsilon_0)$, $g \in SO_{2m}(F)$, *is one for* $\tau \cdot Ad(\varepsilon_0)$.

Proof. Let $\varepsilon^{\vee} = \mathrm{diag}(\bar{\varepsilon}, \ldots, \bar{\varepsilon}) \in GL_{2m}(F)$. Suppose $\{\gamma'\}$ is represented by Y^{-1}. It then follows from (3.4) that

$$X\varepsilon^{\vee} X' = \varepsilon Y + \tilde{\theta}(\varepsilon Y)$$

or

$$(X\varepsilon_0)(X\varepsilon_0)' = \varepsilon Y + \tilde{\theta}(\varepsilon Y).$$

Thus $N_\theta((\varepsilon Y)^{-1})$ consists of $SO_{2m}(F)$-conjugacy classes which are defined by

$$I - (X\varepsilon_0)'(\varepsilon Y)^{-1}(X\varepsilon_0), \tag{7.1.1}$$

as X changes, satisfying (3.4) for Y. But now $\varepsilon^{\vee} = \varepsilon_0 \tilde{\theta}(\varepsilon_0)$ implies the equality of (7.1.1) with

$$\varepsilon_0^{-1}(I - X'Y^{-1}X)\varepsilon_0,$$

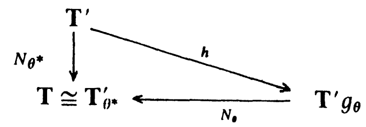
completing the lemma.

We finally specify $\Delta(\{\gamma\}, \{\varepsilon\gamma'\}) = \omega'(\varepsilon)\, \kappa(\{\gamma\}, \{\gamma'\})$ if $\{\gamma\} \in N_\theta(\{(\gamma')^{-1}\})$ and zero otherwise, where the notation is as in Section 2. Unless $\{\gamma\}$ is regular semisimple we set $\kappa \equiv 1$. For a regular semisimple $\{\gamma\}$ it will be defined as follows.

Assume first that $n = 2m$. We shall freely use definitions, notation, and results from Section 5 of the present paper, as well as those in Sections 3 and 4 of [19].

Let $\mathbf{T}$ be a Cartan subgroup of SO_n defined over F. By Lemma 3.3.B of [19], choose a θ^*-stable Cartan subgroup $\mathbf{T}'$ of $\mathbf{G}' = GL_n$, defined over F, so that the attached isomorphism $\mathbf{T} \to \mathbf{T}'_{\theta^*}$ is also defined over F. It induces the *image map* $\mathcal{A}_{\mathbf{G}/\mathbf{G}'}$, introduced in [19, 23], between the semisimple conjugacy classes in $\mathbf{G} = SO_n$ and θ^*-semisimple θ^*-conjugacy classes in $\mathbf{G}' = GL_n$.

Let $g_\theta = \mathrm{diag}(1, -1, \ldots, 1, -1)$ be as in Section 5. Then $\theta^* = \mathrm{Int}(g_\theta) \cdot \theta$ and we have the diagram



which commutes when restricted to strongly θ^*-regular elements in $\mathbf{T}'$ since

$$t\theta^*(t) = -(tg_\theta)\theta(tg_\theta),$$

using Lemma 5.20 and Corollary 5.21. Here $h(t) = tg_\theta$, $t \in \mathbf{T}'$.

The inverse map h^{-1}, then induces the map m of §3.1 of [19] between strongly θ-regular and strongly θ^*-regular conjugacy class in $\mathbf{G}' = GL_n$. Thus we shall also use m to denote h^{-1}. Observe that we are abusing our notation by using N_θ also as a map between $\mathbf{T}'g_\theta$ and $\mathbf{T}'_{\theta^*}$.

