

Local Coefficients as Mellin Transforms of Bessel Functions: Towards a General Stability

Freydoon Shahidi

1 Introduction

Recent breakthrough [7] in transferring generic cuspidal representations of $SO(2n + 1)$ to automorphic forms on $GL(2n)$ is attained by applying recent converse theorems of Cogdell and Piatetski-Shapiro [8, 10, 11] to analytic properties of L-functions obtained in [9, 12, 16, 23, 25, 26, 27, 33]. Strong transfer was later established in [13], as well as in [17]. The main obstacle in extending these results to other classical groups is the lack of stability of root numbers, which so far has only been established in the case of $SO(2n + 1)$ (cf. [9]). In fact, to apply converse theorems using functional equations, we need to show that root numbers, defined in [27] for the cases in [26] for which the Levi subgroup $M = GL_1 \times M'$ in which M' is a classical group of the same type as G , when twisted by highly ramified characters, depend only on the central character of the inducing representation (main theorem of [9, page 437] in the case of $SO(2n + 1)$).

What seems to be the most promising approach to establishing stability is that of [9] where the authors express their γ -function as a Mellin transform of an *incomplete (partial)* Bessel function of the representation on M' , from which, after a careful analysis of the asymptotics of corresponding Bessel functions, stability follows. (We refer to [15] for a proof of stability in the case of $GL(n)$.) On the other hand, to define γ -functions from our method [27], we have to define a family of local coefficients [23, 27], which then define γ -functions inductively [27, Theorem 3.5].

In view of triviality of L-functions under highly ramified twists [30] and the above discussion, the problem of stability of root numbers is that of local coefficients. In this paper, we address the case of self-associate maximal parabolic subgroups in general. These constitute the bulk of the cases in [21, 26] and particularly all the cases which are of interest to us in functoriality. Then what we prove is that the reciprocal of every local coefficient attached to a self-associate case is equal to the product of a 1-dimensional (abelian) γ -function and a Mellin transform of an incomplete Bessel function of the inducing representation, at least when $\omega_\pi^{-1}(w_0\omega_\pi)$ is ramified (Theorem 6.2). Since abelian γ -functions are stable, the problem reduces to that of the Mellin transform. This puts the general case in the same footing as that of $SO(2n + 1)$ [9]. The next step is to show, as in [9], that at least in many cases of interest every Bessel function can be written as a sum of two functions, one depending on the central character, while the other is a *smooth* function depending on representation. This should follow from asymptotic behavior of Bessel functions [2, 3, 4, 9].

Instead of stating our general result, Theorem 6.2, which is a vast generalization of a result of Soudry [32, Lemma 4.5], we explain several special cases of it which are quite important, but easier to explain. Let \mathbf{G} be one of the three classical groups SO_{2n+3} , SO_{2n+2} , or Sp_{2n+2} . We are interested in the parabolic subgroups \mathbf{P} whose Levi subgroups are $GL_1 \times SO_{2n+1}$, $GL_1 \times SO_{2n}$, and $GL_1 \times Sp_{2n}$, respectively. These are the cases which give the γ -factors attached to standard L-functions of SO_{2n+1} , SO_{2n} , or Sp_{2n} , twisted by a character, through the theory of local coefficients [23, 27], respectively. Let σ denote a χ -generic irreducible admissible representation of either $SO_{2n+1}(\mathbb{F})$, $SO_{2n}(\mathbb{F})$, or $Sp_{2n}(\mathbb{F})$. Let $\eta \in \widehat{\mathbb{F}}^*$ and for a fixed $s \in \mathbb{C}$, let $C(s, \eta \otimes \sigma)$ be the corresponding local coefficient [23, 27]. If $\psi_{\mathbb{F}}$ is the additive character defining χ (see Section 2), let $\gamma(2s, \eta^2, \psi_{\mathbb{F}})$ be the (abelian) γ -function attached to η^2 (see (6.3)). Next, define

$$w = \begin{pmatrix} & & 1 \\ & -I_{2n-1} & \\ 1 & & \end{pmatrix}, \tag{1.1}$$

if $\mathbf{G} = SO_{2n+3}$;

$$w = \begin{pmatrix} & & 1 \\ & -K_{2n-2} & \\ 1 & & \end{pmatrix}, \tag{1.2}$$

where

$$K_{2n-2} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in GL_{2n-2} \tag{1.3}$$

if $G = SO_{2n+2}$; while

$$w = \begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix} \tag{1.4}$$

for $G = Sp_{2n+2}$.

Finally, fix a vector \tilde{v} in the space of σ for which $W_{\tilde{v}}(e) = 1$, where $W_{\tilde{v}}$ is the corresponding Whittaker function in a fixed Whittaker model of σ . Let \bar{N}_0 be an open compact subgroup of \bar{N} , the opposite of N , the F -points of unipotent radical of P . Let $j_{\tilde{v}, \bar{N}_0}(m) = j_{\tilde{v}, \bar{N}_0}(m, \omega^{-d-f})$ be the incomplete Bessel function attached to \tilde{v} and \bar{N}_0 by (6.21) and (6.24), where d and f are conductors of ψ_F and η^2 , respectively. It is very easy to check that our incomplete Bessel functions are precisely those of [9]. Our Propositions 7.2 and 7.3 can be simply stated as follows.

Theorem 1.1. Assume that η^2 is ramified. Let $\delta = 1/2, 1, 0$ according as $G = SO_{2n+3}, SO_{2n+2}$ or Sp_{2n+2} , respectively. Then

$$C(s, \eta \otimes \sigma)^{-1} = \eta(-1) \gamma(2s, \eta^2, \psi_F)^{-1} \times \int_{F^*} j_{\tilde{v}, \bar{N}_0} \left(\begin{pmatrix} h & & \\ & I_\ell & \\ & & h^{-1} \end{pmatrix} w \right) \eta(h) |h|^{s-n+\delta} d^*h, \tag{1.5}$$

where $\ell = 2n - 1, 2n - 2$, or $2n - 2$, respectively. Here \bar{N}_0 can be replaced by any larger open compact subgroup of \bar{N} . We replace $j_{\tilde{v}, \bar{N}_0}$ with $j'_{\tilde{v}, \bar{N}_0}$ defined by (7.24) and (7.25) when $G = Sp_{2n+2}$. □

Corollary 1.2. Assume that η^2 is ramified.

(a) Let $\gamma(s, \sigma \otimes \eta, \psi_F)$ be the γ -function attached to $\eta \otimes \sigma$ as in [27]. Then

$$\gamma(s, \sigma \otimes \eta, \psi_F)^{-1} = \eta(-1) \int_{F^*} j_{\tilde{v}, \overline{N}_0} \left(\begin{pmatrix} \mathfrak{h} & & \\ & I_{2n-1} & \\ & & \mathfrak{h}^{-1} \end{pmatrix} w \right) \eta(\mathfrak{h}) |\mathfrak{h}|^{s-n+1/2} d^* \mathfrak{h} \tag{1.6}$$

if σ is a representation of $SO_{2n+1}(F)$.

(b) Suppose that σ is a representation of either $SO_{2n}(F)$ or $Sp_{2n}(F)$. Then

$$\begin{aligned} &\gamma(s, \sigma \otimes \eta, \psi_F)^{-1} \\ &= \eta(-1) \gamma(2s, \eta^2, \psi_F)^{-1} \int_{F^*} j_{\tilde{v}, \overline{N}_0} \left(\begin{pmatrix} \mathfrak{h} & & \\ & I_{2n-2} & \\ & & \mathfrak{h}^{-1} \end{pmatrix} w \right) \eta(\mathfrak{h}) |\mathfrak{h}|^{s-n+\delta} d^* \mathfrak{h}, \end{aligned} \tag{1.7}$$

where w and δ are as before. We again use $j_{\tilde{v}, \overline{N}_0}'$ for $\mathbf{G} = Sp_{2n+2}$. □

We should point out that our general results are obtained under certain natural assumptions (Assumptions 4.1 and 5.1), whose validity are naturally available in these important cases. We expect Assumption 4.1 to be true in general (as well as Assumption 3.1, an assumption which is not necessary for our Theorem 6.2 and its consequences). Assumption 5.1 is inessential and can be removed by enlarging the defining groups (Proposition 5.4).

Our main theorem (Theorem 6.2) is a generalization of a result of Soudry (see [32, Lemma 4.5] for $GL_2 \times GL_2$) as well as Cogdell and Piatetski-Shapiro for $SO(2n + 1)$ [9], to a very general setting, namely, for local coefficients attached to every self-associate pair (\mathbf{G}, \mathbf{M}) in the generality of the class of all quasisplit groups \mathbf{G} (cf. [23, 26, 27]). Our approach is completely different from either of them.

The final project of establishing functoriality from generic cusp forms of split classical groups to $GL(n)$, which has now been taken up by the authors of [7], is readily in hand, since stability now seems to be an immediate consequence of our Propositions 7.2 and 7.3 along the lines of [9].

There are other cases of functoriality which can be established as soon as stability is proved using analogues of our Propositions 7.2 and 7.3, which we hope to deduce as special cases of Theorem 6.2, (6.39). Notable among them are transfers from $GSpin_{2n}(\mathbb{A})$ and $GSpin_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$. The well-known case of generic transfer from $GSp_4(\mathbb{A})$ to

$GL_4(\mathbb{A})$ which is still unavailable, as well as Kim's exterior square transfer [18] from $GL_4(\mathbb{A})$, a double cover of $GSpin_6(\mathbb{A})$, to $GL_6(\mathbb{A})$, are among special cases of these. Here \mathbb{A} is the ring of adèles of a number field. We plan to establish these transfers in future papers.

2 Preliminaries

Let F be a non-archimedean local field of characteristic zero. Denote by \mathcal{O} its ring of integer and let \mathcal{P} be its unique maximal ideal. Fix a uniformizing parameter ϖ generating \mathcal{P} . Let $q = [\mathcal{O} : \mathcal{P}]$ be the residue class degree. We fix a valuation $|\cdot| = |\cdot|_F$ normalized by $|\varpi| = q^{-1}$.

Let \mathbf{G} be a quasisplit connected reductive algebraic group over F . Fix a Borel subgroup $\mathbf{B} = \mathbf{T}\mathbf{U}$ over F , with unipotent radical \mathbf{U} and a maximal torus \mathbf{T} . Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be an F -parabolic subgroup standard with respect to \mathbf{B} , that is, $\mathbf{N} \subseteq \mathbf{U}$. Choose the Levi subgroup \mathbf{M} , uniquely by requiring $\mathbf{M} \supset \mathbf{T}$. Let \mathbf{A}_0 be the maximal split subtorus of \mathbf{T} . The choice of \mathbf{U} determines a set of simple roots Δ for \mathbf{A}_0 . If \mathbf{A} is the split component of \mathbf{M} , that is, the maximal split subtorus of the connected component of the center of \mathbf{M} , then $\mathbf{A} \subset \mathbf{A}_0 \subset \mathbf{T}$. Let W and $W_{\mathbf{M}}$ be the Weyl groups of \mathbf{A}_0 in \mathbf{G} and \mathbf{M} , respectively.

For each algebraic group \mathbf{H} over F , let $H = \mathbf{H}(F)$ be the group of its F -rational points. We then have $G, B, T, U, P, M, N, \dots$

Let χ be a nondegenerate character of $\mathbf{U} = \mathbf{U}(F)$ (cf. [20, 27, 31]). We still use χ to denote $\chi|_{\mathbf{U}_{\mathbf{M}}}$, where $\mathbf{U}_{\mathbf{M}} = \mathbf{U} \cap \mathbf{M}$.

To fix our Weyl group representatives, we need to review and reformulate the notion of compatibility from [27, pages 282–283]. Let ψ_F be a nontrivial character of F . Choice of χ points to an F -splitting [20], that is, a collection of root vectors $\{X_{\alpha'}\}$, one for each (nonrestricted) simple root α' of \mathbf{T} , which is preserved under the action of $\Gamma = \text{Gal}(\bar{F}/F)$ (cf. [20, 27]). More precisely, $\{X_{\alpha'}\}$ determines a map from

$$\mathbf{u} \xrightarrow{\phi} \prod \mathbb{G}_a, \tag{2.1}$$

where the product is over all the simple roots of \mathbf{T} , leading to

$$\chi(\mathbf{u}) = \psi_F \left(\sum_{\alpha'} x_{\alpha'} \right), \tag{2.2}$$

where $\phi(\mathbf{u}) = (x_{\alpha'})_{\alpha'}$. The reverse process is now clear as well. The splitting also fixes the natural homomorphisms from the usual simply connected rank one groups into \mathbf{G}

by means of which we will choose our Weyl group representatives w_α , $\alpha \in \Delta$ (cf. [25, page 989]). We will then choose a representative for each $\tilde{w} \in W$ by means of a reduced decomposition and the choices of w_α made before. It will not depend on the decomposition. It is now clear that every element \tilde{w} in the Weyl group is now compatible with χ (cf. [27]), and we can now assume our $\chi = \chi_0$ in the notation of [27] and set $a = 1$ in [27, equation (3.11)]. From now on, whenever a representative is mentioned for a $\tilde{w} \in W$, it will be the representative w chosen as above.

Let π be χ -nondegenerate irreducible admissible representation of $M = \mathbf{M}(F)$.

To proceed, let

$$\mathfrak{a} = \text{Hom} (X(\mathbf{M})_F, \mathbb{R}) \tag{2.3}$$

be the real Lie algebra of \mathbf{A} . Here $X(\mathbf{M})_F$ is the group of F -rational characters of \mathbf{M} . Let $H_P: M \rightarrow \mathfrak{a}$ be as usual defined by

$$q^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F, \tag{2.4}$$

for all $\chi \in X(\mathbf{M})_F$. If $\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$ denotes the dual of \mathfrak{a} , we let $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$.

Given $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let

$$I(\nu, \pi) = \text{Ind}_{MN \uparrow G} \pi \otimes q^{\langle \nu, H_P(\cdot) \rangle} \otimes \mathbf{1} \tag{2.5}$$

be the corresponding induced representation and denote by $V(\nu, \pi)$ its space.

Throughout this paper, we assume that \mathbf{P} is maximal and let α be the unique simple root of \mathbf{A}_0 in \mathbf{N} (with abuse of terminology). If ρ_P is half the sum of roots in \mathbf{N} , we let $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ as in [26]. Finally, let $\Sigma = \Delta \setminus \{\alpha\}$ denote the subset of simple roots generating \mathbf{M} , that is, $\mathbf{M} = \mathbf{M}_{\Sigma}$.

