

Twists of a General Class of L -Functions by Highly Ramified Characters

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Abstract. It is shown that given a local L -function defined by Langlands-Shahidi method, there exists a highly ramified character of the group which when is twisted with the original representation leads to a trivial L -function.

The purpose of this short note is to prove a general lemma on twists by highly ramified characters of all the L -functions which are obtained from the Langlands-Shahidi method [2], [4], [5], [6]. The lemma generalizes Proposition 5.1 of [1] to many other L -functions and seems to be useful in applications [3]. The idea of the proof is simple and relies on basic general facts in representation theory. Its bulk, if any, is due to its remarkable generality.

Let F be a p -adic local field of characteristic zero. Denote by O its ring of integers and let P be the unique maximal ideal of O . Let q be the cardinality of the field O/P and normalize an absolute value $|\cdot|$ on F such that the absolute value of a generator of P equals q^{-1} .

Let G be a quasisplit connected reductive algebraic group over F . Fix a Borel subgroup $B = TU$ of G with unipotent radical U and a maximal torus T . Let A_0 be the maximal split torus of T . Denote by $W(A_0)$ the Weyl group of A_0 . Let Δ be the set of simple roots of A_0 in U . For each subgroup H of G , we use H to denote the group of F -points of H .

Let P be a maximal parabolic subgroup of G such that $N \subset U$. Let χ be the generic character of U defined via a non-trivial character ψ_F of F (cf. [4]). Fix an irreducible admissible χ -generic representation (cf. [4]) π of $M = M(F)$.

Next let r be the adjoint action of ${}^L M$, the L -group of M , on the Lie algebra ${}^L \mathfrak{n}$ of the L -group ${}^L N$ of N . Decompose $r = \bigoplus_{i=1}^m r_i$ according to the order of eigenvalues of ${}^L A$, the L -group of the split center A of M , in ${}^L \mathfrak{n}$ as in [5]. Let $s \in \mathbb{C}$. For each i , $1 \leq i \leq m$, let $L(s, \pi, r_i)$ denote the L -function attached to π and r_i as in [5]. Finally, let $X^*(M)_F$ denote the subgroup of F -rational characters of M . The purpose of this note is to prove the following useful lemma:

Main Lemma 1 *There exists a rational character $\xi \in X^*(M)_F$ of M such that: Given any irreducible admissible χ -generic representation π of $M = M(F)$, there exists a character η of F^* so that $L(s, \pi \otimes (\eta \cdot \xi), r_i) \equiv 1$, $1 \leq i \leq m$. Moreover η can be replaced by any character of F^* whose conductor is larger than that of η .*

The lemma seems to have useful applications in the theory of automorphic forms (e.g. [3]) as soon as the general method developed in [2], [4], [5], [6] is used. In particular, in the special case of $G = GL_{n+m}$, $M = GL_n \times GL_m$, the lemma is equivalent to the Proposition 5.1 of [1] which uses a completely different method.

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The idea of the proof of Lemma 1 is quite simple, although it becomes cumbersome due to its generality.

Let $\theta \subset \Delta$ be a subset of simple roots so that

$$\pi \hookrightarrow \text{Ind}_{M_\theta N_\theta}^M \sigma \otimes \mathbf{1},$$

where $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$, $\mathbf{N}_\theta \subset \mathbf{U} \cap \mathbf{M}$, and σ is an irreducible supercuspidal χ -generic representation of M_θ .

There exists the natural restriction map from $X^*(\mathbf{M})_F$ into $X^*(\mathbf{M}_\theta)_F$, sending $\xi \mapsto \xi_\theta \in X^*(\mathbf{M}_\theta)_F$. Then

$$(\text{Ind}_{M_\theta N_\theta}^M \sigma) \otimes (\eta \cdot \xi) \cong \text{Ind}_{M_\theta N_\theta}^M (\sigma \otimes (\eta \cdot \xi_\theta))$$

and therefore

$$\pi \otimes (\eta \cdot \xi) \hookrightarrow \text{Ind}_{M_\theta N_\theta}^M (\sigma \otimes (\eta \cdot \xi_\theta)).$$

Let \tilde{w}_0 be the longest element in $W(\mathbf{A}_0)$ modulo that of the Weyl group of \mathbf{A}_0 in \mathbf{M} . Then \tilde{w}_0 sends the unique simple root in \mathbf{N} to a negative root, while $\tilde{w}_0(\theta) \subset \Delta$.