We shall collect this discussion as:

Lemma 7.2 *Let* $\mathbf{T}$ *be a Cartan subgroup of* SO_n *defined over* F. *Then there exists a* θ^**-stable Cartan subgroup* $\mathbf{T}'$ *of* $\mathbf{G}' = GL_n$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{T}' & & \\ {\scriptstyle N_{\theta^*}} \downarrow & \searrow^{m^{-1}} & \\ \mathbf{T} \cong \mathbf{T}'_{\theta^*} & \xleftarrow[N_\theta]{} & \mathbf{T}' g_\theta \end{array}$$

commutes for all strongly θ^**-regular elements of* $\mathbf{T}'(F)$. *The isomorphism* $\mathbf{T} \cong \mathbf{T}'_{\theta^*}$ *induces the image map* $\mathscr{A}_{\mathbf{G}/\mathbf{G}'}$. *Moreover, if* $\delta^* \in \mathbf{T}'$ *is strongly* θ^**-regular, then* $Cent_{\theta^*}(\delta^*, \mathbf{G}') = (\mathbf{T}')^{\theta^*}$.

Proof. For the last statement one only needs to observe that $\mathbf{T}'_{\delta^*} = \mathbf{T}'$ (notation as in §3.3 of [19]) and the set of fixed points of $\mathrm{Int}(\delta^*) \cdot \theta^*$ in $\mathbf{T}'$ is $(\mathbf{T}')^{\theta^*}$.

If $d\gamma'$ is a measure on $(\mathbf{T}')^{\theta^*}(F)$, then by Lemmas 5.17 and 5.20, $|D_{\theta^*}(\gamma')|d\gamma'$ is one for θ^*-conjugacy classes of $GL_n(F)$ which meet $\mathbf{T}'(F)$, where the discriminant D_{θ^*}, or more precisely $|D_{\theta^*}|^{1/2}$, is defined as in §4.5 of [19]. Using the map m of Lemma 7.2, it then transfers to a measure for θ-conjugacy classes of $GL_n(F)$ which intersect $\mathbf{T}'(F)g_\theta$. Similarly, if $d\gamma$ is a measure for $\mathbf{T}(F)$, then $|D(\gamma)|d\gamma$ is a measure for semisimple conjugacy classes of $SO_n(F)$, meeting $\mathbf{T}(F)$.

By Lemma 7.2 of this paper and Lemma 4.5.A of [19]

$$|D_{\theta^*}(\gamma')|d\gamma'/|D(\gamma)|d\gamma$$

is a continuous function whenever $\{\gamma\} = N_{\theta^*}(\{\gamma'\})$. Observe that $|D(\gamma^{-1})| = |D(\gamma)|$. This is our function $\kappa(\{\gamma\}, \{\gamma'\})$.

It is now clear, at least formally, that (6.7) equals to $R_{\mathscr{A}}(f,f')$ where $\Delta(\{\gamma\}, \{\gamma'\})$ is as in the beginning of this section.

We shall now address the question of convergence of (6.7), verifying condition (2.6) of Section 2. We only need to study the convergence of the integrals of the form

$$\int_{\mathbf{T}(F)} \Phi(\gamma,f)\Phi_\theta(\mathscr{A}(\{\gamma\}),f')|D(\gamma)|d\gamma,$$

where $\mathbf{T}$ is an elliptic torus in SO_n.

Since $|D(\gamma)|^{1/2}\Phi(\gamma,f)$ is continuous, it remains bounded on $\mathbf{T}(F)$ and consequently we need to study the convergence of

$$\int_{\mathbf{T}(F)} |\Phi_\theta(\mathscr{A}(\{\gamma\}),f')||D(\gamma)|^{1/2}d\gamma$$

or of

$$\int_{\substack{\gamma \in \mathbf{T}(F) \\ N^{\theta^*}(\{\gamma'\}) = \{\gamma^{-1}\}}} |\Phi_\theta(\varepsilon\gamma' g_\theta,f')|\kappa(\{\gamma\},\{\gamma'\})|D(\gamma)|^{1/2}d\gamma$$

for each representative $\varepsilon \in (F^*)^2\backslash F^*$. Appealing to Lemma 7.1, we may drop ε.