Given $s \in \mathbb{C}$, $s \mapsto s\tilde{\alpha}$, identifies \mathbb{C} with a subspace of $\mathfrak{a}_{\mathbb{C}}^*$. We use $I(s, \pi)$ to denote $I(s\tilde{\alpha}, \pi)$. If π is unitary, then $I(\pi) = I(0, \pi)$ is a unitarily induced representation of G .

There exists a unique element $\tilde{w}_0 \in W = W(\mathbf{A}_0)$ such that $\tilde{w}_0(\Sigma) \subset \Delta$ while $\tilde{w}_0(\alpha)$ is a negative root. Let $\mathbf{M}' \supset \mathbf{T}$ be the Levi subgroup generated by $\tilde{w}_0(\Sigma)$ and denote by \mathbf{N}' the unipotent radical of the F -parabolic subgroup $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ which is standard with respect to \mathbf{B} . The parabolic subgroup \mathbf{P} is called self-associate or self-conjugate, if $\tilde{w}_0(\Sigma) = \Sigma$. Then $\tilde{w}_0(\alpha) = -\alpha$, $\mathbf{M}' = \mathbf{M}$, and $\mathbf{N}' = \mathbf{N}$. We let w_0 be the representative for \tilde{w}_0 as prescribed previously.

We now define the corresponding intertwining operator as

$$A(s, \pi)f(g) = \int_{N'} f(w_0^{-1}n'g) \, dn' \quad (g \in G), \tag{2.6}$$

where $f \in V(s, \pi)$ (cf. [26, 27]).

Let $\lambda_\chi(s, \pi)$ be the canonical Whittaker functional attached to $I(s, \pi)$ as in [23, 27]. More precisely, fix a Whittaker functional λ for π . Then for each $v \in \mathcal{H}(\pi)$, the space of π , $W_v(g) = \langle \pi(g)v, \lambda \rangle$ will define a Whittaker function. The collection of all $W_v, v \in \mathcal{H}(\pi)$, will define a Whittaker model for π . It is unique up to scalar multiplication. We set

$$\lambda_\chi(s, \pi)(f) = \int_{N'} \langle f(w_0^{-1}n'), \lambda \rangle \overline{\chi(n')} \, dn'. \tag{2.7}$$

Observe that integration must be over N' and not N as f transforms according to π . Then $\lambda_\chi(-s, w_0(\pi)) = \lambda_\chi(\tilde{w}_0(s\tilde{\alpha}), w_0(\pi))$ is one for $I(\tilde{w}_0(s\tilde{\alpha}), w_0(\pi))$ and the corresponding local coefficient $C(s, \pi) = C_\chi(s\tilde{\alpha}, \pi, w_0)$ attached to $\chi, s\tilde{\alpha}, \pi$, and w_0 in [23, 27] is defined by

$$\lambda_\chi(s, \pi) = C(s, \pi)\lambda_\chi(-s, w_0(\pi))A(s, \pi). \tag{2.8}$$

Next, let ${}^L N$ be the L-group of N defined as in [5]. Denote by ${}^L n$ its (complex) Lie algebra and let r be the adjoint action of ${}^L M$ on ${}^L n$. Decompose $r = \bigoplus_{i=1}^m r_i$ to its irreducible subrepresentations, indexed by values $\langle \tilde{\alpha}, \beta \rangle$ as β ranges among the positive roots of T . More precisely, $X_{\beta^\vee} \in {}^L n$ lies in the space of r_i if and only if $\langle \tilde{\alpha}, \beta \rangle = i$. Here X_{β^\vee} is a root vector attached to the coroot β^\vee , considered as a root of the L-group. Moreover, $\langle \cdot, \cdot \rangle$ denotes the Killing form, that is, for every pair of positive roots γ and δ of T , $\langle \gamma, \delta \rangle = 2(\gamma, \delta)/(\delta, \delta) = (\gamma, \delta^\vee)$, where δ^\vee is the coroot $2\delta/(\delta, \delta)$ attached to δ (cf. [21, 26]).

Finally, let for each $i, 1 \leq i \leq m$, $\gamma(s, \pi, r_i, \psi_F)$ be the corresponding γ -function defined inductively in [27, Theorem 3.5]. Then [27, Theorem 3.5, equation (3.11)] states that

$$C(s, \pi) = \lambda_G(\psi_F, w_0)^{-1} \prod_{i=1}^m \gamma(is, \pi, \tilde{r}_i, \overline{\psi}_F). \tag{2.9}$$

Observe that, in view of the earlier discussion on compatibility, we may assume that $\alpha = 1$ in [27, equation (3.11)]. Throughout the rest of the paper, we assume that P is self-conjugate, that is, $N = N'$, and therefore $P = P'$ and $M = M'$. This covers most of the interesting cases. The purpose of this paper is to express $C(s, \pi)^{-1}$ as a product of

the Mellin transform of the Bessel function of π with an abelian γ -function, the first step in proving that each $C(s, \pi)$ —and then consequently inductively [9, Propositions 4.1 and 5.1] each $\gamma(s, \pi, r_i, \psi_F)$ —is stable, that is, they depend only on the central character of π if π is replaced by a highly ramified twist of π (in the sense of [15] and [19, Proposition 2.1], and as in [7]). This is necessary in establishing functoriality from classical groups to appropriate $GL(m)$ as in [7], where the case of $SO(2n + 1)$, $m = 2n$, was established. We refer to [18] for another important application.

3 Another look at Bessel functions

In this section, we revisit the theory of Bessel functions for generic representations of a quasisplit group [2, 3, 4, 9]. Here, we approach this theory from a slightly different angle, which we are naturally led to encounter in our method. We keep the notation from Section 2, that is, our group is denoted by G and so on. But later we use this theory only for representations of $M = \mathbf{M}(F)$. Let σ be an irreducible admissible χ -generic representation of $G = \mathbf{G}(F)$. We continue to assume that our Weyl group elements, chosen through the choice of splitting, are compatible with χ . Let θ be an F -automorphism of G preserving the Γ -splitting $(\mathbf{B}, \mathbf{T}, \{X\})$ discussed earlier (cf. [20, 27, 31]). Then $\theta: \mathcal{U} \rightarrow \mathcal{U}$. Let $\chi^\theta = \chi \cdot \theta$. We assume that $\chi^\theta = \chi$. Let W_ν be a Whittaker function in a fixed Whittaker model $W(\sigma)$ defined by a χ -Whittaker functional λ , that is, $W_\nu(g) = \lambda(\sigma(g)\nu)$, $g \in G$. Here ν is a vector in the space of σ . We are interested in the following integral, which may be considered as a kind of twisted orbital integral (see Remark 3.2):

$$\begin{aligned} J_{\sigma, \nu}(g) &= \int_{\mathcal{U}'_g \backslash \mathcal{U}} W_\nu(\theta(u)^{-1}gu) \, du \\ &= \int_{\mathcal{U}'_g \backslash \mathcal{U}} W_\nu(gu) \overline{\chi(u)} \, du, \end{aligned} \tag{3.1}$$

where

$$\mathcal{U}'_g = \{u \in \mathcal{U} \mid gug^{-1} \in \mathcal{U} \text{ and } \chi(gug^{-1}) = \chi(u)\}. \tag{3.2}$$

Observe that $\mathcal{U}'_g \supset \mathcal{U}_g$, the centralizer of g in \mathcal{U} . Moreover, \mathcal{U}'_g is generated by those simple roots which under conjugation by g go to simple ones with no coordinate changes. We assume for the moment that the integral converges for the given W_ν , though the question of convergence is a very interesting and delicate one [2, 3, 4, 9]. Now observe that, given g , $\nu \mapsto J_{\sigma, \nu}(g)$ defines another χ -Whittaker functional for σ . Thus by the uniqueness of

Whittaker functionals

$$J_{\sigma, \nu}(g) = J_{\sigma}(g)W_{\nu}(e) \quad (3.3)$$

with a function $J_{\sigma}(g)$ depending only on the class of σ . Observe that this requires that $J_{\sigma, \nu}(g)$ exist for all $\nu, g \in G$. Now we discuss the case of interest to us. Our reductive group is the Levi factor \mathbf{M} fixed in Section 2. The χ -generic representation σ is π and we fix the Γ -splitting of \mathbf{G} determined by χ as in Section 2. We recall that χ is a generic character of \mathbf{U} . We then choose our Weyl group representatives for elements of W again as before using this splitting. Notice that $(\mathbf{B} \cap \mathbf{M}, \mathbf{T}, \{\chi_{\alpha'}\}_{\alpha' \in \Sigma'})$ is a Γ -splitting for \mathbf{M} . Here Σ' denotes the set of nonrestricted simple roots of \mathbf{T} , restricting to Σ . We set $\theta = \text{Int}(w_0)|_{\mathbf{M}}$. Since w_0 and χ are compatible, $\chi^{\theta} = \chi$ (see Section 2). Given W_{ν} in a χ -Whittaker model of π with $W_{\nu}(e) = 1$, we recall the Bessel function

$$J_{\pi}(m) = \int_{\mathbf{U}'_{\mathbf{M}, m} \setminus \mathbf{U}_{\mathbf{M}}} W_{\nu}(mu) \overline{\chi(u)} \, du \quad (m \in M), \quad (3.4)$$

where $\mathbf{U}'_{\mathbf{M}, m}$ means $(\mathbf{U}_{\mathbf{M}})'_m$. The theory of Bessel functions is still incomplete. In the case of simply laced split groups, thanks to the efforts of Baruch [4], we know the existence of a subspace $W^0(\pi)$ of $W(\pi)$ for which (3.4) converges in a very simple sense [4, Theorem 6.7]. More precisely, for each $W \in W^0(\pi)$ and every $m \in M$, there exists a compact subset $C \subset \mathbf{U}_{\mathbf{M}}$ such that $W(mu) \neq 0$ implies that $u \in \mathbf{U}'_{\mathbf{M}, m}C$. Baruch's theorem [4, Theorem 6.7] is stronger than this. We assume the natural extension of this to any quasisplit \mathbf{M} . It will only be used to justify the definition of an incomplete Bessel function and will not be needed for our main theorem.

Assumption 3.1. There exists a nonzero subspace $W^0(\pi)$ of $W(\pi)$ such that for each $W \in W^0(\pi)$ and $m \in M$, there exists a compact subset $C \subset \mathbf{U}_{\mathbf{M}}$ so that $W(mu) \neq 0$ implies $u \in \mathbf{U}'_{\mathbf{M}, m}C$. In particular, (3.4) converges for every $W \in W^0(\pi)$. \square

Remark 3.2. Observe that the functions in $W^0(\pi)$ play the role of smooth functions of compact support, and Bessel functions play that of (twisted) orbital integrals, when (twisted) conjugation by $\mathbf{U}_{\mathbf{M}}$ is replaced by (twisted) conjugation in the corresponding disconnected subgroup.

Remark 3.3. In the case of $\text{SO}(2n+1)$, Assumption 3.1 is the corollary of [9, Proposition 4.2, page 450]. Notice that $\text{SO}(2n+1)$ is not simply laced.

4 An assumption

For the purposes of this paper we are interested in all those $n \in N$ such that $w_0^{-1}n \in P\bar{N}$ (cf. [29]). For such n , write

$$w_0^{-1}n = mn'\bar{n}, \tag{4.1}$$

where $m \in M$, $n' \in N$, and $\bar{n} \in \bar{N}$. The decomposition is clearly unique. Moreover, if $u \in U_M$ centralizes n , that is, $unu^{-1} = n$, then $w_0(u)mu^{-1} = m$, where $w_0(u) = w_0^{-1}uw_0$. Consequently, if

$$U_{M,n} = \{u \in U_M \mid unu^{-1} = n\}, \tag{4.2}$$

then

$$U_{M,n} \subset U'_{M,m} \tag{4.3}$$

(see equation (3.1)) since χ and w_0 are compatible and therefore $\chi(w_0(u)) = \chi(u)$. Let dn be a right-invariant measure for N . The set of all n which do not satisfy (4.1) is of measure zero with respect to dn . In fact, let $\bar{N}_1 = \bar{N} \cap Pw_0N$. Then \bar{N}_1 is an open subset of \bar{N} and therefore $\dim \bar{N}_1 = \dim \bar{N}$. Now suppose that \bar{n} is in \bar{N} , but not in \bar{N}_1 . Then $\bar{n} \in Pw'N$ for which $\tilde{N} = w'Nw'^{-1} \cap P$ is a nontrivial subgroup, where $w' \in W$ which is the Weyl group of T as well. Let $N' = w'^{-1}\tilde{N}w' \subset N$. Let $N'_1 = N' \setminus N$. Then $Pw'N = Pw'N'_1$, where the choice of representatives for N'_1 is unimportant. Then $\dim N'_1 < \dim N$, since $\dim N' > 0$. Set

$$\bar{N}_2 = \{\bar{n} \in \bar{N} \mid \bar{n} \in Pw'N'_1\}. \tag{4.4}$$

Then $P\bar{N}_2 \subset Pw'N'_1$ and therefore

$$\dim P + \dim \bar{N}_2 \leq \dim P + \dim N'_1. \tag{4.5}$$

Then this immediately implies that $\dim \bar{N}_2 < \dim N$. Although $U'_{M,m}$ is defined in terms of χ , it only depends on m and not the choice of χ . As explained before, it is generated by those simple roots in U_M which under $\text{Int}(m)$ go to simple roots with no coordinate changes. It can be interpreted as a twisted centralizer of m . The following assumption is quite natural, necessary to our purposes, and must be true in general. Here we verify it in many cases of interest to us. The general case will be addressed in a future paper.