Decompose \tilde{w}_0 as $\tilde{w}_0 = \tilde{w}_{n-1} \cdots \tilde{w}_1$ with respect to θ as in Lemma 2.1.1 of [6]. Then for each j there exists a unique root $\alpha_j \in \Delta$ such that $\tilde{w}_j(\alpha_j) < 0$. For each j , $2 \leq j \leq n-1$, let $\tilde{w}_j = \tilde{w}_{j-1} \cdots \tilde{w}_1$ with $\tilde{w}_1 = 1$. Let $\theta_{j+1} = \tilde{w}_j(\theta_j)$, $\theta_1 = \theta$ and for each j , let $\Omega_j = \theta_j \cup \{\alpha_j\}$. Then \mathbf{M}_{Ω_j} contains \mathbf{M}_{θ_j} as a maximal Levi subgroup. Moreover $\tilde{w}_j: \mathbf{M}_\theta \cong \mathbf{M}_{\theta_j}$ and $\tilde{w}_j: \mathbf{A}_\theta \cong \mathbf{A}_{\theta_j}$, their split components, and $\tilde{w}_j(\sigma) = \sigma_j$ becomes an irreducible supercuspidal representation of M_{θ_j} . Finally, if

$$\gamma_i(s, \pi, \psi_F, \tilde{w}_0) = \varepsilon(s, \pi, r_i, \psi_F) L(1-s, \pi, \tilde{r}_i) / L(s, \pi, r_i),$$

then by Theorem 3.5, part 3, of [5],

$$\gamma_i(s, \pi, \psi_F, \tilde{w}_0) = \prod_{j \in S_i} \gamma_{i(j)}(s, \tilde{w}_j(\sigma), \psi_F, \tilde{w}_j),$$

where $i(j)$ and S_i are as in [5] and the factors on the right are defined the same way for the triple $(M_{\Omega_j}, M_{\theta_j}, \sigma_j)$. Here $\varepsilon(s, \pi, r_i, \psi_F)$ is the corresponding root number defined in [5].

A similar identity holds if π and σ are replaced by $\pi \otimes (\eta \cdot \xi)$ and $\sigma \otimes (\eta \cdot \xi_\theta)$, respectively.

To prove the main lemma it would be enough to show that η and ξ can be chosen in such a way that each $\gamma_{i(j)}(s, \tilde{w}_j(\sigma \otimes (\eta \cdot \xi_\theta)), \psi_F, \tilde{w}_j)$ becomes a monomial in q^{-s} for each j . For, then the L -functions $L(s, \pi \otimes (\eta \cdot \xi_\theta), r_i) \equiv 1$ for tempered π and consequently the same holds for arbitrary π by further induction and analytic continuation.

But $\gamma_{i(j)}(s, \tilde{w}_j(\sigma \otimes (\eta \cdot \xi_\theta)), \psi_F, \tilde{w}_j)$ becomes a monomial in q^{-s} as soon as the representation of M_{Ω_j} induced from any unramified twist of the representation $\tilde{w}_j(\sigma \otimes (\eta \cdot \xi_\theta))$ of M_{θ_j} is irreducible. More precisely, the irreducibility implies that the local coefficient $C_{\tilde{\chi}}(s\tilde{\alpha}_j, \tilde{w}_j(\tilde{\sigma} \otimes (\eta^{-1} \cdot \xi_\theta)), \tilde{w}_j)$ (cf. [5]) is a monomial in q^{-s} which then implies the same fact about $\gamma_{i(j)}(s, \tilde{w}_j(\sigma \cdot \xi_\theta), \psi_F, \tilde{w}_j)$. Here one only needs to use Proposition 7.3 of [5] which implies that no cancellations take place among factors of γ 's appearing in the local coefficient (Theorem 3.5 of [5], equation (3.11)).

For representations of M_{Ω_j} , induced from unramified twists of $\bar{w}_j(\sigma \otimes (\eta \cdot \xi_{\theta})) = \sigma_j \otimes (\eta \cdot \xi_{\theta_j})$, to become irreducible, it is enough to have

$$(1) \quad \bar{w}_j(\sigma'_j \otimes (\eta \cdot \xi_{\theta_j})) \not\cong \sigma'_j \otimes (\eta \cdot \xi_{\theta_j}),$$

where σ'_j denotes an arbitrary unramified twist of $\sigma_j = \bar{w}_j(\sigma)$ and $\xi_{\theta_j} = \bar{w}_j(\xi_{\theta})$. Observe that we may assume $\bar{w}_j(\mathbf{A}_{\theta_j}) = \mathbf{A}_{\theta_j}$.

Now, by taking central characters in (1), it is enough to show that there exists a choice of ξ and η such that

$$\bar{w}_j(\omega'_j(a)\eta(\xi_{\theta_j}(\bar{w}_j(a)))) = \omega'_j(a)\eta(\xi_{\theta_j}(a))$$

or

$$(2) \quad \eta \cdot \xi_{\theta_j}(\bar{w}_j(a)a^{-1}) = \omega'_j(a\bar{w}_j(a^{-1}))$$

does not hold for all $a \in A_{\theta_j}$, where ω'_j is the central character of σ'_j . Moreover the same is true if η is replaced with another character of higher conductor than η .