By the definition of $\kappa(\{\gamma\}, \{\gamma'\})$ we must therefore consider

$$\int_{N^{\theta^*}(\{\gamma'\}) = \{\gamma\}} |\Phi_{\theta^*}(\gamma', R_{g_\theta} f')| \, |D_{\theta^*}(\gamma')| D(\gamma)|^{-1/2} d\gamma',$$

where the integral is over those θ^*-conjugacy classes of $GL_n(F)$ in $\mathcal{N} g_\theta^{-1}$ which meet $\mathbf{T}'(F)$. The reader must observe that the function $\kappa(\{\gamma\}, \{\gamma'\})$ has to be introduced precisely because the integration is originally over $\mathcal{N} \subset \mathscr{C}'$ rather then $\mathscr{C}$.

By Lemma 5.16 and Proposition 5.9, the integration is over a compact set. The convergence now follows from the continuity of the functions

$$|D_{\theta^*}(\gamma')|^{1/2} \Phi_{\theta^*}(\gamma', R_{g_\theta} f') \tag{7.1}$$

and

$$|D_{\theta^*}(\gamma')|^{1/2} / |D(\gamma)|^{1/2} \tag{7.2}$$

where we use Lemma 4.5.A of [19] for the second one.

A similar set of arguments applies if $n > 2m$ as soon as we identify SO_{2m} as the subgroup

$$I_{(n/2)-m} \times SO_{2m}$$

of the standard Levi subgroup $GL_{(n/2)-m} \times SO_{2m}$ of SO_n. The function κ is now defined accordingly. We have therefore proved.

Proposition 7.3 *Suppose $n \geqq 2m$. Then the intertwining operator $A(s, \tau \otimes \tau', w_0)$ has a pole at $s = 0$ if and only if $R_{\mathscr{A}}(f, f') \neq 0$ for some f and f', where f is a matrix coefficient of τ and $f' \in C_c^\infty(GL_n(F))$ defines one for τ'.*

Let us now observe that

Lemma 7.4 *$R_{\mathscr{A}} \equiv 0$ unless $\omega' = 1$.*

Proof. We need to show that if $\omega' \neq 1$, then $R_{\mathscr{A}} \equiv 0$. Choose a discrete series representation $\tau^\vee$ of $GO_{2m}(F)$ and realize $I(\tau \otimes \tau') \subset I(\tau^\vee \otimes \tau')$, the representation of $GO_{2m+2n}(F)$ induced from $GO_{2m}(F) \times GL_n(F)$. It then follows from general facts about intertwining operators that $R_{\mathscr{A}}$ also determines residues for the standard intertwining operators of $GO_{2m+2n}(F)$. But since n is even, $\tau^\vee \otimes \tau'$ is ramified if and only if $\tau' \cong \tilde{\tau}'$, and $\omega' = 1$. Thus if $\omega' \neq 1$, the intertwining operator for $GO_{2m+2n}(F)$ will not have a pole at $s = 0$ and consequently $R_{\mathscr{A}} = 0$.

Corollary 7.5. *If $R_{\mathscr{A}} \neq 0$, then $\omega' = 1$.*

We also have

Proposition 7.6 *Suppose $\tau' \cong \tilde{\tau}'$ and $\omega' = 1$. Then $I(\tau \otimes \tau')$ is irreducible if and only if $I(\tau^\vee \otimes \tau')$ is.*

Proof. Write $\tau^\vee | SO_{2m}(F) = \oplus_i m_i \tau_i$, $m_i \in \mathbf{Z}^+$. If $I(\tau \otimes \tau')$ is reducible, then so is each $I(\tau_i \otimes \tau')$ (cf. [43] for such arguments). Consequently,

$$\lim_{s \to 0} sA(s, \tau_i \otimes \tau', w_0) = 0$$

for all i. But by the definition of intertwining operators the same is true for $A(s, \tau^\vee \otimes \tau', w_0)$. Since $\tau^\vee \otimes \tau'$ is ramified, $I(s, \tau^\vee \otimes \tau')$ must also be reducible. The converse is Lemma 7.4.