Assumption 4.1. Except for a set of measure zero with respect to dn ,

$$U_{M,n} = U'_{M,m}, \tag{4.6}$$

where $w_0^{-1}n = mn'\bar{n}$ as in (4.1). □

Assume that $n \in N$ satisfies (4.1). Let $M_m^{w_0}$ denote the $\text{Int}(w_0)$ -twisted centralizer of m in M , that is,

$$M_m^{w_0} = \{m_1 \in M \mid w_0(m_1)mm_1^{-1} = m\}. \tag{4.7}$$

Then it is easily seen that

$$M_n \subset M_m^{w_0}, \tag{4.8}$$

where

$$M_n = \{m_1 \in M \mid m_1nm_1^{-1} = n\}. \tag{4.9}$$

Notice that for any $x \in G$,

$$w_0^{-1}xnx^{-1} = w_0(x)mx^{-1} \cdot xn'x^{-1} \cdot x\bar{n}x^{-1}, \tag{4.10}$$

where $w_0(x) = w_0^{-1}xw_0$. Thus, if n satisfies (4.1), then so does every member of the intersection of its conjugacy class under G with N . The corresponding m -, n' -, and \bar{n} -components are then given by the above decomposition. Observe that for m , we need to consider its $\text{Int}(w_0)$ -twisted conjugacy class. There are a good number of examples in which, not only $M_n \neq M_m^{w_0}$, but in fact $[M_m^{w_0} : M_n] = \infty$, even on a big open subset of N . (See the first example below.) What our assumption states is that, if we instead look at centralizers and twisted centralizers in U_M , we in fact have the equality $U_{M,n} = U'_{M,m}$. Our first example is of the above type in which $[M_m^{w_0} : M_n] = \infty$. But we will show that Assumption 4.1 still holds. We could choose an example from either [14] or [28]. For simplicity we consider the case of split $G = SO_{6r}$ from [28] with the standard Levi subgroup $M = GL_{2r} \times SO_{2r}$. To make matters even easier, we will assume that r is even. The subgroup N consists of

$$N = \left\{ n = n(X, Y) = \begin{pmatrix} I_{2r} & X & Y \\ 0 & I_{2r} & X' \\ 0 & 0 & I_{2r} \end{pmatrix} \mid Y + \tilde{\theta}(Y) = XX', X + \tilde{\theta}(X') = 0 \right\}, \tag{4.11}$$

where $w_{2r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tilde{\theta}(Y) = w_{2r}^{-1} \cdot {}^t Y \cdot w_{2r}$, $Y \in M_{2r}(F)$. Moreover, $w_0 = \text{diag}(w_{2r}, w_{2r}, w_{2r}) \cdot w_{6r} \in \text{SO}_{6r}(F)$ since r is even, and $n \in \mathbf{N}$ satisfies (4.1), if $Y \in \text{GL}_{2r}(F)$. The element w_0 may not be the precise representative discussed in Section 2. But that is irrelevant. Then [28, Lemma 3.1] implies that

$$m = \text{diag}(\tilde{\theta}(Y^{-1}), I_{2r} - X'Y^{-1}X, Y). \tag{4.12}$$

Moreover, by [28, Corollary 5.7]

$$n(X, Y) \longmapsto I_{2r} - X'Y^{-1}X \tag{4.13}$$

is a surjection onto $\text{SO}_{2r}(F)$. It can be easily checked that if $X \in \text{GL}_{2r}(F)$, then for $n = n(X, Y)$

$$M_n \cong \{g \in \text{GL}_{2r}(F) \mid gY\tilde{\theta}(g) = Y\}. \tag{4.14}$$

In fact, the element $(g, g', \theta(g^{-1}))$ is in the centralizer of n in M if and only if $gY\tilde{\theta}(g) = Y$ and $gXg'^{-1} = X$. But since $X \in \text{GL}_{2r}(F)$, g' is uniquely determined by g equal to $g' = X^{-1}gX$. Thus, the centralizer M_n of n in M is isomorphic to the subgroup of all $g \in \text{GL}_{2r}(F)$ such that $gY\tilde{\theta}(g) = Y$. Observe that in view of the relation satisfied by X and Y , the fact that $X^{-1}gX \in \text{SO}_{2r}(F)$ is automatic and therefore does not put any new restriction on the set of all $g \in \text{GL}_{2r}(F)$ satisfying $gY\tilde{\theta}(g) = Y$. On the other hand,

$$M_m^{w_0} \cong M_n \times M_k, \tag{4.15}$$

where M_k is the centralizer of $k = I_{2r} - X'Y^{-1}X$ in $\text{SO}_{2r}(F)$. Observe that for a regular semisimple element k , M_k is a Cartan subgroup and therefore $[M_m^{w_0} : M_n] = \infty$ for an open dense subset of N . On the other hand, all those m that belong to the open cell $(B \cap M)w_M u_M$, will have a trivial U_M -twisted centralizer. (Here w_M is the longest element of W_M .) In fact, suppose $m = u_1 t w_M u_2$, $u_1, u_2 \in U_M$, $t \in T$, and assume that for $u \in U_M$

$$m u m^{-1} = u_0 \in U_M. \tag{4.16}$$

Then

$$u_0^{-1} u_1 t w_M u_2 = u_1 t w_M u_2 u^{-1}, \tag{4.17}$$

which implies $u = u_0 = 1$. Consequently, for such an open dense subset of N

$$U_{M,n} = U'_{M,m}. \tag{4.18}$$

Now we attend to three other examples. They are quite important and in fact are our main motivation for this paper.

Case 1. For a given positive integer n , let \mathbf{G} be the split group SO_{2n+3} . The parabolic subgroup $\mathbf{P} = \mathbf{M}\mathbf{N}$ is the one with $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{2n+1}$. The element w_0 is simply

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{2n+1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}_{2n+1}(F). \tag{4.19}$$

We will provide more detail on how such representatives are chosen for the next two more delicate cases.

To continue, we borrow the notation from [9]. If $t \in F^{2n+1}$ is a row vector, we use $t^* \in F^{2n+1}$ to denote the dual column to it. More precisely, if $t = (t_1, \dots, t_{2n+1})$, then ${}^t(t^*) = (t_{2n+1}, \dots, t_1)$. Then $\langle t, t \rangle = tt^* = \sum_{i=1}^{2n+1} t_i t_{2n+2-i}$ is the defining quadratic form for $\mathrm{SO}_{2n+1}(F)$. An arbitrary element in $\mathbf{N} = \mathbf{N}(F)$ is of the form

$$n = n(t) = \begin{pmatrix} 1 & t & -\frac{1}{2}\langle t, t \rangle \\ 0 & I_{2n+1} & -t^* \\ 0 & 0 & 1 \end{pmatrix} \tag{4.20}$$

for t as above. We use $\mathrm{diag}(a, k, a^{-1})$, $a \in F^*$, $k \in \mathrm{SO}_{2n+1}(F)$, to denote an arbitrary element of \mathbf{M} . The following lemma is a consequence of a straightforward calculation.

Lemma 4.2. Assume that $w_0^{-1}n(t) \in \mathbf{P}\bar{\mathbf{N}}$, $t \in F^{2n+1}$. Write $w_0^{-1}n(t) = mn'\bar{n}$ as in (4.1). Then $\langle t, t \rangle = tt^* \neq 0$. Write $m = \mathrm{diag}(a, k, a^{-1})$. Then $a = -2/\langle t, t \rangle$ and

$$k = -(I_{2n+1} - 2t^*t/\langle t, t \rangle). \tag{4.21}$$

□

Proof. Consider the decomposition

$$w_0^{-1}n(t) = \begin{pmatrix} a & ax & -\frac{axx^*}{2} \\ 0 & k & -kx^* \\ 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & I_{2n+1} & 0 \\ -\frac{1}{2}y^*y & -y^* & I \end{pmatrix}, \tag{4.22}$$

where $n(t)$ is as in (4.20). This then implies, among other things, that $a = -2/\langle t, t \rangle$ and

$$k(I_{2n+1} - 2t^*t/\langle t, t \rangle) = -I_{2n+1}. \tag{4.23}$$

We therefore only need to show that

$$(I_{2n+1} - 2t^*t/\langle t, t \rangle) \tag{4.24}$$

is equal to its own inverse. Observe that if

$$w_{2n+1} = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix} \in GL_{2n+1}(F) \tag{4.25}$$

is the permutation matrix defining $O_{2n+1}(F)$, then

$$w_{2n+1} \cdot {}^t(I_{2n+1} - 2t^*t/\langle t, t \rangle) \cdot w_{2n+1}^{-1} = I_{2n+1} - 2t^*t/\langle t, t \rangle. \tag{4.26}$$

But the first term, k being in $O_{2n+1}(F)$, equals to

$$(I_{2n+1} - 2t^*t/\langle t, t \rangle)^{-1}. \tag{4.27}$$

■

We now verify [Assumption 4.1](#) in this case. We do this while also determining a set of representative for the adjoint action of AU_M on N , which will be needed in one of the main propositions of this paper. Let $t = (t_1, \dots, t_{2n-1}) \in F^{2n-1}$ and let $x(t)$, as in [9], denote the corresponding element in U_M , the principal unipotent subgroup of $SO_{2n+1}(F)$. In fact, for the sake of clarity, throughout this section and [Section 7](#), while we use $n(t)$ and $n(t, T)$ (in [Case 3](#)) to denote the elements of N , we employ $x(t)$ and $x(t, t_0)$ (in [Case 3](#)) to denote the corresponding unipotent elements in the corresponding immediate lower rank groups. Every element in N can be represented by a column matrix $\alpha = {}^t(\alpha_1, \dots, \alpha_{2n+1}) \in F^{2n+1}$. Let $B = {}^t(b, 0, \dots, 0, c) \in F^{2n+1}$.

Lemma 4.3. Suppose that α_{2n+1} and $\langle \alpha, \alpha \rangle$ are both in F^* . Then there exist $t \in F^{2n-1}$, and b, c in F^* , such that

$$x(t)B = \alpha. \tag{4.28}$$

Consequently, except for a set of measure zero, each orbit of N under the action of U_M can be represented by a pair $(b, c) \in (F^*)^2$. □

Proof. It is easy to see that (4.28) is equivalent to the following set of equations

$$\begin{aligned}
 b - c\langle t, t \rangle / 2 &= \alpha_1 \\
 -ct_1 &= \alpha_2 \\
 -ct_2 &= \alpha_3 \\
 &\vdots \\
 -ct_{2n-1} &= \alpha_{2n} \\
 c &= \alpha_{2n+1}.
 \end{aligned}
 \tag{4.29}$$

Under the assumption that $\alpha_{2n+1} \neq 0$, this leads to

$$\begin{aligned}
 t_i &= -\alpha_{i+1} \alpha_{2n+1}^{-1} \quad (1 \leq i \leq 2n - 1), \\
 c &= \alpha_{2n+1}.
 \end{aligned}
 \tag{4.30}$$

Finally,

$$b = \alpha_1 + \alpha_{2n+1} \langle t, t \rangle / 2.
 \tag{4.31}$$

A straightforward calculation then shows that

$$b = \langle \alpha, \alpha \rangle (2\alpha_{2n+1})^{-1}
 \tag{4.32}$$

which is in F^* by our assumptions. This completes the proof of the lemma. ■

Now suppose that $n \in \mathbb{N}$ is of the form $n = n(\alpha w_{2n+1})$ with α as in Lemma 4.3. Then we can choose $u \in U_M$ such that $uB = \alpha$, where B is as in Lemma 4.3, that is, $B = {}^t(b, 0, \dots, 0, c)$, with b and c in F^* . Using (4.21) we determine k satisfying (4.1) for n as

$$-u(I - 2B \cdot {}^tB w_{2n+1} / \langle B, B \rangle) u^{-1},
 \tag{4.33}$$

which using $\langle B, B \rangle = 2bc$ equals

$$u \begin{pmatrix} & & bc^{-1} \\ & -I_{2n-1} & \\ cb^{-1} & & \end{pmatrix} u^{-1}
 \tag{4.34}$$

or

$$u \begin{pmatrix} bc^{-1} & & \\ & I_{2n-1} & \\ & & cb^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix} u^{-1}. \tag{4.35}$$

It is then immediate that [Assumption 4.1](#) is valid for all such n whose complement is of course of measure zero. In fact, we only need to consider this choice of representative for the conjugacy class under conjugation by U_M to see the equality $U_{M,n}$ and $U'_{M,m}$. Finally, we point out that as it can be easily observed, again for almost all n , given a conjugacy class n of N under the action of AU_M , the AU_M - $\text{Int}(w_0)$ -twisted conjugacy class of the corresponding m can be represented by an element,

$$\begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix}, \tag{4.36}$$

with $h \in F^*$. The choice of h is unique. This is precisely because under conjugation by A , cb^{-1} , and bc^{-1} remain unchanged. We collect this information as follows.

Proposition 4.4. Suppose that $n \in N$ satisfying [\(4.1\)](#) is represented by the column vector α for which $\langle \alpha, \alpha \rangle$ and α_{2n+1} are nonzero. Then

- (a) $U_{M,n} = U'_{M,m}$;
- (b) the AU_M - $\text{Int}(w_0)$ -twisted conjugacy class of m , $w_0^{-1}n = mn'\bar{n}$, can be represented by $m = \text{diag}(a, k, a^{-1})$ with $a = -h^{-1}$ and

$$k = \begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix} \tag{4.37}$$

for a unique $h \in F^*$. □

Remark 4.5. In the notation of [\[9\]](#),

$$\begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} = \underline{h},$$

$$\begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix} = \underline{\beta}. \tag{4.38}$$

This is quite important and is one of our main motivating factors in pursuing this approach.

Case 2. Now we address the case of an even split special orthogonal group. We let $G = SO_{2n+2}$ and consider the parabolic subgroup $P = MN$ for which $M = GL_1 \times SO_{2n}$. We first recall that G is the connected component of the subgroup of $g \in GL_{2n+2}$ satisfying

$$w_{2n+2} {}^t g^{-1} w_{2n+2} = g, \tag{4.39}$$

where $w_{2n+2} \in GL_{2n+2}$ is

$$w_{2n+2} = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & & 1 & \\ 1 & & & \end{pmatrix}. \tag{4.40}$$

An arbitrary element $n \in N$ will again be given by a row vector $t \in F^{2n}$, as in the odd case, by

$$n = n(t) = \begin{pmatrix} 1 & t & -\frac{1}{2}\langle t, t \rangle \\ & I_{2n} & -t^* \\ & & 1 \end{pmatrix}, \tag{4.41}$$

where $\langle t, t \rangle = tt^*$. The Weyl group element $\tilde{w}_0 = \tilde{w}_G \tilde{w}_M$ will send $e_1 \mapsto -e_1, e_i \mapsto e_i, 2 \leq i \leq n, e_{n+1} \mapsto -e_{n+1}$. With the usual Bourbaki notation,

$$\tilde{w}_0 = \tilde{w}_{\alpha_1} \tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_{n-1}} \tilde{w}_{\alpha_{n+1}} \tilde{w}_{\alpha_{n-1}} \cdots \tilde{w}_{\alpha_1} \tag{4.42}$$

is a reduced decomposition, where $\tilde{w}_{\alpha_{n+1}}$ sends $e_n \mapsto -e_n$ and $e_{n+1} \mapsto -e_{n+1}$. Taking the image of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ inside each rank one group generated by each α_i by means of the homomorphisms from SL_2 into SO_{2n+2} (cf. [25]), which are determined by the standard splitting, as representatives w_{α_i} for each \tilde{w}_{α_i} , we get

$$\begin{aligned} w_0 &= w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{n+1}} \cdots w_{\alpha_1} \\ &= \begin{pmatrix} & & & 1 \\ & & -K_{2n} & \\ & & & \\ 1 & & & \end{pmatrix}, \end{aligned} \tag{4.43}$$

The analogue of Lemma 4.3 is also valid. It can be verified exactly the same way as Lemma 4.3 and consequently we only state the result for the sake of book-keeping without any details.

