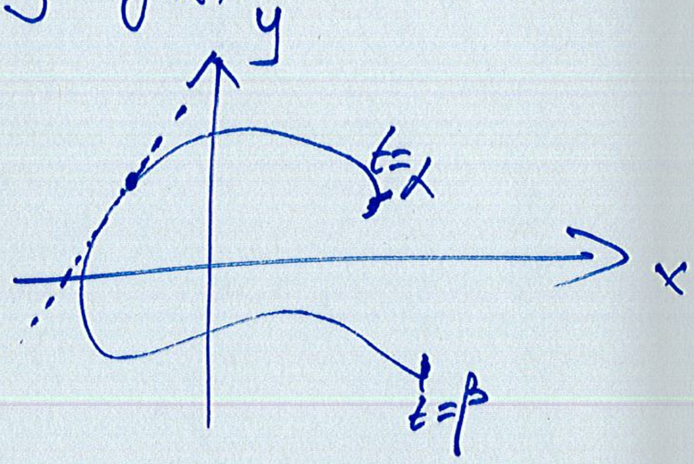
Review for chap 10

* Parametric equations: $\left(y = F(x) \right)$

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \alpha \le t \le \beta$$

(i) slope of the tangent line



$$= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{g'(t)}{f'(t)}$$

(ii) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{g'(t)}{f'(t)}\right) = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{\frac{dx}{dt}}$$

Ex. $x = 2\cos t$, $y = 2\sin t$, $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

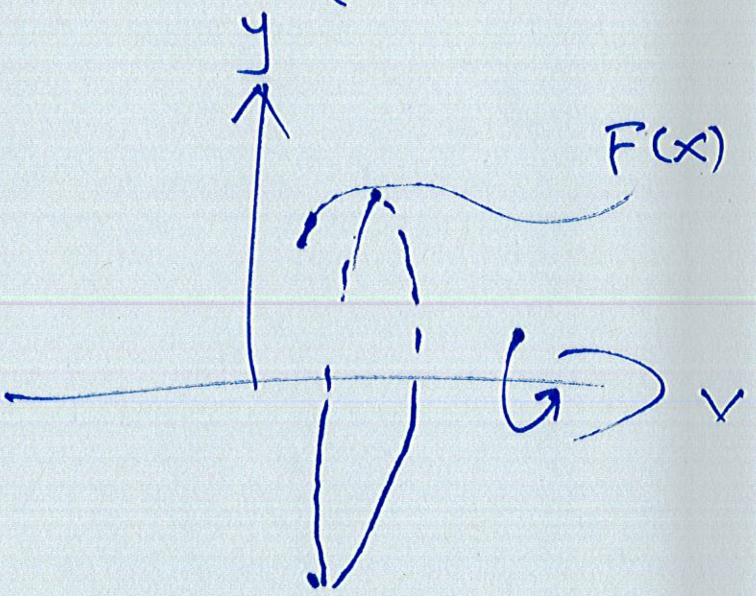
$$\frac{dy}{dx} = \frac{2\cos t}{-2\sin t} = -\cot t, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-\cot t)}{-2\sin t}.$$

Arc Length: $\left(y = F(x)\right)$ $\qquad \begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \alpha \le t \le \beta.$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$\underline{\underline{x = f(t)}}$ $\quad \underline{\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt} \quad \left(= \int_\alpha^\beta ds \right)$

Surface area:

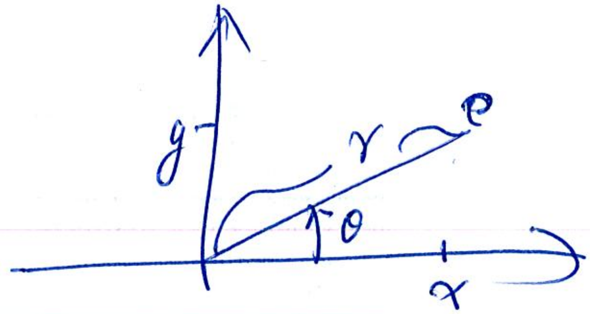


$$S = \int_a^b 2\pi y \, ds$$

$\underline{\underline{x = f(t)}}$ $\quad \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