Next, for each $G = \mathbf{G}(F)$-conjugacy class $\{\mathbf{G}_\gamma\}$ of centralizers of elements of G, $\gamma \in G$, we fix a left-invariant measure dh_γ. If $\gamma_1 = h^{-1}\gamma h$, $\gamma_1 \in G$, $h \in \mathbf{G} = \mathbf{G}(\bar{F})$, then $\mathbf{G}_{\gamma_1} = h^{-1}\mathbf{G}_\gamma h$. The measure dh_γ is defined by a left-invariant form of highest degree on $\mathbf{G}_\gamma$. We transport this form by means of $\mathrm{Int}(h)$ to get one on $\mathbf{G}_{\gamma_1}$. We then attach a Haar measure to $\mathbf{G}_{\gamma_1}(F)$ using this form as in Section (1.4) of [23]. The measure on G is then the one attached to $\{\mathbf{G}_e\}$. Similarly we fix dg on θ-twisted centralizers of elements in $GL_n(F)$.

Fix a singular elliptic element $\sigma \in G$. Let $\mathscr{U}_{\mathbf{G}^0_\sigma}$ be the set of equivalence classes of elements of $\mathscr{U}_{G^0_\sigma}$ (cf. Section 2) for which u is equivalent to u', u, $u' \in \mathscr{U}_{G^0_\sigma}$, if and only if $u' = h_0^{-1} u h_0$ for some $h_0 \in \mathbf{G}^0_\sigma$. When $\mathbf{G}_\sigma$ is connected this is the same as stable conjugation defined in [18].

For each $u \in \mathscr{U}_{\mathbf{G}^0_\sigma}$ set

$$R^u_{\{\mathbf{G}_\sigma\}}(f,f') = \sum_{\{\mathbf{G}_\sigma\}} \sum_{\hat{\sigma}_1 \in \mathbf{Z}_{\sigma_1}(F)'} \sum_{u_1} c_{\sigma_1 u_1} \boldsymbol{\Phi}(\{\hat{\sigma}_1 u_1\}, f) \boldsymbol{\Phi}_\theta(\mathscr{A}(\{\hat{\sigma}_1 u_1\}), f').$$

The outer sum runs over $\mathbf{G}(F)$-conjugacy classes of centralizers of elements of the form $\sigma_1 = h^{-1}\sigma h$, $h \in \mathbf{G}(\bar{F})$. The sum over u_1 is over unipotent conjugacy classes in $G^0_{\sigma_1}$, which are of the form $h^{-1}u'h$, where u' belongs to the class of $u \in \mathscr{U}_{\mathbf{G}^0_\sigma}$. Observe that given $u_1 \in \mathscr{U}_{G^0_{\sigma_1}}$, $\mathbf{G}_{\hat{\sigma}_1 u_1}$ is the same for any $\hat{\sigma}_1 \in \mathbf{Z}_{\sigma_1}(F)'$. Moreover, the measure for each $\mathbf{G}_{\sigma_1 u_1} = \mathbf{G}_{\hat{\sigma}_1 u_1}$ is uniquely determined by the one attached to $\mathbf{G}_{\sigma u}$. The coefficient $c_{\sigma_1 u_1}$ is the inverse of the index $[N(\mathbf{Z}_{\sigma_1}(F)') \cap \mathbf{G}_{u_1}(F) \colon \mathbf{G}_{\sigma_1 u_1}(F)]$ (cf. Section 2).