Lemma 4.7. Let $\alpha = {}^t(\alpha_1, \dots, \alpha_{2n})$ and $B = {}^t(b, 0, \dots, 0, c) \in F^{2n}$. Suppose that α_{2n} and $\langle \alpha, \alpha \rangle = \sum_{i=1}^{2n} \alpha_i \alpha_{2n+1-i}$ are both in F^* . Then there exist $t \in F^{2n-2}$, and b, c in F^* such that

$$\chi(t)B = \alpha. \tag{4.49}$$

Consequently, except for a set of measure zero, each orbit of N under the action of U_M can be presented by a pair $(b, c) \in (F^*)^2$. More precisely, we can take $c = \alpha_{2n}$ and $b = \langle \alpha, \alpha \rangle (2\alpha_n)^{-1}$. \square

We conclude by stating the analogue of Proposition 4.4 in this case. The arguments are precisely as in Proposition 4.4.

Proposition 4.8. Suppose that $n \in N$ satisfying (4.1) is represented by the column vector $\alpha \in F^{2n}$ for which $\langle \alpha, \alpha \rangle$ and α_{2n} are nonzero. Then

- (a) $U_{M,n} = U'_{M,m}$;
- (b) the AU_M -Int(w_0)-twisted conjugacy class of m , $w_0^{-1}n = mn'\bar{n}$, can be represented by $m = \text{diag}(a, k, a^{-1})$ with $a = -h^{-1}$ and

$$k = \begin{pmatrix} h & & & \\ & I_{2n-2} & & \\ & & h^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & \\ & -K_{2n-2} & & \\ 1 & & & \\ & & & \end{pmatrix} \tag{4.50}$$

for a unique $h \in F^*$. Here K_{2n-2} is defined by (4.44) with $2n$ replaced by $2n - 2$. \square

Case 3. We now attend to our final case, that of a symplectic group. We let $G = Sp_{2n+2}$ and consider the parabolic subgroup $P = MN$ for which $M = GL_1 \times Sp_{2n}$.

The group G is the set of all $g \in GL_{2n+2}$ for which

$$J_{n+1} {}^t g^{-1} J_{n+1}^{-1} = g, \tag{4.51}$$

where

$$J_{n+1} = \begin{pmatrix} 0 & w_{n+1} \\ -w_{n+1} & 0 \end{pmatrix} \in GL_{2n+2}, \tag{4.52}$$

with w_{n+1} the second diagonal identity as before. Given a row vector $t \in F^{2n}$ and $T \in F$, we can define the matrix

$$n(t, T) = \begin{pmatrix} 1 & t & T \\ 0 & I_{2n} & t^* \\ 0 & 0 & 1 \end{pmatrix} \in N \tag{4.53}$$

to represent a general member of N , where

$$t^* = J_n \cdot {}^t t. \tag{4.54}$$

Observe that $tt^* = 0$ and that there is no relation between t and T . The Weyl group element $\tilde{w}_0 = \tilde{w}_G \tilde{w}_M$ will send $e_1 \mapsto -e_1$ but $e_i \mapsto e_i$ for all other i . It can be easily shown that

$$\tilde{w}_0 = \tilde{w}_{\alpha_1} \tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n+1}} \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n-1}} \cdots \tilde{w}_{\alpha_1} \tag{4.55}$$

is a reduced decomposition of \tilde{w}_0 . If we again, as in [Case 2](#), take the images of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ inside each rank one group generated by each α_i by means of the homomorphism from SL_2 into Sp_{2n+2} (cf. [\[25\]](#)), which are determined by the standard splitting, as representative w_{α_i} for \tilde{w}_{α_i} , we get

$$\begin{aligned} w_0 &= w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_n} w_{\alpha_{n+1}} w_{\alpha_n} \cdots w_{\alpha_1} \\ &= \begin{pmatrix} & & -1 \\ & -I_{2n} & \\ 1 & & \end{pmatrix} \in Sp_{2n+2}(F). \end{aligned} \tag{4.56}$$

A little bit of tedious calculation again shows the following lemma.

Lemma 4.9. Assume $w_0^{-1}n(t, T) \in P\bar{N}$, $t \in F^{2n}$, $T \in F$. Write $w_0^{-1}n(t, T) = mn'\bar{n}$ as in [\(4.1\)](#). Then $T \neq 0$. Write $m = \text{diag}(a, k, a^{-1})$, $a \in F^*$, $k \in Sp_{2n}(F)$. Then $a = -T^{-1}$ and

$$k = -(I + t^*t/T), \tag{4.57}$$

where $t^* = J_n \cdot {}^t t$. □

Proof. This time we need the factorization

$$\begin{pmatrix} & & 1 \\ & -I_{2n} & \\ -1 & & \end{pmatrix} \begin{pmatrix} 1 & t & T \\ 0 & I_{2n} & t' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax & aX \\ 0 & k & kx' \\ 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & I_{2n} & 0 \\ Y & y' & 1 \end{pmatrix}. \tag{4.58}$$

Again we only need to show that

$$(I - t^*t/T)^{-1} = I + t^*t/T. \tag{4.59}$$

But this follows immediately using the fact that $I - t^*t/T \in \text{Sp}_{2n}(F)$ as before. As before, we now proceed to verify [Assumption 4.1](#) and determine a set of representatives for the adjoint action of AU_M on N . Every element in N is determined by a column matrix $\alpha = {}^t(\alpha_1, \dots, \alpha_{2n}) \in F^{2n}$ and $\alpha_0 \in F$. Let $B = {}^t(b, 0, 0, \dots, c) \in F^{2n}$. ■

Lemma 4.10. Suppose that α_{2n} and α_0 are both in F^* . Then there exist $t = (t_1, \dots, t_{2n-2}) \in F^{2n-2}$, $t_0 \in F$, and, b and c in F^* such that

$$\chi(t, t_0)B = \alpha \tag{4.60}$$

and $bc = \alpha_0$. Consequently, except for a set of measure zero, each orbit of N under the action of U_M can be presented by a pair $(b, c) \in F^*$. □

Proof. As in [Lemma 4.3](#) we need to solve the system

$$\begin{aligned} b + ct_0 &= \alpha_1 \\ ct_{2n-2} &= \alpha_2 \\ &\vdots \\ ct_n &= \alpha_n \\ -ct_{n-1} &= \alpha_{n+1} \\ &\vdots \\ -ct_1 &= \alpha_{2n-1} \\ c &= \alpha_{2n}. \end{aligned} \tag{4.61}$$

Since $\alpha_{2n} \neq 0$, this implies that $c = \alpha_{2n}$ and

$$\begin{aligned} t_i &= \alpha_{2n-i}\alpha_{2n}^{-1} \quad (n \leq i \leq 2n - 2), \\ t_i &= -\alpha_{2n-i}\alpha_{2n}^{-1} \quad (1 \leq i \leq n - 1). \end{aligned} \tag{4.62}$$

Moreover, under the assumption $bc = \alpha_0 \neq 0$, $b = \alpha_0\alpha_{2n}^{-1}$. Finally, we get

$$t_0 = (\alpha_1 - \alpha_0\alpha_{2n}^{-1})\alpha_{2n}^{-1} \tag{4.63}$$

to complete the lemma. ■

Now suppose that $n \in N$ is of the form $n({}^t\alpha_{J_n}, \alpha_0)$. Then we can choose $u \in U_M$ such that $uB = \alpha$ as in Lemma 4.10. We observe that under adjoint action of U_M , α_0 does not change. Using (4.57) we determine k as

$$-u(I + B \cdot {}^tB_{J_n}/\alpha_0)u^{-1}, \tag{4.64}$$

which, using $bc = \alpha_0$, equals

$$u \begin{pmatrix} 0 & 0 & -bc^{-1} \\ 0 & -I_{2n-2} & 0 \\ cb^{-1} & 0 & -2 \end{pmatrix} u^{-1} \tag{4.65}$$

or

$$u \begin{pmatrix} & & -bc^{-1} \\ & -I_{2n-2} & \\ cb^{-1} & & \end{pmatrix} \begin{pmatrix} 1 & 0 & -2bc^{-1} \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} u^{-1}. \tag{4.66}$$

As before, we can now again observe that the AU_M -Int(w_0)-twisted conjugacy class of m can be represented by an element

$$\begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.67}$$

with $h \in F^*$. The choice of h is unique for the same reason as in Case 1. In fact, conjugating $n({}^t\alpha_{J_n}, \alpha_0)$ with $a = \text{diag}(t, 1, \dots, 1, t^{-1}) \in A$ will change α to $t\alpha$ and α_0 to $t^2\alpha_0$. This results in changing b and c to tb and tc , leaving bc^{-1} invariant. We collect this information as follows.

Proposition 4.11. Suppose that $n \in N$ satisfying (4.1) is represented by the vector $\alpha \in F^{2n}$ and $\alpha_0 \in F$ for which α_{2n} and α_0 are nonzero. Then

- (a) $U_{M,n} = U'_{M,m}$;
- (b) the AU_M -Int(w_0)-twisted conjugacy class of m , $w_0^{-1}n = mn'\bar{n}$, can be represented by an element $m = \text{diag}(a, k, a^{-1})$ with $a = -h^{-1}$ and

$$k = \begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.68}$$

for a unique $h \in F^*$. Observe that

$$\chi \left(\begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 1. \quad (4.69) \quad \square$$

Remark 4.12. Later on, we will need to evaluate an incomplete Bessel function of the representation at elements of type (4.68). We will observe later that even though

$$\begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.70)$$

will drop out if we consider the full Bessel function, the situation is not the same with an incomplete one. But it is still very close to it ([Proposition 7.3](#)).

5 A reduction step

We continue to assume that \mathbf{G} and $\mathbf{P} = \mathbf{MN}$ are as in earlier sections, that is, \mathbf{P} is a self-associate maximal F -parabolic subgroup of \mathbf{G} satisfying $\mathbf{N} \subset \mathbf{U}$ and $\mathbf{T} \subset \mathbf{M}$ for a fixed Borel subgroup $\mathbf{B} = \mathbf{TU}$ defined over F . Let α be the corresponding simple root. Finally, let $Z_{\mathbf{G}}$ and $Z_{\mathbf{M}}$ be the centers of \mathbf{G} and \mathbf{M} , respectively. To carry out our calculations in [Section 6](#) we need the following assumption.

Assumption 5.1. There exists an injection α^\vee from F^* into $Z_{\mathbf{G}} \backslash Z_{\mathbf{M}}$ such that $\alpha'(\alpha^\vee(t)) = t$, $t \in F^*$, for any root α' of \mathbf{T} which restricts to α . Set $Z_{\mathbf{M}}^0 = \alpha^\vee(F^*)$. \square

The assumption is clearly false even for SL_2 , but not for GL_2 . On the other hand, it is valid in each of the three crucial cases ($n \geq 2$ for $\mathbf{G} = \mathrm{Sp}_{2n}$) considered in [Section 4](#) since $Z_{\mathbf{G}} \backslash Z_{\mathbf{M}}$ is precisely

$$A = \{ \mathrm{diag}(t, 1, \dots, 1, t^{-1}) \mid t \in F^* \}. \quad (5.1)$$

We can rectify this difficulty by proving the validity of [Assumption 5.1](#) when $Z_{\mathbf{G}}$ is a cohomologically trivial torus ([Lemma 5.2](#)). On the other hand, since local coefficients depend only on the derived group of \mathbf{G} , we can replace \mathbf{G} by a larger group, sharing the same derived group as \mathbf{G} , for which the lemma and therefore [Assumption 5.1](#) are valid ([Proposition 5.4](#)).

Lemma 5.2. Suppose that Z_G is a torus for which $H^1(F, Z_G) = 1$. Then there exists an injection α^\vee from F^* into $Z_G \backslash Z_M$ such that $\alpha'(\alpha^\vee(t)) = t$, $t \in F^*$, for any root α' of \mathbf{T} which restricts to α . \square

Proof. Consider the exact sequence

$$0 \longrightarrow Z_G \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}_{\text{ad}} \longrightarrow 0. \tag{5.2}$$

As a group over \bar{F} , we identify \mathbf{G} with its defining split group. Similarly for \mathbf{T} and Z_G . Since $Z_G \backslash \mathbf{G}$ is an adjoint Chevalley group over \bar{F} having \mathbf{T}_{ad} as a maximal torus, $X_*(\mathbf{T}_{\text{ad}})$, the lattice of cocharacters of \mathbf{T}_{ad} , equals to the lattice of its coweights. Thus there exists $\alpha_0^\vee \in X_*(\mathbf{T}_{\text{ad}})$ such that

$$\alpha'(\alpha_0^\vee(t)) = t, \tag{5.3}$$

but

$$\beta'(\alpha_0^\vee(t)) = 1, \tag{5.4}$$

where β' is any simple root of \mathbf{T} , different from α' . Here $t \in \mathbb{G}_m = GL_1$. This then implies $\alpha_0^\vee(t) \in Z_M$.