* polar coordinates (special case)

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



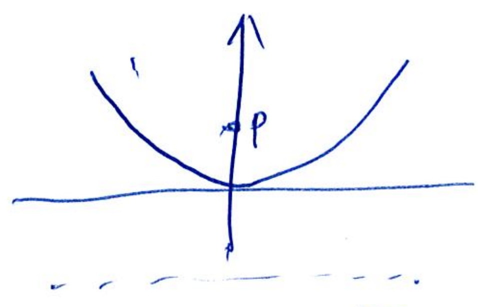
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

How to compute tangent for curve $r = f(\theta)$?

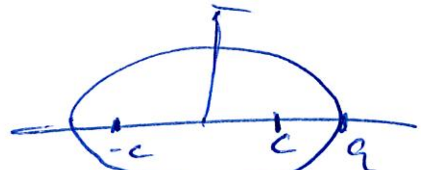
$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

* Conic sections.

parabolas : $y = \frac{1}{4p}x^2$



ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a \geq b)$

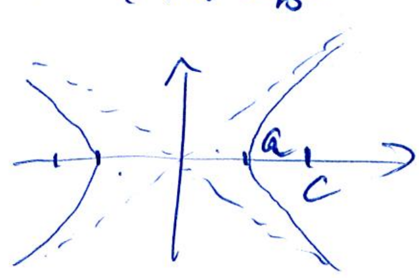


vertices at $(\pm a, 0)$

foci at $(\pm c, 0)$ with $c = \sqrt{a^2-b^2}$

hyperbolas: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

foci at $(\pm c, 0)$ $c = \sqrt{a^2+b^2}$



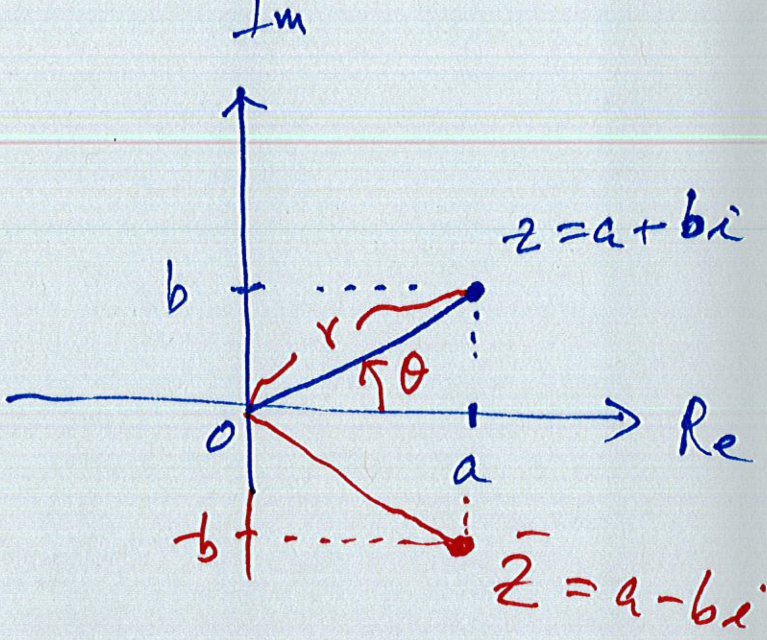
Appendix H: Complex #s

$z = a + bi$

$i = \sqrt{-1}, \quad i^2 = -1$

a: real part

b: Imaginary part



$|z| = \sqrt{a^2 + b^2}$

Complex conjugate of z:

$\bar{z} = a - bi$

Polar form of

$z = a + bi = r\cos\theta + r\sin\theta\, i$

$\underset{\text{Euler}}{=} re^{i\theta} \quad \left(\begin{array}{l} r = \sqrt{a^2+b^2} \\ \tan\theta = \frac{b}{a} \end{array} \right)$