On the other hand for a regular elliptic σ we have the Weyl integration formula applied to $\boldsymbol{\Phi}_\theta(\mathscr{A}(\{\hat{\sigma}\}), f')$ which we denote by

$$R_{\mathbf{G}}(f,f') = \sum_{\{\mathbf{G}^0_\sigma\}} |W(\mathbf{G}^0_\sigma)|^{-1} \int_{\hat{\sigma} \in \mathbf{G}^0_\sigma(F)} \boldsymbol{\Phi}(\{\hat{\sigma}\}, f) \boldsymbol{\Phi}_\theta(\mathscr{A}(\{\hat{\sigma}\}), f') |D(\hat{\sigma})| d\hat{\sigma}.$$

The sum is over conjugacy classes of elliptic tori in $\mathbf{G}$.

With this notation, then $R_{\mathscr{A}}(f,f')$ can be written (cf. Sect. 2) as (f is cuspidal):

$$R_{\mathscr{A}}(f,f') = R_{\mathbf{G}}(f,f') + \sum_{\{\mathbf{G}_\sigma\}} \sum_{u \in \mathscr{U}_{\mathbf{G}^0_\sigma}} R^u_{\{\mathbf{G}_\sigma\}}(f,f').$$

Now observe that changing the measure $dh_{\sigma u}$ on each $G_{\sigma u}$, which is possible independent of one another, will change $R^u_{\{\mathbf{G}_\sigma\}}$ (as well as $R_{\mathbf{G}}$) by an arbitrary positive constant, independent of each other. On the other hand if $I(\tau \otimes \tau')$, with $\tilde{\tau}' \cong \tau'$, is reducible, then $R_{\mathscr{A}} = 0$ independent of any choices of measure. Thus:

Lemma 7.7 *$R_{\mathscr{A}} = 0$ if and only if $R_G = 0$ as well as $R^u_{\{G_\sigma\}} = 0$ for all singular elliptic σ and $u \in \mathscr{U}_{G^0_\sigma}$.*

We can now state our final result as:

Theorem 7.8 *Suppose $n \geqq 2m$.*

a) *The induced representation $I(\tau \otimes \tau')$ is reducible if and only if $\tau' \cong \tilde{\tau}'$ and $R_G(f,f') = 0$ as well as $R^u_{\{G_\sigma\}}(f,f') = 0$ for all singular elliptic σ and $u \in \mathscr{U}_{G^0_\sigma}$, and for every matrix coefficient f of τ and every $f' \in C^\infty_c(GL_n(F))$ defining one for τ'. If $\tau' \cong \tilde{\tau}'$ and $I(\tau \otimes \tau')$ is irreducible, then $\omega' = 1$.*
b) *Suppose τ is supercuspidal and generic (cf. [33]). Moreover assume $\tau' \cong \tilde{\tau}'$ and that $I(\tau \otimes \tau')$ is irreducible. Then the points of reducibility for $I(s, \tau \otimes \tau')$, $s \in \mathbb{R}$, are exactly at $s = \pm 1/2$ or ± 1.*

8. The case $n < 2m$

Now assume $n < 2m$. Embed SO_n as the subgroup $I_{m-(n/2)} \times SO_n$ of the standard Levi subgroup $GL_{m-(n/2)} \times SO_n$ of SO_{2m}.

Let $\tilde{N}_\theta: \tilde{\mathscr{N}} \to \tilde{\mathscr{C}}$ be the norm correspondence from θ-conjugacy classes in $GL_n(F)$ to conjugacy classes in $SO_n(F)$. If $N_\theta: \mathscr{N} \to \mathscr{C}$ denotes the norm correspondence into conjugacy classes in $SO_{2m}(F)$, then $\mathscr{N} \supset \tilde{\mathscr{N}}$. Moreover if $\mathscr{C}^\vee$ is the subset of $\mathscr{C}$ consisting of those conjugacy classes which meet $SO_n(F)$, then $N_\theta: \mathscr{N} \to \mathscr{C}^\vee$.