Since Z_G is a torus, (5.2) splits and therefore

$$X_*(\mathbf{T}) \simeq X_*(\mathbf{T}_{\text{ad}}) \oplus X_*(Z_G). \tag{5.5}$$

Let $\alpha^\vee = (\alpha_0^\vee, 1)$. Then $\alpha'(\alpha^\vee(t)) = t$, while $\beta'(\alpha^\vee(t)) = 1$ for α' and β' as before. The roots which restrict to α are in the same Γ -orbit of a fixed one, say, α' . Thus for each $\sigma \in \Gamma = \text{Gal}(\bar{F}/F)$,

$$(\sigma\alpha')(\alpha^\vee(t)) = \sigma(\alpha'(\sigma^{-1}(\alpha^\vee(t)))) = t. \tag{5.6}$$

Changing t to $\sigma(t)$, we get

$$\alpha'(\sigma^{-1}(\alpha^\vee(\sigma(t))))(\alpha^\vee(t))^{-1} = 1. \tag{5.7}$$

Changing α' in its Γ -orbit, it is clear that (5.7) is valid for any α' which restricts to α .

Similarly, (5.7) is valid for any β' , that is, any simple root of \mathbf{T} which does not restrict to α . Consequently, there exists $z_\sigma(t) \in \mathbf{Z}_G$ such that

$$\sigma(\alpha^\vee(\sigma^{-1}(t))) = \alpha^\vee(t)z_\sigma(t) \tag{5.8}$$

for each $\sigma \in \Gamma$. It is easily checked that the class of

$$\sigma \longmapsto z_\sigma(t) \tag{5.9}$$

is in $H^1(F, \mathbf{Z}_G) = 1$. Choose $\mathfrak{a} = \mathfrak{a}(t) \in \mathbf{Z}_G$ such that $z_\sigma = \mathfrak{a}\sigma(\mathfrak{a})^{-1}$. Thus

$$\sigma(\alpha^\vee(\sigma^{-1}(t))\mathfrak{a}(t)) = \alpha^\vee(t)\mathfrak{a}(t). \tag{5.10}$$

Taking $t \in F^*$, $\alpha^\vee(t)\mathfrak{a}(t) \in \mathbf{Z}_M = \mathbf{Z}_M(F)$. Observe that α^\vee sets up an injection from \mathbb{G}_m into $\mathbf{Z}_G \backslash \mathbf{Z}_M$. Moreover, $(\mathbf{Z}_G \backslash \mathbf{Z}_M)(F) \simeq \mathbf{Z}_G \backslash \mathbf{Z}_M$. Thus α^\vee gives an injection from F^* into $\mathbf{Z}_G \backslash \mathbf{Z}_M$ satisfying $\alpha'(\alpha^\vee(t)) = t$, $t \in F^*$, as desired. ■

Now, we consider arbitrary \mathbf{Z}_G . We first imbed \mathbf{G} in another quasisplit connected reductive group \mathbf{G}' with the same derived group as \mathbf{G} , but with a connected center. We do exactly as in [22, Lemma 2.1]. We take a free Γ -module Q satisfying

$$Q \longrightarrow X(\mathbf{Z}_G) \longrightarrow 0 \tag{5.11}$$

and choose a torus \mathbf{Z}' whose character module (over \bar{F}) $X(\mathbf{Z}') = Q$. Then $\mathbf{Z}_G \subset \mathbf{Z}'$. We then define

$$\mathbf{G}' = (\mathbf{Z}' \times \mathbf{G}_D) / \mathbf{Z}_G \cap \mathbf{G}_D, \tag{5.12}$$

where \mathbf{G}_D is the derived group of \mathbf{G} . We can therefore assume that \mathbf{G} has a connected center, that is, a torus \mathbf{Z}_G . We are done unless \mathbf{Z}_G has a nonzero first Galois cohomology. For that we use the following well-known lemma for which we like to thank Wentang Kuo.

Lemma 5.3. Let \mathbf{T} be a torus over F . Then there exists an F -torus $\tilde{\mathbf{T}}$ with $\mathbf{T} \subset \tilde{\mathbf{T}}$ and $H^1(F, \tilde{\mathbf{T}}) = 1$. □

Proof. Let L/F be a Galois extension over which \mathbf{T} splits. Let $\tilde{\mathbf{T}} = \text{Res}_{L/F} \mathbf{T}$. Clearly $\mathbf{T} \subset \tilde{\mathbf{T}}$. By Shapiro's lemma,

$$H^1(F, \tilde{\mathbf{T}}) = H^1(L, \mathbf{T}) \tag{5.13}$$

which is trivial since \mathbf{T} splits over L . ■

Arguing as above, we have the following proposition.

Proposition 5.4. Let \mathbf{G} be a connected reductive group over F . Then there exists a connected reductive group $\tilde{\mathbf{G}}$ over F , whose center is a torus $Z_{\tilde{\mathbf{G}}}$, satisfying

- (1) $\mathbf{G}_D = \tilde{\mathbf{G}}_D$ and $Z_{\mathbf{G}} \subset Z_{\tilde{\mathbf{G}}}$,
- (2) $H^1(F, Z_{\tilde{\mathbf{G}}}) = 1$,
- (3) $\mathbf{G} \subset \tilde{\mathbf{G}}$. Moreover if \mathbf{G} is quasisplit, then so is $\tilde{\mathbf{G}}$. □

6 Local coefficients as Mellin transforms of Bessel functions

We prove the main result of this paper. More precisely, we show that under Assumptions 4.1 and 5.1, and up to an abelian γ -function, the reciprocal $C(s, \pi)^{-1}$ of the local coefficient $C(s, \pi)$, defined by (2.8), can be written as a Mellin transform, over orbits of N under conjugation by $Z_M^0 U_M$, of an incomplete Bessel function of the representation. Since γ -functions from our method are inductively defined by local coefficients [27], and in view of the important paper of Cogdell and Piatetski-Shapiro [9], this may be considered as the first step towards the proof of stability of γ -functions under twisting by highly ramified characters in generality of our method. This will have important applications to transfer of automorphic forms by means of Langlands functoriality [7]. Assumption 5.1 can be dropped as soon as \mathbf{G} is enlarged to satisfy Proposition 5.4, which will not change the value of $C(s, \pi)$ as discussed in Section 5. Of course, \mathbf{M} and consequently π will need to be changed accordingly, but nothing else. The extension of π will not be unique. But $C(s, \pi)$ will remain the same. (See, e.g., [1].) We continue to assume that $\mathbf{P} = \mathbf{M}\mathbf{N}$ is self-associate. The representation π of \mathbf{M} will be any irreducible admissible χ -generic representation. As explained in Section 2, χ will determine an F -splitting $\{X_{\alpha'}\}$ which in turn determines a choice of representatives for w_0 . It will be compatible with χ . The Bessel function J_π is defined by (3.4), whenever it converges. We use ω_π and ω_{π_s} to denote central characters of π and $\pi_s = \pi \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot) \rangle}$, respectively, that is, $\omega_\pi = \pi|_{Z_M}$. Similarly for ω_{π_s} . Then

$$\omega_{\pi_s}(z) = \omega_\pi(z)q^{\langle s\tilde{\alpha}, H_M(z) \rangle} \quad (z \in Z_M). \tag{6.1}$$

The character $w_0(\omega_\pi)$ is then defined by

$$w_0(\omega_\pi)(z) = \omega_\pi(w_0^{-1}zw_0). \tag{6.2}$$

Similarly for $w_0(\omega_{\pi_s})$. A quick check of the Levi subgroup $GL_n \times GL_n$ inside GL_{2n} shows that $w_0(\omega_\pi)$ need not equal ω_π^{-1} . Let ψ_F be the nontrivial additive character of F which

was fixed to define χ as in Section 2. Given a (unitary) character θ of F^* and a complex number $\nu \in \mathbb{C}$, we recall the formal definition of the corresponding abelian γ -function

$$\gamma(\nu, \theta, \psi_F) = \int_{F^*} \theta^{-1}(t) |t|^{1-\nu} \overline{\psi}_F(t) d^*t, \quad (6.3)$$

where the integration is a principal value integral which stabilizes for $|t|$ large. In other words, the integral needs to be taken only over a compact set $|t| \leq \kappa$ for some $\kappa > 0$. In what follows we will be more precise and explain in detail the definition and conventions involved in the definition of $\gamma(\nu, \theta, \psi_F)$. The γ -function $\gamma(\nu, \theta, \psi_F)$ then defines the root number $\epsilon(\nu, \theta, \psi_F)$ through

$$\gamma(\nu, \theta, \psi_F) = \epsilon(\nu, \theta, \psi_F) L(\theta^{-1}, 1 - \nu) / L(\theta, \nu), \quad (6.4)$$

where the L-functions are those of Hecke-Tate, satisfying

$$L_{\widehat{\varphi}}(\theta^{-1}, 1 - \nu) = \gamma(\nu, \theta, \psi_F) L_{\varphi}(\theta, \nu) \quad (6.5)$$

for every $\varphi \in C_c^\infty(F)$. The Fourier transform $\widehat{\varphi}$ is defined by

$$\widehat{\varphi}(t) = \int_F \varphi(tx) \overline{\psi}_F(x) dx, \quad (6.6)$$

where dx is a self-dual measure, that is, that $\widehat{\widehat{\varphi}}(x) = \varphi(-x)$. It is then easily seen that

$$\gamma(\nu, \theta, \psi_F) \gamma(1 - \nu, \theta^{-1}, \overline{\psi}_F) = 1. \quad (6.7)$$

Therefore,

$$\gamma(\nu, \theta, \psi_F)^{-1} = \int_{F^*} \theta(t) |t|^\nu \psi_F(t) d^*t \quad (6.8)$$

which converges for $\operatorname{Re}(\nu) > 0$.

Remark 6.1. We can get these L-functions and γ -factors using our method by considering the standard parabolic subgroup of $SL_2(F)$ with the character, which sends $\operatorname{diag}(t, t^{-1})$ to $\theta(t)|t|^s$ and

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \psi_F(x). \quad (6.9)$$

The corresponding local coefficient is then precisely $\gamma(s, \theta, \psi_F)$. In the notation of Langlands L-functions and in terms of representations of L-groups

$$\gamma(s, \theta, \psi_F) = \gamma((\theta, s), \tilde{r}_1, \bar{\psi}_F), \tag{6.10}$$

where (θ, s) denotes the 1-dimensional representation $\theta(t)|t|^s$ of the diagonal subgroup of $SL_2(F)$ and \tilde{r}_1 is the contragredient of the adjoint action r_1 of ${}^L T$ on ${}^L \mathfrak{n}$, the L-group of T on that of the complex Lie algebra of 2×2 unipotent matrices. This equality agrees with calculations and conventions in [21] and [23, 26, 27], and must be used whenever (2.9) involves a 1-dimensional γ -function. Next, assume that $n \in N$ satisfies (4.1), that is,

$$w_0^{-1}n = mn'n, \tag{6.11}$$

where $m \in M$, $n' \in N$, and $\bar{n} \in \bar{N}$. Our Mellin transform will eventually be an integration over orbits of N under conjugation by $Z_M^0 U_M$. If n satisfies (4.1), then so does every element in its orbit. In particular, given the orbit attached to n , m will belong to a $Z_M^0 U_M$ -Int(w_0)-twisted conjugacy class since

$$w_0^{-1}xn x^{-1} = w_0(x)mx^{-1} \cdot xn'x^{-1} \cdot x\bar{n}x^{-1}, \tag{6.12}$$

where $w_0(x) = w_0^{-1}xw_0$, $x \in G$. Next, note that an arbitrary element \bar{n}_1 of \bar{N} is of the form $w_0 n_1 w_0^{-1}$ for some $n_1 \in N$. We define a character χ' of \bar{N} by

$$\chi'(\bar{n}_1) = \chi(w_0^{-1}\bar{n}_1 w_0). \tag{6.13}$$

Then

$$\chi'(zu\bar{n}_1 u^{-1}z^{-1}) = \chi(\text{Int}(w_0(z))n_1), \tag{6.14}$$

where $\text{Int}(x)y = xyx^{-1}$ for any x and y in G , $z \in Z_M^0$, and $u \in U_M$. Here we are identifying the Weyl group of $T = T(F)$ with that of A_0 . If $n_1 = \prod_{\alpha'} \exp(x_{\alpha'} X_{\alpha'}) n'_1$, where the product is over all the nonrestricted roots of T restricting to α and n'_1 is in the derived group of U , then by (6.14)

$$\chi'(zu\bar{n}_1 u^{-1}z^{-1}) = \psi_F(\alpha'(w_0(z))x_\alpha). \tag{6.15}$$

Here

$$x_\alpha = \sum_{\alpha'} x_{\alpha'} \in F. \tag{6.16}$$

Since α' 's make one Γ -orbit, the value of $\alpha'(w_0(z))$ does not depend on the choice of α' . Next, observe that under [Assumption 5.1](#), we can consider $\omega_{\pi_s}|_{Z_M^0}$ (as well as $\omega_{\pi}|_{Z_M^0}$) as a character of F^* by means of

$$\omega_{\pi_s}(t) = \omega_{\pi_s}(\alpha^\vee(t)). \tag{6.17}$$

As usual, we extend ω_{π_s} to F by $\omega_{\pi_s}(0) = 0$. We define

$$w_0(\omega_{\pi_s})(t) = \omega_{\pi_s}(w_0^{-1}zw_0), \tag{6.18}$$

where $z = \alpha^\vee(t)$, since $w_0^{-1}\alpha^\vee(t)w_0 \equiv \alpha^\vee(t^{-1}) \pmod{Z_G}$ (see [\(6.28\)](#) below). Given n satisfying [\(4.1\)](#), we use \mathfrak{n} , \mathfrak{m} , and $\bar{\mathfrak{n}}$ to denote a set of representatives for each of their orbits when $Z_M^0 U_M$ acts by conjugation on n . Let \dot{x}_α denote the element of F attached to $\bar{\mathfrak{n}}$ in [\(6.16\)](#). We will now fix a measure $d\mathfrak{n}$ on $Z_M^0 U_M \backslash N$ by

$$d\mathfrak{n} = q^{(2\rho, H_M(z))} dz du d\mathfrak{n}, \tag{6.19}$$

where du is a right invariant measure on U_M . In fact, the modulus character for the measure $d\mathfrak{n}$ is simply

$$d(zunu^{-1}z^{-1})/d\mathfrak{n} = d(znz^{-1})/d\mathfrak{n} \tag{6.20}$$

which simply equals $q^{(2\rho, H_M(z))}$. To formulate our main theorem, we need to introduce an incomplete (partial) Bessel function (cf. [\[9\]](#)). Let \bar{N}_0 be an open compact subgroup of \bar{N} . We assume \bar{N}_0 is chosen so that $\alpha^\vee(t)\bar{N}_0\alpha^\vee(t)^{-1}$ depends only on $|t|$ for all $t \in F^*$. Clearly \bar{N} has an exhaustive sequence of such subgroups. Let φ denote the characteristic function of \bar{N}_0 and denote by $\varphi_{|t|}$ the one for $\alpha^\vee(t)\bar{N}_0\alpha^\vee(t)^{-1}$. Fix $W_v \in W(\pi_s)$. Choose $n \in N$ such that $w_0^{-1}n$ satisfies [\(4.1\)](#). Write $w_0^{-1}n = m\mathfrak{n}'\bar{\mathfrak{n}}$. Given $z \in Z_M^0$, define

$$j_{v, \bar{N}_0}(m, z) = \int_{U_{M, n} \backslash U_M} W_v(mu^{-1})\varphi(zu\bar{\mathfrak{n}}u^{-1}z^{-1})\chi(u) du. \tag{6.21}$$