Consider the exact sequence

$$0 \rightarrow \mathbf{A}'_{\theta_j} \rightarrow \mathbf{A}_{\theta_j} \rightarrow \bar{w}_j(\mathbf{A}_{\theta_j})\mathbf{A}_{\theta_j}^{-1} \rightarrow 0$$

in which the one before last arrow is defined by

$$a \mapsto \bar{w}_j(a)a^{-1}.$$

Its kernel \mathbf{A}'_{θ_j} consists of all a with $\bar{w}_j(a) = a$. It contains \mathbf{A}_{Ω_j} . Set

$$\mathbf{A}_{\theta_j}^1 = \bar{w}_j(\mathbf{A}_{\theta_j})\mathbf{A}_{\theta_j}^{-1} = \{a \in \mathbf{A}_{\theta_j} \mid \bar{w}_j(a) = a^{-1}\},$$

where the last equality is easy to check. Observe that

$$\mathbf{A}_{\theta_j}^1 \cong \mathbf{A}_{\theta_j} / \mathbf{A}'_{\theta_j}$$

is connected.

We first specify ξ . Let \mathfrak{n} be the Lie algebra of \mathbf{N} and set

$$\xi(m) = \det(\text{Ad}(m) \mid \mathfrak{n}),$$

$m \in \mathbf{M}$. Then $\xi \in X^*(\mathbf{M})$ and defines $\xi_{\theta} \in X^*(\mathbf{M}_{\theta})$ by restriction. Define $\xi_j = \xi_{\theta_j} \in X^*(\mathbf{M}_{\theta_j})$ as before by

$$\xi_j(\bar{w}_j(m)) = \xi_{\theta}(m),$$

$m \in \mathbf{M}_{\theta}$. We need:

Lemma 2 $\xi_j \mid \mathbf{A}_{\theta_j}^1 \neq 1$.

Proof Suppose $\xi_j(\tilde{w}_j(a_j)) = \xi_j(a_j), \forall a_j \in \mathbf{A}_{\theta_j}$ which makes sense since we have assumed $\tilde{w}_j(\mathbf{A}_{\theta_j}) = \mathbf{A}_{\theta_j}$. Then

$$\begin{aligned} \xi_j(m_j) &= \xi_j(\tilde{w}_j(m)) \\ &= \det(\text{Ad}(m) | \mathfrak{n}) \\ &= \det(\text{Ad}(m_j) | \mathfrak{n}_j), \end{aligned}$$

where $\mathfrak{n}_j = \tilde{w}_j(\mathfrak{n}), \forall m_j = \tilde{w}_j(m) \in \mathbf{M}_{\theta_j}$. We therefore have

$$\left\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, \tilde{w}_j(a_j) \right\rangle = \left\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, a_j \right\rangle \quad (\forall a_j \in \mathbf{A}_{\theta_j})$$

or

$$\left\langle \sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha), a_j \right\rangle = \left\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, a_j \right\rangle.$$

Thus

$$\sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha) = \sum_{X_\alpha \in \mathfrak{n}_j} \alpha.$$

Write:

$$\alpha = \alpha_j + \sum_{\beta \in \theta_j} m_{\beta}^{\alpha} \beta,$$

with $m_{\beta}^{\alpha} \in \mathbb{Z}$. Then

$$\sum_{X_\alpha \in \mathfrak{n}_j} \alpha = n_j \alpha_j + \sum_{\alpha} \sum_{\beta \in \theta_j} m_{\beta}^{\alpha} \beta,$$

where n_j is a positive integer. On the other hand

$$\sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha) = n_j \tilde{w}_j(\alpha_j) + \sum_{\alpha} \sum_{\beta \in \theta_j} m_{\beta}^{\alpha} \tilde{w}_j(\beta).$$

But $\tilde{w}_j(\alpha_j) = -\alpha_j$ and $\tilde{w}_j(\beta) \neq \pm \alpha_j$. Equality

$$n_j \alpha_j + \sum_{\alpha} \sum_{\beta} m_{\beta}^{\alpha} \beta = -n_j \alpha_j + \sum_{\alpha} \sum_{\beta} m_{\beta}^{\alpha} \tilde{w}_j(\beta)$$

is impossible since α_j and $\beta \in \Delta$ and $n_j > 0$. This proves the lemma.

Corollary $\xi_j: \mathbf{A}_{\theta_j}^1 \rightarrow \mathbb{G}_m$ is onto.

Proof By Lemma 2 this is a non-constant morphism of a connected variety (of dimension 1) into \mathbb{G}_m .

Proof of Main Lemma 1 By Corollary $\xi_j(\tilde{w}_j(A_{\theta_j})A_{\theta_j}^{-1})$ is open in F^* . Choose $\ell_j \in \mathbb{Z}^+$ such that $1 + P^{\ell_j} \subset \xi_j(\tilde{w}_j(A_{\theta_j})A_{\theta_j}^{-1})$. Take η with conductor larger than the larger of ℓ_j and

the conductor of ω_j , the central character of σ_j , which is the same for all σ'_j , for all j . This completes the proof.

Corollary of Lemma 1 *There exist a positive integer N such that for every character η of F^* with conductor larger than N , the function*

$$\gamma(s, \pi_\eta, r_i, \psi_F) = \varepsilon(s, \pi_\eta, r_i, \psi_F) L(1-s, \pi_\eta, \tilde{r}_i) / L(s, \pi_\eta, r_i),$$

$\pi_\eta = \pi \otimes (\eta \cdot \xi)$, is a monomial in q^{-s} .

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