

Final Exam – Part 1 (Due April 29)

1 Rules

This is part of the take-home final exam. All students are expected to complete the project **independently**.

2 The problem

Consider the partial differential equation

$$u_t = u^2 - u_x, \quad x \in (0, 1), \quad t \in (0, \frac{1}{2}], \quad (1)$$

with the boundary condition $u(0, t)$ and initial condition $u(x, 0)$ given by the exact solution

$$u(x, t) = \frac{\sin^2(\pi(x - t))}{1 - t \sin^2(\pi(x - t))}. \quad (2)$$

Let $x_j = j/N$, $j = 0, 1, \dots, N$.

2.1 Space discretization:

Using a fourth-order central difference scheme for the interior points and one-sided third-order approximation at the boundary, one arrives to a system of ODEs.

2.2 Time discretization:

Use the following three splitting schemes to solve this system of ODEs:

1. Take $A_1 u = u^2$ and $A_2 u = -u_x$. Use the simple splitting scheme with corrected boundary conditions.
2. Take $A_1 u = -u_x$ and $A_2 u = u^2$. Use the simple splitting scheme with corrected boundary conditions.
3. Transform the equation (1) to an equation with homogeneous boundary conditions and use the Strang's splitting method.

The reaction step, whenever possible, is to be solved exactly, while the advection step (and the reaction step in the third scheme) is to be solved by using the classical fourth-order Runge-Kutta method.

2.3 The task:

For each of the three splitting schemes, compute an approximate solution at $t = \frac{1}{2}$ with time steps $\tau = 1/20, 1/40, 1/80, 1/160$ and $N = \frac{2}{\tau}$.

Produce a table (as the Table 1.1 in P. 346 of Hundsdorfer and Verwer's book) which list the errors at the final time $t = \frac{1}{2}$ and estimate the convergence rates of the three splitting schemes. Comment on your results.