Although $n \mapsto m$ in [\(4.1\)](#) is not an injection in general, [Assumption 4.1](#) implies that there is a bijection between the U_M -conjugacy class of n and the U_M -twisted conjugacy class of m . Consequently, the class of m determines that of n and therefore $\bar{\mathfrak{n}}$ uniquely. This justifies the dependence only on m (rather than $\bar{\mathfrak{n}}$) in the definition of $j_{v, \bar{N}_0}(m, z)$. Assume that W_v satisfies [Assumption 3.1](#), and further that [Assumption 4.1](#) is valid. We can then assume that u^{-1} belongs to a compact subset of U_M modulo $U_{M, \bar{\mathfrak{n}}} = U_{M, n} = U'_{M, m}$. The

first equality follows from the bijection $n \mapsto \bar{n}$ in (4.1). Enlarging \bar{N}_0 , depending both on z and n (and therefore m), we see that

$$\lim_{\bar{N}_0} j_{v, \bar{N}_0}(m, z) = J_{\pi_s}(m)W_v(e), \tag{6.22}$$

where J_{π_s} is the Bessel function of π_s . In particular, (6.21) converges. The limit will be achieved for \bar{N}_0 sufficiently large. Given $z \in Z_M^0$, we consequently call $j_{v, \bar{N}_0}(m, z)$ an *incomplete* or *partial* Bessel function (cf. [9]). Observe that

$$j_{v, \bar{N}_0}(m, z) = j_{v, z^{-1}\bar{N}_0z}(m, 1). \tag{6.23}$$

Finally, assume $z = \alpha^\vee(t)$, where $t = y^{-1} \cdot \dot{x}_\alpha$, $y \in F^*$, and with the above notation, define

$$j_{v, \bar{N}_0}(\dot{m}, y) = j_{v, \bar{N}_0}(\dot{m}, \alpha^\vee(y^{-1} \cdot \dot{x}_\alpha)), \tag{6.24}$$

where as before \dot{n} , \dot{m} , and $\bar{\dot{n}}$ ($w_0^{-1}\dot{n} = \dot{m}\bar{\dot{n}}$) denote a set of representatives for each of their orbits when $Z_M^0U_M$ acts by conjugation on \mathfrak{n} (twisted conjugation for m), and \dot{x}_α is the element of F attached to $\bar{\dot{n}}$ in (6.16). We now claim that, given $y \in F^*$,

$$j_{v, \bar{N}_0}(\dot{m}, y)\omega_{\pi_s}^{-1}(\dot{x}_\alpha)w_0(\omega_{\pi_s})(\dot{x}_\alpha)q^{\langle \rho, H_M(\dot{m}) \rangle} d\dot{n} \tag{6.25}$$

is a well-defined measure on $Z_M^0U_M \setminus N$, through $w_0^{-1}\dot{n} = \dot{m}\bar{\dot{n}}$. We first rewrite $j_{v, \bar{N}_0}(\dot{m}, y)$ as

$$\begin{aligned} & j_{v, \bar{N}_0}(\dot{m}, y) \\ &= \int_{U_{M, \dot{n}} \setminus U_M} W_v(\dot{m}u^{-1})\varphi(u\alpha^\vee(y)^{-1}\alpha^\vee(\dot{x}_\alpha)\bar{\dot{n}}\alpha^\vee(\dot{x}_\alpha)^{-1}\alpha^\vee(y)u^{-1})\chi(u) du. \end{aligned} \tag{6.26}$$

Consequently, changing \dot{n} to $z_1u_1\dot{n}u_1^{-1}z_1^{-1}$, $z_1 = \alpha^\vee(t_1)$, $t_1 \in F^*$,

$$\begin{aligned} & j_{v, \bar{N}_0}(w_0(z_1u_1)\dot{m}u_1^{-1}z_1^{-1}, y) \\ &= \omega_{\pi_s}(w_0(z_1))\omega_{\pi_s}^{-1}(z_1) \int_{U_{M, \dot{n}} \setminus U_M} W_v(\dot{m}u^{-1})\varphi(u\alpha^\vee(y)^{-1}\alpha^\vee(t_1^{-1} \cdot \dot{x}_\alpha)z_1\bar{\dot{n}}z_1^{-1} \\ & \quad \times \alpha^\vee(t_1^{-1} \cdot \dot{x}_\alpha)^{-1}\alpha^\vee(y)u^{-1})\chi(u) du \end{aligned} \tag{6.27}$$

since in (6.26) \dot{x}_α will change to $t_1^{-1} \cdot \dot{x}_\alpha$ as \dot{n} changes to $z_1 u_1 \dot{n} u_1^{-1} z_1^{-1}$. In fact, since

$$\begin{aligned} \alpha'(w_0(z)) &= \alpha'(z^{-1}) \\ &= \alpha'(\alpha^\vee(t^{-1})) \\ &= t^{-1}, \end{aligned} \tag{6.28}$$

(6.15) can be written as

$$\chi'(zu\bar{n}_1 u^{-1} z^{-1}) = \psi_F(t^{-1} x_\alpha). \tag{6.29}$$

Thus changing \dot{n} to $z_1 u_1 \dot{n} u_1^{-1} z_1^{-1}$ will change \dot{x}_α to $t_1^{-1} \cdot \dot{x}_\alpha$, where $z_1 = \alpha^\vee(t_1)$. Moreover, $\omega_{\pi_s}^{-1}(\dot{x}_\alpha)w_0(\omega_{\pi_s})(\dot{x}_\alpha)$ will be multiplied by $\omega_{\pi_s}(t_1)w_0(\omega_{\pi_s})(t_1^{-1})$. Now, since $z_1 = \alpha^\vee(t_1)$, (6.27) immediately implies that

$$j_{\mathfrak{v}, \bar{N}_0}(w_0(z_1 u_1) \dot{n} u_1^{-1} z_1^{-1}, \mathfrak{y}) = \omega_{\pi_s}(w_0(z_1))\omega_{\pi_s}^{-1}(z_1)j_{\mathfrak{v}, \bar{N}_0}(\dot{n}, \mathfrak{y}). \tag{6.30}$$

Note that

$$\omega_{\pi_s}(\alpha^\vee(t_1)^{-1})w_0(\omega_{\pi_s})(\alpha^\vee(t_1)) = \omega_{\pi_s}(t_1^{-1})w_0(\omega_{\pi_s})(t_1). \tag{6.31}$$

Finally, by (6.19),

$$q^{\langle \rho, H_M(w_0(z_1) \dot{n} z_1^{-1}) \rangle} d(z_1 \dot{n} z_1^{-1}) \tag{6.32}$$

equals $q^{\langle \rho, H_M(\dot{n}) \rangle} d\dot{n}$. Thus (6.25) remains unchanged under changing \dot{n} to $z_1 u_1 \dot{n} u_1^{-1} z_1^{-1}$ and our claim follows. Finally, we use \tilde{v} to denote the vector in the space of π , which goes to $v = \tilde{v} \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot) \rangle}$. We use $j_{\tilde{v}, \bar{N}_0}$ to denote the corresponding partial Bessel function defined by (6.21) and (6.24). Next, observe that $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1} \rho_P$ may be considered as a character of M and in fact one of Z_M^0 . Consequently, we can compute

$$q^{\langle s\tilde{\alpha}, H_M(z) \rangle} = |\tilde{\alpha}(\alpha^\vee(t))|^s \tag{6.33}$$

for $z = \alpha^\vee(t)$. Since

$$t \mapsto |\tilde{\alpha}(\alpha^\vee(t))| \tag{6.34}$$

is an unramified character of F^* , we can define $\langle \tilde{\alpha}, \alpha^\vee \rangle \in \mathbb{C}$ such that

$$|\tilde{\alpha}(\alpha^\vee(t))| = |t|^{\langle \tilde{\alpha}, \alpha^\vee \rangle}, \tag{6.35}$$

and therefore,

$$q^{\langle s\tilde{\alpha}, H_M(z) \rangle} = |t|^{\langle \tilde{\alpha}, \alpha^\vee \rangle s}. \tag{6.36}$$

Recall that if $H_\alpha(t)$ is the standard coroot at α , then $\tilde{\alpha}(H_\alpha(t)) = t$, while $\tilde{\alpha}(H_\beta(t)) = 1$ for every other simple root β and therefore $\tilde{\alpha}$ is a fundamental weight. Now consider a few examples: if $G = GL_{2n}$ and $M = GL_n \times GL_n$, then it is easily seen that for

$$\alpha^\vee(t) = \text{diag}(\overbrace{t, t, \dots, t}^{n \text{ times}}, 1, \dots, 1), \tag{6.37}$$

$\langle \tilde{\alpha}, \alpha^\vee \rangle = n/2$ (check [24]). On the other hand, for each of the three cases considered in Section 4, $\tilde{\alpha} = e_1$ and $\langle \tilde{\alpha}, \alpha^\vee \rangle = 1$, for $\alpha^\vee(t) = \text{diag}(t, 1, \dots, 1, t^{-1})$. We can now state the main result of our paper as follows.

Theorem 6.2. Suppose that Assumptions 4.1 and 5.1 are both valid. Then

$$\begin{aligned} C(s, \pi)^{-1} &= \int_{|y| \leq \kappa_0} |y|^{2\langle \tilde{\alpha}, \alpha^\vee \rangle s} \omega_\pi(y) (w_0 \omega_\pi^{-1})(y) \psi_F(y) \\ &\quad \times \int_{Z_M^0 \backslash U_M \backslash N} j_{\tilde{\nu}, \overline{N}_0}(\mathfrak{m}, y) \omega_{\pi_s}^{-1}(\dot{x}_\alpha) (w_0 \omega_{\pi_s})(\dot{x}_\alpha) q^{\langle s\tilde{\alpha} + \rho, H_M(\mathfrak{m}) \rangle} d\mathfrak{n} d^*y, \end{aligned} \tag{6.38}$$

where κ_0 is a positive constant depending only on conductors of ω_π and ψ_F , \overline{N}_0 is a sufficiently large open compact subgroup of \overline{N} for which $\alpha^\vee(t)\overline{N}_0\alpha^\vee(t)^{-1}$ depends only on $|t|$ for all $t \in F^*$, and $W_{\tilde{\nu}}(e) = 1$. The constant κ_0 and the open compact subgroup \overline{N}_0 can be replaced by any larger value and open compact subgroup of the same type, respectively. The incomplete Bessel function $j_{\tilde{\nu}, \overline{N}_0}(\mathfrak{m}, y)$ is defined by (6.21) and (6.24). Moreover, assume that $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified. Then

$$\begin{aligned} C(s, \pi)^{-1} &= \gamma(2\langle \tilde{\alpha}, \alpha^\vee \rangle s, \omega_\pi(w_0\omega_\pi^{-1}), \psi_F)^{-1} \\ &\quad \times \int_{Z_M^0 \backslash U_M \backslash N} j_{\tilde{\nu}, \overline{N}_0}(\mathfrak{m}) \omega_{\pi_s}^{-1}(\dot{x}_\alpha) (w_0 \omega_{\pi_s})(\dot{x}_\alpha) q^{\langle s\tilde{\alpha} + \rho, H_M(\mathfrak{m}) \rangle} d\mathfrak{n}, \end{aligned} \tag{6.39}$$

where $j_{\tilde{\nu}, \overline{N}_0}(\mathfrak{m}) = j_{\tilde{\nu}, \overline{N}_0}(\mathfrak{m}, y_0)$ with $\text{ord}_F(y_0) = -d - f$, where d and f are conductors of ψ_F and $\omega_\pi^{-1} \cdot (w_0 \omega_\pi)$, respectively. The choice of y_0 is irrelevant. Particularly, the integral in (6.39) is independent of the choice of $\tilde{\nu}$ and \overline{N}_0 so long as $W_{\tilde{\nu}}(e) = 1$ and \overline{N}_0 is sufficiently large. □

Remark 6.3. In the case of $GL_2 \times GL_2$ inside GL_4 , (6.38) is basically the content of an important lemma of Soudry [32, Lemma 4.5].

Remark 6.4. Assumption 3.1, although being necessary to motivate the definition of an incomplete Bessel function by means of equation (6.21), is not necessary for the proof of Theorem 6.2. The convergence of (6.21) for arbitrary W_v follows from Fubini’s theorem as a biproduct of our proof.