The residue is then $R_{\mathscr{A}^\vee}(f,f')$, where $\mathscr{A}^\vee = \mathscr{A}|\mathscr{C}^\vee$. Since $n < 2m$, no regular elliptic conjugacy class can belong to $\mathscr{C}^\vee$ and therefore $R_{\mathscr{A}^\vee}$ will only run over centralizers of singular elliptic elements and from such conjugacy classes it only picks up those which intersect $SO_n(F)$. We need to show that $\mathscr{A}^\vee$ is again one to finite. This follows from:

Lemma 8.1 *The norm correspondence N_θ has finite fibers.*

Proof. The method of Lemma 5.10 again applies. In fact, we may assume X is a row reduced echelon matrix of rank r. Then X sends F^n to $F^n X = F^r \subset F^{2m}$ and $F^n = F^r \oplus F^{n-r}$ with X sending F^r isomorphically to a subspace of F^{2m}. Moreover

$$\mathbf{v}\theta(Y^{-1})Y^{-1}X = -\mathbf{v}X(I - X'Y^{-1}X)$$

for $\mathbf{v}$ in F^r. On the other hand the left kernel of X is F^{n-r} on which $\theta(Y^{-1})Y^{-1}$ acts like $-I$ by (3.4). Therefore again $\{I - X'Y^{-1}X\}$ determines the semisimple part of $\theta(Y^{-1})Y^{-1}$ uniquely.

Applying Lemmas 7.4 and 7.7, we have:

Theorem 8.1 *Suppose $n < 2m$.*

a) *The induced representation $I(\tau \otimes \tau')$ is reducible if and only if $\tau' \cong \tilde{\tau}'$ and $R^u_{\{G_\sigma\}} = 0$ for every singular elliptic element $\sigma \in SO_{2m}(F)$ and every $u \in \mathscr{U}_{G^0_\sigma}$ for* *which the conjugacy class of σ in $SO_{2m}(\bar{F})$ intersects $SO_n(F)$. The domain of each $R^u_{\{G_\sigma\}}$ is the product of the matrix coefficients of τ and the space of functions in $C_c^\infty(GL_n(F))$ which determine matrix coefficients of τ'. Moreover, if $\tau' \cong \tilde{\tau}'$ and $I(\tau \otimes \tau')$ is irreducible, then $\omega' = 1$.*

b) *Suppose τ is supercuspidal and generic. Moreover assume $\tau' \cong \tilde{\tau}'$ and that $I(\tau \otimes \tau')$ is irreducible according to part a). Then the points of reducibility for $I(s, \tau \otimes \tau')$, $s \in \mathbb{R}$, are exactly at $s = \pm 1/2$ or ± 1.*

9. Relation with endoscopy

Suppose $n = 2m$. By [5, 6, 12, 13] it is clear that f and f' can be chosen so that

$$R_G(f, f') = \sum_{\{G^0_\sigma\}} |W(G^0_\sigma)|^{-1} \int_{\hat{\sigma} \in G^0_\sigma(F)} \chi_\tau(\hat{\sigma}) \chi^\theta_{\tau'}(\mathscr{A}(\{\hat{\sigma}\})) |D(\hat{\sigma})| d\hat{\sigma},$$

where χ_τ is the character of τ and

$$\chi^\theta_{\tau'}(\mathscr{A}(\{\hat{\sigma}\})) = \sum_{\varepsilon \in (F^*)^2 \backslash F^*} \omega'(\varepsilon) \chi^\theta_{\tau'}(\varepsilon N_\theta^{-1}(\hat{\sigma})). \tag{9.1}$$

Here $\chi^\theta_{\tau'}$ is the character of a representation T' of $GL_n(F) \rtimes \{1, \theta\}$, the F-points of the disconnected group $GL_n \rtimes \{1, \theta\}$, weighted by the function κ. The representation T' is an extension of τ' to $GL_n(F) \rtimes \{1, \theta\}$ by fixing an isomorphism between τ' and $\tilde{\tau}'$. In view of Corollary 7.6 we may assume that $\omega' \equiv 1$ in (9.1).

Thus R_G defines a pairing between characters χ_τ and $\chi^\theta_{\tau'}$, and its nonvanishing points to τ' coming from τ. We expect this to be equivalent to the θ-twisted endoscopic transfer [1, 19, 20] between SO_n and GL_n.