$z \cdot \bar{z} = (a+bi)\cdot(a-bi) = a^2 - (bi)^2 = a^2 + b^2 = |z|^2$

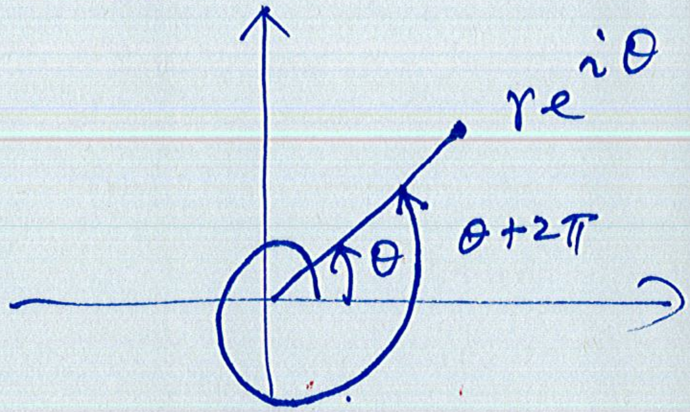
$= \bar{z} \cdot z$

Ex. $z = \dfrac{2-i}{1+3i} = \dfrac{2-i}{1+3i} \cdot \dfrac{1-3i}{1-3i}$

$= \dfrac{2 - i - 6i + 3i^2}{1^2 + 3^2} = \dfrac{1}{10}(-1 - 7i) = -\dfrac{1}{10} - \dfrac{7}{10}i$

$$z = re^{i\theta} = re^{i(\theta + 2k\pi)}$$

$\cancel{\sqrt{z}}$

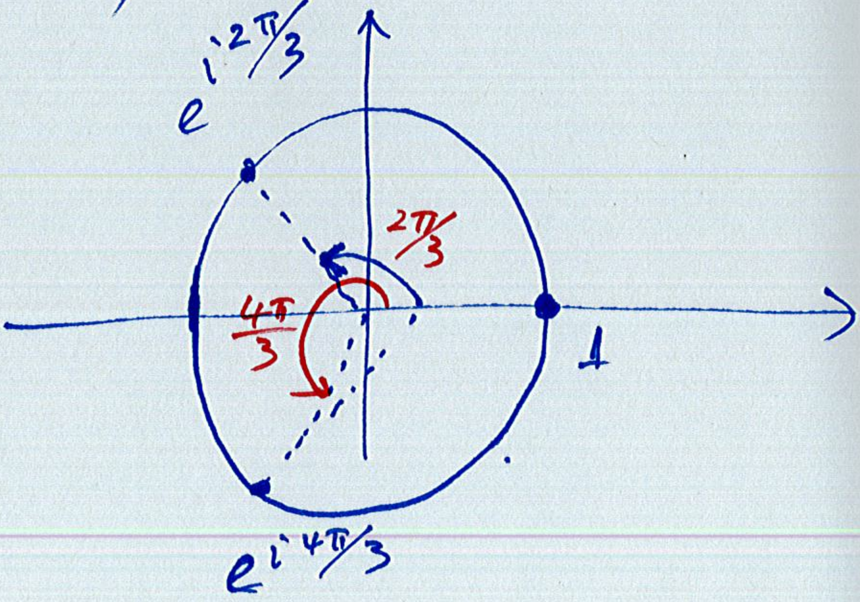


$$z^{\frac{1}{2}} = \left(re^{i\theta}\right)^{\frac{1}{2}}$$

$$= r^{\frac{1}{2}} e^{i\theta/2}$$

$$z^{\frac{1}{n}} = \left(re^{i(\theta+2k\pi)}\right)^{\frac{1}{n}} = \sqrt[n]{r}\; e^{i\,\frac{\theta+2k\pi}{n}}, \quad k = 0, 1, \cdots, n-1$$