Proof. We need to compute the expression

$$\lambda_\chi(-s, w_0(\pi))A(s, \pi), \tag{6.40}$$

which is equal to

$$C(s, \pi)^{-1}\lambda_\chi(s, \pi) \tag{6.41}$$

by (2.8), for an appropriate function $f \in V(s, \pi)$. If we replace f by $R_{w_0^{-1}}f$, this means to evaluate

$$\int_{\bar{N}} \langle A(s, \pi)f(\bar{n}_1), \lambda \rangle \bar{\chi}(w_0^{-1}\bar{n}_1w_0) d\bar{n}_1 = \int_{\bar{N}} \langle A(s, \pi)f(\bar{n}_1), \lambda \rangle \bar{\chi}'(\bar{n}_1) d\bar{n}_1, \tag{6.42}$$

where $A(s, \pi)$ is defined by (2.6). From the general theory of Whittaker functionals (cf. [6, 23]), the integral over \bar{N} stabilizes and therefore we may replace \bar{N} by an open compact subgroup of it, say, \bar{N}_0 , which can be enlarged arbitrarily. The choice of \bar{N}_0 does not depend on s . (See Step (2) of the proof of [6, Lemma 2.2, page 214] in the unramified case. The general case follows precisely the same steps.) The choice of f will not effect the result and the matters become a lot simpler if we assume that f has a compact support in $\bar{P}\bar{N}$ modulo P which from now on we will assume to be the case. Moreover, we will assume \bar{N}_0 to contain the support of f modulo P . If we assume $\text{Re}(s) > 0$, in which case we can use the absolute convergence of intertwining operators, then (6.42) can be written as

$$\int_{\bar{N}_0} \left\langle \int_N f(w_0^{-1}n\bar{n}_1) dn, \lambda \right\rangle \bar{\chi}'(\bar{n}_1) d\bar{n}_1 \tag{6.43}$$

or

$$\int_{\bar{N}_0} \left\langle \int_N \pi_s(m)f(\bar{n}\bar{n}_1) q^{\langle \rho_P, H_M(m) \rangle} dn, \lambda \right\rangle \bar{\chi}'(\bar{n}_1) d\bar{n}_1, \tag{6.44}$$

where the integral over N is over those n for which $w_0^{-1}n$ satisfy (4.1). Under the assumption that \bar{N}_0 contains the support of f modulo P , we can change \bar{n}_1 to $(\bar{n})^{-1}\bar{n}_1$, to get

$$\int_{\bar{N}_0} \left\langle \int_N \pi_s(m)f(\bar{n}_1), \lambda \right\rangle q^{\langle \rho_P, H_M(m) \rangle} \chi'(\bar{n})\bar{\chi}'(\bar{n}_1) dn d\bar{n}_1. \tag{6.45}$$

Since $\text{Re}(s) > 0$, we can apply Fubini's theorem to write this as

$$\int_{\mathbb{N}} \left\langle \pi_s(\mathfrak{m}) \int_{\overline{\mathbb{N}}_0} f(\bar{n}_1) \overline{\chi'}(\bar{n}_1) d\bar{n}_1, \lambda \right\rangle q^{\langle \rho, H_M(\mathfrak{m}) \rangle} \chi'(\bar{n}) dn. \tag{6.46}$$

Let

$$v = \int_{\overline{\mathbb{N}}_0} f(\bar{n}_1) \overline{\chi'}(\bar{n}_1) d\bar{n}_1. \tag{6.47}$$

It is independent of $\overline{\mathbb{N}}_0$, since the support of f modulo P is contained in $\overline{\mathbb{N}}_0$. Formula (6.46) is now equal to

$$\int_{\mathbb{N}} \langle \pi_s(\mathfrak{m})v, \lambda \rangle \varphi(\bar{n}) q^{\langle \rho, H_M(\mathfrak{m}) \rangle} \chi'(\bar{n}) dn, \tag{6.48}$$

where φ is the characteristic function of $\overline{\mathbb{N}}_0$. Observe that (2.7), that is, the left-hand side of (2.8), when evaluated at $R_{w_0^{-1}} f$, equals

$$\int_{\overline{\mathbb{N}}_0} \langle f(\bar{n}_1), \lambda \rangle \overline{\chi'}(\bar{n}_1) d\bar{n}_1 \tag{6.49}$$

or

$$\left\langle \int_{\overline{\mathbb{N}}_0} f(\bar{n}_1) \overline{\chi'}(\bar{n}_1) d\bar{n}_1, \lambda \right\rangle = W_v(e). \tag{6.50}$$

From now on we assume $W_v(e) = 1$. Consequently, (2.8) and (6.48) imply that

$$C(s, \pi)^{-1} = \int_{\mathbb{N}} \langle \pi_s(\mathfrak{m})v, \lambda \rangle \varphi(\bar{n}) q^{\langle \rho, H_M(\mathfrak{m}) \rangle} \chi'(\bar{n}) dn. \tag{6.51}$$

Finally, we write (6.51) as

$$C(s, \pi)^{-1} = \int_{\mathbb{N}} W_v(\mathfrak{m}) \varphi(\bar{n}) q^{\langle \rho, H_M(\mathfrak{m}) \rangle} \chi'(\bar{n}) dn, \tag{6.52}$$

where

$$W_v(\mathfrak{m}) = \langle \pi_s(\mathfrak{m})v, \lambda \rangle \tag{6.53}$$

gives the value of the corresponding element $W_v \in W(\pi_s)$ at \mathfrak{m} . We now start by expanding the right-hand side of (6.52) by integrating over orbits of \mathbb{N} under conjugation by U_M . Since sending n to unu^{-1} will send \mathfrak{m} to $w_0(u)\mathfrak{m}u^{-1}$, while \bar{n} goes to $u\bar{n}u^{-1}$, the

right-hand side of (6.52) now becomes

$$C(s, \pi)^{-1} = \int_{U_M \setminus N} \left(\int_{U_{M, \bar{n}} \setminus U_M} W_v(\mu u^{-1}) \varphi(u \bar{n} u^{-1}) \chi(u) \right. \\ \left. \times \chi'(u \bar{n} u^{-1}) du \right) q^{\langle \rho, H_M(\bar{m}) \rangle} d\bar{n}, \quad (6.54)$$

using the compatibility of χ and w_0 , that is, $\chi(w_0(u)) = \chi(u)$. Here we need to use $U_{M, \bar{n}} = U'_{M, \bar{m}}$ which is valid by Assumption 4.1. Finally, since $\chi'(u \bar{n} u^{-1}) = \chi'(\bar{n})$, (6.54) can be written as

$$C(s, \pi)^{-1} = \int_{U_M \setminus N} \left(\int_{U_{M, \bar{n}} \setminus U_M} W_v(\mu u^{-1}) \varphi(u \bar{n} u^{-1}) \chi(u) du \right) \\ \times \chi'(\bar{n}) q^{\langle \rho, H_M(\bar{m}) \rangle} d\bar{n}. \quad (6.55)$$

Now we incorporate conjugation over Z_M^0 as well and using (6.19) and (6.55) conclude that

$$C(s, \pi)^{-1} \\ = \int_{Z_M^0 U_M \setminus N} \int_{Z_M^0} \left(\int_{U_{M, \bar{n}} \setminus U_M} W_v(w_0(z) \dot{m} z^{-1} u^{-1}) \varphi(z u \bar{n} u^{-1} z^{-1}) \chi(u) du \right) \\ \times \chi'(z \bar{n} z^{-1}) q^{\langle \rho, H_M(w_0(z) \dot{m} z^{-1}) \rangle} q^{\langle 2\rho, H_M(z) \rangle} dz d\dot{n}. \quad (6.56)$$

Here, \dot{n} , \dot{m} , and \bar{n} are the sets of representatives under $Z_M^0 U_M$ -conjugation (twisted conjugation for m), satisfying $w_0^{-1} \dot{n} = \dot{m} \bar{n}$, for each $n \in N$ satisfying (2.8). Since

$$q^{\langle \rho, H_M(w_0(z) \dot{m} z^{-1}) \rangle} = q^{-\langle 2\rho, H_M(z) \rangle} q^{\langle \rho, H_M(\dot{m}) \rangle}, \quad (6.57)$$

using (6.15), (6.56) can be written as

$$\int_{Z_M^0 U_M \setminus N} \int_{Z_M^0} \left(\int_{U_{M, \bar{n}} \setminus U_M} W_v(\dot{m} u^{-1}) \varphi(z u \bar{n} u^{-1} z^{-1}) \chi(u) du \right) \\ \times \omega_{\pi_s}(w_0(z) z^{-1}) \psi_F(\alpha'(w_0(z)) \dot{\chi}_\alpha) q^{\langle \rho, H_M(\dot{m}) \rangle} dz d\dot{n}. \quad (6.58)$$

Invoking Assumption 5.1, (6.28) implies that

$$\omega_{\pi_s}(w_0(z) z^{-1}) \psi_F(\alpha'(w_0(z)) \dot{\chi}_\alpha) = \omega_{\pi_s}(t^{-1})(w_0 \omega_{\pi_s}^{-1})(t^{-1}) \psi_F(t^{-1} \dot{\chi}_\alpha), \quad (6.59)$$

where $z = \alpha^\vee(t)$. Using this equality and (6.21), and assuming $\text{Re}(s) \gg 0$, we can now apply Fubini's theorem to (6.58) so as to write it as

$$\int_{Z_M^0} \omega_{\pi_s}(t^{-1}\dot{x}_\alpha)(w_0\omega_{\pi_s}^{-1})(t^{-1}\dot{x}_\alpha)\psi_F(t^{-1}\dot{x}_\alpha) \times \int_{Z_M^0 \cup_M \setminus N} j_{v, \bar{N}_0}(\dot{m}, z)\omega_{\pi_s}^{-1}(\dot{x}_\alpha)(w_0\omega_{\pi_s})(\dot{x}_\alpha)q^{\langle \rho, H_M(\dot{m}) \rangle} d\dot{n} d^*(t^{-1}\dot{x}_\alpha). \tag{6.60}$$

Setting $y = t^{-1}\dot{x}_\alpha$ and using (6.24), (6.60) equals

$$\int_{F^*} \omega_{\pi_s}(y)(w_0\omega_{\pi_s}^{-1})(y)\psi_F(y) \times \int_{Z_M^0 \cup_M \setminus N} j_{v, \bar{N}_0}(\dot{m}, y)\omega_{\pi_s}^{-1}(\dot{x}_\alpha)(w_0\omega_{\pi_s})(\dot{x}_\alpha)q^{\langle \rho, H_M(\dot{m}) \rangle} d\dot{n} d^*y. \tag{6.61}$$

Finally, we claim that under our assumption on \bar{N}_0 , $j_{v, \bar{N}_0}(\dot{m}, y)$ depends only on $|y|$. This follows from

$$j_{v, \bar{N}_0}(\dot{m}, y) = \int_{U_{M, n} \setminus U_M} W_v(\dot{m}u^{-1})\varphi_{|y|}(u\alpha^\vee(\dot{x}_\alpha)\bar{n}\alpha^\vee(\dot{x}_\alpha)^{-1}u^{-1})\chi(u) du, \tag{6.62}$$

in which, we recall that $\varphi_{|y|}$ is the characteristic function of $\alpha^\vee(y)\bar{N}_0\alpha^\vee(y)^{-1}$, which depends only on $|y|$ by our assumption on \bar{N}_0 . Let

$$\theta(|y|) = \int_{Z_M^0 \cup_M \setminus N} j_{v, \bar{N}_0}(\dot{m}, y)\omega_{\pi_s}^{-1}(\dot{x}_\alpha)(w_0\omega_{\pi_s})(\dot{x}_\alpha)q^{\langle \rho, H_M(\dot{m}) \rangle} d\dot{n}. \tag{6.63}$$

Equation (6.61) is now clearly

$$\int_{F^*} \omega_{\pi_s}(y)(w_0\omega_{\pi_s}^{-1})(y)\psi_F(y)\theta(|y|) d^*y. \tag{6.64}$$

It is well known that the integral over F^* can be achieved by integrating over P^{-f-d-1} , where f and d are conductors of $\omega_\pi \cdot (w_0\omega_\pi^{-1})$ and ψ_F , respectively. The first assertion of Theorem 6.2 is now proved. For (6.39), we only need to use the well-known fact that if $\eta \in \widehat{F}^*$ is ramified

$$\int_{O^*} \eta(x)\psi_F(xy) d^*x \neq 0 \tag{6.65}$$

if and only if $\text{ord}_F(y) = -d - f$. ■

Corollary 6.5. Let W be a Whittaker function in the Whittaker model $W(\pi)$ of π and fix an open compact subgroup $\bar{N}_0 \subset \bar{N}$ as in [Theorem 6.2](#). Let φ be its characteristic function. Then the integral

$$\int_{U_{M,n} \backslash U_M} W(mu^{-1}) \varphi(u\bar{n}u^{-1}) \chi(u) du \tag{6.66}$$

is absolutely convergent for every triple (n, m, \bar{n}) satisfying $w_0^{-1}n = mn'\bar{n}$. □

Proof. This follows from applying Fubini’s theorem to the convergence of [\(6.54\)](#) for $\text{Re}(s) \gg 0$ and the fact that [\(6.66\)](#) is equal to $q^{(s\bar{\alpha}, H_M(m))}$ times $j_{\bar{v}, \bar{N}_0}(m, 1)$, if $W = W_{\bar{v}}$. ■

7 Classical groups

We now apply [Theorem 6.2](#) to the three cases of classical groups we discussed in [Section 4](#). It is remarkable that in the case of $SO_{2n+1}(F)$ ([Proposition 7.2.a](#)), this leads to the basically same formulas as those of Cogdell and Piatetski-Shaprio [\[9\]](#). On the other hand, our [Propositions 7.2.b](#) and [7.3](#) establish new formulas for γ -functions for $GL_1(F) \times SO_{2n}(F)$ and $GL_1(F) \times Sp_{2n}(F)$, respectively, which are strikingly similar to the case of $SO_{2n+1}(F)$. In each case $Z_G \backslash Z_M \simeq A$. Moreover, while [Assumptions 4.1](#) and [5.1](#) are valid in all three cases, [Assumption 3.1](#), though not necessary for these results, needs to be verified, at least for representatives given by [Propositions 4.4](#), [4.8](#), and [4.11](#), but only for $M = GL_1 \times Sp_{2n}$ inside $G = Sp_{2n+2}$, thanks to [\[4, 9\]](#). It is quite easy to see that, using the representatives given in above propositions, our incomplete Bessel functions are precise analogues of those in [\[9\]](#). In view of [Propositions 4.4](#), [4.8](#), and [4.11](#), we have a set of representatives for each AU_M -Int(w_0)-twisted conjugacy class of m by means of a well-defined element k in $SO_{2n+1}(F)$, $SO_{2n}(F)$, or $Sp_{2n}(F)$. To apply [Theorem 6.2](#), we need to determine $\dot{\chi}_\alpha$ as well as the measure $q^{(\rho, H_M(m))} dn$ in each case. We first address the two cases of special orthogonal groups, that is, $G = SO_{2n+3}$ and SO_{2n+2} . Recall that $M = GL_1(F) \times SO_{2n+1}(F)$ or $M = GL_1(F) \times SO_{2n}(F)$, respectively. The representation $\pi = \eta \otimes \sigma$, where η is a character of F^* and σ is an irreducible admissible χ -generic representation of $SO_{2n+1}(F)$ or $SO_{2n}(F)$, respectively. As in [Section 4](#), we need to consider the decomposition

$$\begin{aligned} w_0^{-1}n(t) &= mn'\bar{n} \\ &= \begin{pmatrix} a & ax & -\frac{axx^*}{2} \\ 0 & k & -kx^* \\ 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & I_{2n+1} & 0 \\ -\frac{1}{2}y^*y & -y^* & 1 \end{pmatrix} \end{aligned} \tag{7.1}$$

for $G = SO_{2n+3}$, where w_0 is defined as in Section 4 in each case. For $G = SO_{2n+2}$, the only change is I_{2n+1} to I_{2n} in the second matrix. Recall from Lemmas 4.2 and 4.6, or simply upon using (7.1), that $a = -2/\langle t, t \rangle$, $\langle t, t \rangle = tt^*$. Moreover, (7.1) implies that $y^* = -at = 2t/\langle t, t \rangle$ in both cases. We need to compute \dot{x}_α from $w_0^{-1}\bar{n}w_0$. It is easily found to equal the first coordinate of the vector y^* . Our representatives from Propositions 4.4 and 4.8 correspond to $t = (c, 0, \dots, 0, b)$, $(b, c) \in (F^*)^2$, which lead to

$$\dot{x}_\alpha = b^{-1}, \tag{7.2}$$

using $\langle t, t \rangle = 2bc$. Moreover,

$$a = -2/\langle t, t \rangle = -b^{-1}c^{-1}. \tag{7.3}$$

As suggested by Propositions 4.4, 4.8, and 4.11, in all three cases we need to consider the subgroup H of the corresponding classical groups $SO_{2n+1}(F)$, $SO_{2n}(F)$, and $Sp_{2n}(F)$ defined as

$$H = \{ \underline{h} = \text{diag}(h, 1, \dots, 1, h^{-1}) \mid h \in F^* \}, \tag{7.4}$$

following the lead from Cogdell and Piatetski-Shapiro’s notation in [9]. The measure $d\underline{h} = d^*h$ is simply the multiplicative measure of F . We need to relate this to our invariant measure $q^{(\rho, H_M(\dot{m}))} d\dot{n}$, if we want to get expressions similar to those of [9, Proposition 4.1]. We have the following lemma.