We therefore define

Definition 9.1 *A self-dual irreducible supercuspidal representation τ' of $GL_n(F)$ is called the θ-twisted endoscopic transfer of a discrete series representation τ of $SO_n(F)$ if $R_G(f, f') \neq 0$ for some matrix coefficient f of τ and $f' \in C_c^\infty(GL_n(F))$ defining one for τ'.*

Remark 9.2. Due to the theory of endoscopy [21, 23, 24] one expects $R_G \neq 0$ not only for τ but for every other representation in its L-packet.

Proposition 9.3 *Assume $n = 2m$. Let δ and δ_1 be elements of $SO_n(F)$ satisfying $\delta_1 = h^{-1}\delta h$ with $h \in SO_n(\bar{F})$. Let Y and Y_1 be elements of $GL_n(F)$ such that $\delta = I - X'Y^{-1}X$ and $\delta_1 = I - X'_1 Y_2^{-1} X_1$, where X, Y, X_1, and Y_1 are determied by means of Corollary 5.7. Then Y and Y_1 are in the same $GL_n(\bar{F})$-θ-conjugacy class.*

Proof. Observe that $h^{-1}X'Y^{-1}Xh \in M_n(F)$. Clearly $h^{-1}Xh$ is a projection and moreover

$$\begin{aligned} \ker(h^{-1}X'Y^{-1}Xh) &= \ker(h^{-1}Y^{-1}Xh) \\ &= \ker(Y^{-1}Xh) \\ &= h^{-1}(\ker(X)). \end{aligned}$$

Thus $h(\ker(X_1)) = \ker(X)$ and consequently $X_1 = h^{-1}Xh$. In particular $h^{-1}Xh \in M_n(F)$ and

$$h: \ker(X_1) \otimes_F \bar{F} \to \ker(X) \otimes_F \bar{F}$$

is defined over F. Kernels are calculated in F^n.

By construction $h^{-1}Y^{-1}h = Y_1^{-1}$ on the right image of projection $X_1 = h^{-1}Xh$. Moreover since both $h^{-1}Y^{-1}h$ and Y_1^{-1} satisfy the condition of Corollary 5.7, they both send $\ker(X_1)$ and $Im(X_1)$ onto $\ker(-X'_1)$ and $Im(-X'_1)$, respectively.

Choose $g \in GL_n(\bar{F})$, satisfying $gX_1 = X_1 g = X_1$, such that

$$\theta(g)^{-1}h^{-1}Y^{-1}hg|\ker(X_1) \otimes_F \bar{F}$$

is defined over F. This is possible since $h^{-1}Yh|ker(X_1) \otimes_F \bar{F}$ is $\tilde{\theta}$-skew-symmetric whose θ-twisted conjugacy class always has a rational representative (cf. Lemma 5.8 and Proposition 5.9). This completes the proposition since $\theta(g)^{-1}h^{-1}Y^{-1}hg$ and $h^{-1}Y^{-1}h$ have same restrictions to the image of X_1.

Corollary 9.4 *Suppose δ and δ_1 are $SO_n(F)$-conjugate. Then Y and Y_1 are $GL_n(F) - \theta$-conjugate.*

Corollary 9.5 *Let δ be a singular elliptic element in $SO_n(F)$. Determine $X \in M_n(F)$ and $Y \in GL_n(F)$ attached to δ as in Proposition 5.9, i.e. $\delta = I - X'Y^{-1}X$. Assume $\delta = h^{-1}\delta_0 h$ with $\delta_0 = diag(-I_{r_0/2}, I_{n-r_0}, -I_{r_0/2})$, r_0 even, and $h \in SO_n(\bar{F})$. Then the θ-twisted centralizer of Y^{-1}* in *$GL_n(\bar{F})$ is conjugate to $Sp_{n-r_0} \times O_{r_0}$* under *$h$.*

Proof. One needs to observe that for δ_0, the corresponding X determined by Proposition 5.9 is in fact X_0 of Lemma 5.1.