$$\text{Ex. } z^3 = 1 = e^{i(0+2k\pi)}$$



$$\Rightarrow z = e^{i2k\pi/3}$$

$$k=0: \quad z = e^{i0} = 1$$

$$k=1: \quad z = e^{i2\pi/3} =$$

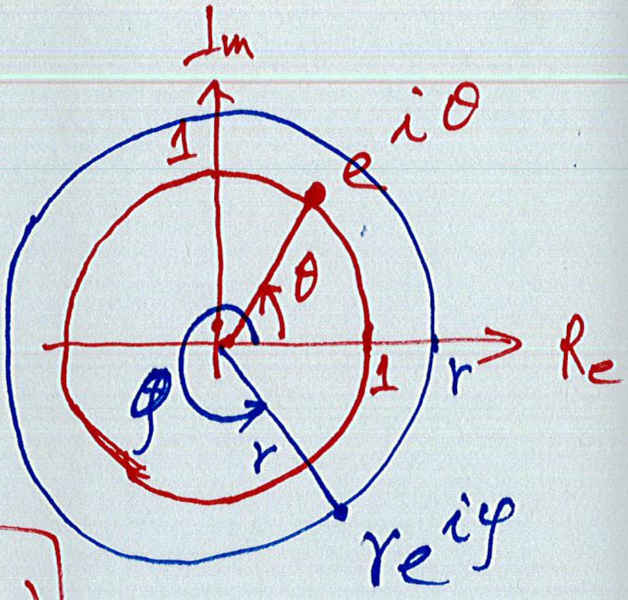
$$k=2 \quad z = e^{i4\pi/3}$$

Euler formula:

$$e^{i\theta} \equiv \cos\theta + \sin\theta \, i$$

$\downarrow$

complex exponential



$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$$\Rightarrow \boxed{z_1 \cdot z_2 = r_1 \cdot r_2 \, e^{i(\theta_1 + \theta_2)}}$$

Ex. $z_1 = 1 + \sqrt{3}\, i$

$r = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow z_1 = 1 + \sqrt{3}\, i$

$= 2 e^{i\frac{\pi}{3}}$

$z_2 = 1 + i$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

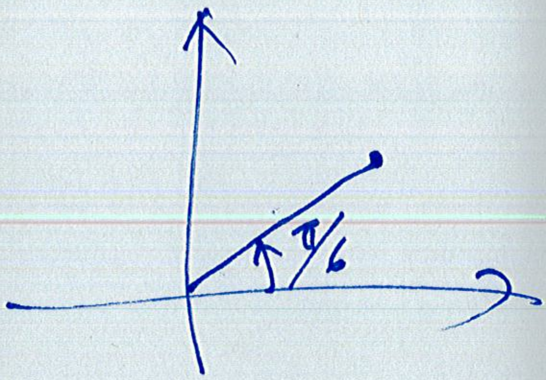
$\tan\theta = \frac{1}{1} \Rightarrow \theta = \frac{\pi}{4}$

$z_2 = 1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z_1 \cdot z_2 = 2\sqrt{2}\, e^{i\left(\frac{\pi}{3} + \frac{\pi}{4}\right)}$

Ex. $z = \sqrt{3} + i$

$z^{\frac{1}{4}} = ?$



$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$z = \sqrt{3} + i = 2\, e^{i\left(\frac{\pi}{6} + 2k\pi\right)}$

$z^{\frac{1}{4}} = \sqrt[4]{2}\, e^{i\left(\frac{\pi}{6} + 2k\pi\right)/4}$

$k=0: \quad z_1 = \sqrt[4]{2}\, e^{i\frac{\pi}{24}}$

$k=1: \quad z_2 = \sqrt[4]{2}\, e^{i\left(\frac{\pi}{6} + \frac{\pi}{2}\right)}$

$k=2: \quad \vdots$

$k=3: \quad \vdots$

Ex. $(\sqrt{3} + i)\, z = 1 + i$, write z in complex exponential.

$\Rightarrow z = \frac{1+i}{\sqrt{3}+i} = \cdots = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2 e^{i\pi/6}} = \frac{1}{\sqrt{2}}\, e^{i\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}$