Lemma 7.1. Assume $G = SO_{2n+3}$ or SO_{2n+2} . Parametrize the cosets of $AU_M \backslash N$ by elements \underline{h} of H according to Propositions 4.4 and 4.8, respectively. Let $\rho = \rho_p$ and let the simple root α be as before, that is, $\alpha = e_1 - e_2$ in both cases. Then

$$q^{(\rho, H_M(\dot{m}))} d\dot{n} = |\underline{h}|^{-\langle \rho, \alpha \rangle + 1} d\underline{h}. \tag{7.5}$$

Consequently, $q^{(\rho, H_M(\dot{m}))} d\dot{n}$ equals $|\underline{h}|^{-n+1/2} d\underline{h}$ or $|\underline{h}|^{-n+1} d\underline{h}$, according as $G = SO_{2n+3}$ or SO_{2n+2} , respectively. □

Proof. As discussed before, every coset of N under AU_M , satisfying (4.1), can be represented by an element $x((1, 0, \dots, 0, h))$, $h \in F^*$, $(1, 0, \dots, 0, h) \in F^{2n+1}$ or F^{2n} according as $G = SO_{2n+3}$ or $G = SO_{2n+2}$. In fact, with the $(b, 0, \dots, 0, c)$ -representative discussed

earlier, $b = h$ and $c = 1$. Then, by (7.3), $a = -b^{-1}c^{-1} = -h^{-1}$ and therefore,

$$\begin{aligned} q^{\langle \rho, H_M(\dot{m}) \rangle} &= |\rho(\dot{m})| \\ &= |\rho(H_\alpha(h^{-1}))| \\ &= |h|^{-\langle \rho, \alpha \rangle}. \end{aligned} \tag{7.6}$$

The measure is now being invariant and is just the additive measure $dh = |h| d^*h$. The lemma follows. ■

To use Theorem 6.2 we need to compute

$$j_{\tilde{v}, \bar{N}_0}(\dot{m}) \omega_{\pi_s}^{-1}(\dot{\chi}_\alpha) (w_0 \omega_{\pi_s})(\dot{\chi}_\alpha) q^{\langle s\tilde{\alpha}, H_M(\dot{m}) \rangle}, \tag{7.7}$$

where $j_{\tilde{v}, \bar{N}_0}(\dot{m})$ is as in Theorem (6.39). Since $\pi = \sigma \otimes \eta$, by Propositions 4.4 and 4.8

$$\begin{aligned} j_{\tilde{v}, \bar{N}_0}(\dot{m}) &= \eta(a) j_{\tilde{v}, \bar{N}_0} \left(\begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix} \right) \\ &= \eta(-b^{-1}c^{-1}) j_{\tilde{v}, \bar{N}_0} \left(\begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -I_{2n-1} & & \\ 1 & & \end{pmatrix} \right), \end{aligned} \tag{7.8}$$

if $G = SO_{2n+3}$, and

$$j_{\tilde{v}, \bar{N}_0}(\dot{m}) = \eta(-b^{-1}c^{-1}) j_{\tilde{v}, \bar{N}_0} \left(\begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ -K_{2n-2} & & \\ 1 & & \end{pmatrix} \right), \tag{7.9}$$

if $G = SO_{2n+2}$. Next, observe that the rest of the product in (7.7) is now equal to

$$\omega_{\pi_s}^{-1}(b^{-1}) (w_0 \omega_{\pi_s})(b^{-1}) q^{\langle s\tilde{\alpha}, H_M(\dot{m}) \rangle} = \eta(b^2) |b^2|^s |b^{-1}c^{-1}|^s, \tag{7.10}$$

using

$$\begin{aligned} q^{\langle -s\tilde{\alpha}, H_\alpha(b^{-1}) \rangle} &= |\rho(H_\alpha(b))|^{s/\langle \rho, \alpha \rangle} = |b|^s, \\ q^{\langle s\tilde{\alpha}, H_M(\dot{m}) \rangle} &= |\rho(H_\alpha(-b^{-1}c^{-1}))|^{s/\langle \rho, \alpha \rangle} = |b^{-1}c^{-1}|^s. \end{aligned} \tag{7.11}$$

Observe that $h = bc^{-1}$. Putting these together with [Lemma 7.1](#), we have the following proposition.

Proposition 7.2. (a) Suppose that $\mathbf{G} = \mathrm{SO}_{2n+3}$ and $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{2n+1}$. Let $\eta \in \widehat{\mathbf{F}}^*$ and fix an irreducible admissible χ -generic representation σ of $\mathrm{SO}_{2n+1}(\mathbf{F})$. Assume that η^2 is ramified. Then with \tilde{v} and $\overline{\mathbf{N}}_0$ as in [\(6.39\)](#)

$$\begin{aligned} & C(s, \eta \otimes \sigma)^{-1} \\ &= \eta(-1)\gamma(2s, \eta^2, \psi_{\mathbf{F}})^{-1} \\ & \times \int_{\mathbf{F}^*} j_{\tilde{v}, \overline{\mathbf{N}}_0} \left(\begin{pmatrix} h & & & \\ & I_{2n-1} & & \\ & & h^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & -I_{2n-1} & & \\ & & & \\ 1 & & & \end{pmatrix} \right) \eta(h)|h|^{s-n+1/2} d^*h. \end{aligned} \tag{7.12}$$

(b) Suppose that $\mathbf{G} = \mathrm{SO}_{2n+2}$ and $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{SO}_{2n}$. Let $\eta \in \widehat{\mathbf{F}}^*$ and fix an irreducible admissible χ -generic representation σ of $\mathrm{SO}_{2n}(\mathbf{F})$. Assume that η^2 is ramified. Then with \tilde{v} and $\overline{\mathbf{N}}_0$ as in [\(6.39\)](#)

$$\begin{aligned} & C(s, \eta \otimes \sigma)^{-1} \\ &= \eta(-1)\gamma(2s, \eta^2, \psi_{\mathbf{F}})^{-1} \\ & \times \int_{\mathbf{F}^*} j_{\tilde{v}, \overline{\mathbf{N}}_0} \left(\begin{pmatrix} h & & & \\ & I_{2n-2} & & \\ & & h^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & 1 \\ & -K_{2n-2} & & \\ & & & \\ 1 & & & \end{pmatrix} \right) \eta(h)|h|^{s-n+1} d^*h, \end{aligned} \tag{7.13}$$

where

$$K_{2n-2} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}. \tag{7.14}$$

□

We now turn to the case of symplectic groups. Let $\mathbf{G} = \mathrm{Sp}_{2n+2}$ and $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{Sp}_{2n}$. Again $\pi = \eta \otimes \sigma$, where $\eta \in \widehat{\mathbf{F}}^*$, while σ is an irreducible admissible χ -generic representation of $\mathrm{Sp}_{2n}(\mathbf{F})$. To determine $\dot{\chi}_\alpha$, we need, as in [Section 4](#), to consider

the decomposition $w_0^{-1}n(t, T) = mn'\bar{n}$, that is,

$$\begin{pmatrix} & & 1 \\ & -I_{2n} & \\ -1 & & \end{pmatrix} \begin{pmatrix} 1 & t & T \\ 0 & I_{2n} & t' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax & aX \\ 0 & k & kx' \\ 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & I_{2n} & 0 \\ Y & y' & 1 \end{pmatrix}. \tag{7.15}$$

This gives $a = -T^{-1}$, $Y = T^{-1}$, and $y' = -at = t/T$. To proceed for the value of $\dot{\chi}_\alpha$, we calculate

$$w_0^{-1}\bar{n}w_0 = \begin{pmatrix} 1 & -y' & -Y \\ 0 & I_{2n} & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.16}$$

Then $\dot{\chi}_\alpha$ is equal to the first coordinate of $-y'$. We will choose our standard representative of Lemma 4.10 and Proposition 4.11, that is, we take $t = {}^tBJ_n$, $B = {}^t(b, 0, \dots, 0, c) \in F^{2n}$ and $T = bc$. We then get

$$\dot{\chi}_\alpha = b^{-1}. \tag{7.17}$$

Moreover,

$$a = -T^{-1} = -b^{-1}c^{-1} \tag{7.18}$$

upon using Lemma 4.10 and Proposition 4.11. Thus everything is in complete agreement with previous cases. The argument with measures in Lemma 7.1 goes through just the same and we get

$$q^{\langle \rho, H_M(\dot{m}) \rangle} d\dot{n} = |h|^{-n} d\underline{h}, \tag{7.19}$$

since $\langle \rho, \alpha \rangle = n + 1$. To have a formula which looks like the two other cases (7.12) and (7.13), we need to be more careful since now

$$k = \begin{pmatrix} h & & \\ & I_{2n-2} & \\ & & h^{-1} \end{pmatrix} \begin{pmatrix} & & -1 \\ & -I_{2n-2} & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.20}$$

gives, up to an element in F^* , a representative \dot{m} for each $Z_M^0 U_M$ -twisted conjugacy class

of \mathfrak{m} (Proposition 4.11). Let

$$H = \begin{pmatrix} 1 & 0 & -2h \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.21}$$

and observe that H would have been dropped out if we were dealing with the full Bessel function J_π . But, this not being the case, we need to take H into account, though its effect is practically only a change of sign as we explain below. A quick calculation using (7.15) shows that with $t = {}^tBJ_n$, $B = {}^t(h, 0, \dots, 0, 1)$, and $T = h$, $y = {}^t(-1, 0, \dots, 0, -h^{-1})$, $y' = {}^t(-h^{-1}, 0, \dots, 0, 1)$, and $Y = h^{-1}$. This then gives \bar{n} as

$$\bar{n} = \begin{pmatrix} 1 & 0 & 0 \\ y & I_{2n} & 0 \\ Y & y' & 1 \end{pmatrix}. \tag{7.22}$$

Moreover, if

$$H\bar{n}H^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ y_1 & I_{2n} & 0 \\ Y_1 & y'_1 & 1 \end{pmatrix}, \tag{7.23}$$

then $y_1 = {}^t(1, 0, \dots, 0, -h^{-1})$, $y'_1 = {}^t(-h^{-1}, 0, \dots, 0, -1)$, and $Y_1 = Y = h^{-1}$, which is obtained from \bar{n} by only changing the coordinates ± 1 to ∓ 1 . It is then easily checked, using a change of variables in (6.21), that

$$j_{\mathfrak{v}, \bar{N}_0}(\mathfrak{m}, z) = \int_{U_{\mathfrak{M}, \bar{n}} \backslash U_{\mathfrak{M}}} W_{\mathfrak{v}}(\mathfrak{m}H^{-1}u^{-1})\varphi(zuH\bar{n}H^{-1}u^{-1}z^{-1})\chi(u) du. \tag{7.24}$$

Set

$$j'_{\mathfrak{v}, \bar{N}_0}(\mathfrak{m}H^{-1}, z) = j_{\mathfrak{v}, \bar{N}_0}(\mathfrak{m}, z). \tag{7.25}$$

Finally, define $j'_{\mathfrak{v}, \bar{N}_0}(\mathfrak{m}H^{-1})$ by means of (6.24) and a y_0 satisfying $\text{ord}_F(y_0) = -d - f$ as in Theorem 6.2. Our final result is the following proposition.

Proposition 7.3. Suppose $\mathbf{G} = \text{Sp}_{2n+2}$ and $\mathbf{M} = \text{GL}_1 \times \text{Sp}_{2n}$. Let $\eta \in \widehat{F}^*$ and fix an irreducible admissible χ -generic representation σ of $\text{Sp}_{2n}(F)$. Suppose that η^2 is ramified.

Then with \tilde{v} and \overline{N}_0 as in (6.39) and $j'_{\tilde{v}, \overline{N}_0}$ as above

$$\begin{aligned} & C(s, \eta \otimes \sigma)^{-1} \\ &= \eta(-1) \gamma(2s, \eta^2, \psi_F)^{-1} \\ & \quad \times \int_{F^*} j'_{\tilde{v}, \overline{N}_0} \left(\begin{pmatrix} \mathfrak{h} & & & \\ & I_{2n-2} & & \\ & & \mathfrak{h}^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & & & -1 \\ & & & \\ & -I_{2n-2} & & \\ 1 & & & \end{pmatrix} \right) \eta(\mathfrak{h}) |\mathfrak{h}|^{s-n} d^* \mathfrak{h}. \end{aligned} \tag{7.26}$$

□

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Freydoon Shahidi: Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

E-mail address: shahidi@math.purdue.edu