In view of Corollary 9.5 and discussions in [1, 32], we now define

Definition 9.7 *Assume $0 \leqq r_0 < n$ is an even integer. A self-dual irreducible supercuspidal representation τ' of $GL_n(F)$ is said to be θ-twisted endoscopic transfer of a finite set $\{\tau^\vee\}$ of discrete series representations of $SO_{n-r_0+1}(F) \times SO^*_{r_0}(F)$, to be determined by τ, if $R^u_{\{G_\sigma\}}(f, f') \neq 0$ for some σ for which $\sigma_0 = diag(-I_{r_0/2}, I_{n-r_0}, -I_{r_0/2})$, $\sigma = h^{-1}\sigma_0 h, h \in SO_n(\bar{F})$, and some $u \in \mathscr{U}_{G^0_\sigma}$. Here $SO^*_{r_0}$ denotes a possible quasisplit form of SO_{r_0}.*

Remark 9.8. If $R^u_{\{G_\sigma\}}(f, f') \neq 0$, then one expects the L-packet of τ to be the endoscopic transfer [21, 23, 24] of an L-packet $\{\tau_{\mathbf{H}}\}$ of discrete series representations of the F-points of an endoscopic group $\mathbf{H}$ of SO_n. The group $\mathbf{H}$ must then be endoscopic to $SO_{n-r_0+1} \times SO^*_{r_0}$ and the finite set $\{\tau^\vee\}$ must transfer $\{\tau_{\mathbf{H}}\}$.

Remark 9.9. Since τ' is supercuspidal, one expects that either $r_0 = 0$ or $r_0 = n$ (cf. Section 3 of [32]). If $r_0 = 0$, then we must consider $\Phi(\pm e, f) = f(\pm e)$ which can easily be made non-zero. If τ' does not come from $SO_{n+1}(F)$ (as defined in

[32]), then by Propositions 5.1 and 10.1 of [32], one expects that $\Phi_\theta(\mathscr{A}(\{\pm e\}), f') = 0$. Therefore $R_G(f, f')$ is expected to be the only non-zero contribution to the residue. This confirms the equivalence of Definition 9.1 of the present paper and Definition 7.5 of [32].

We can now formulate part a) of our Theorem 7.9 as follows.

Theorem 9.10 *Assume $n = 2m$. Let τ' be an irreducible self-dual supercuspidal representation of $GL_n(F)$. If τ is a discrete series representation of $SO_n(F)$, then $I(\tau \otimes \tau')$ is irreducible if and only if either τ' come from τ through θ-twisted endoscopic transfer from $SO_n(F)$ to $GL_n(F)$, or it comes from a finite set $\{\tau^\vee\}$ of discrete series representations of $SO_{n-r_0+1}(F) \times SO^*_{r_0}(F)$ again through θ-twisted endsocopic transfer. In the second case $\{\tau^\vee\}$, r_0, and $SO^*_{r_0}$ are all determined by τ.*

It is not at all hard to formulate similar statements when $n \neq 2m$.

Remark 9.11. One can now try to determine the local Rankin–Selberg L-function $L(s, \tau \times \tau')$ as in [32] using the theory of endoscopy and the definition of $L(s, \tau', \Lambda^2 \rho_n)$ given in [32].

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Note added in proof

The proof of Lemma 7.4 given here is not valid if the composition of ω' with the multiplier character of $GO_{2m}(F)$ fixes the class of $\tau^\vee$. Consequently, at present, the reader must disregard the sentence "If $\tau' \cong \tilde{\tau}'$ and $I(\tau \otimes \tau')$ is irreducible, then $\omega' = 1$" at the end of part a) of Theorems 7.8 and 8.1, whenever this is the case. It must be pointed out that Lemma 7.4 is completely independent of the rest of the paper and can be completely removed (together with its consequences mentioned above) with no serious effect on the significance of the main results of the paper.