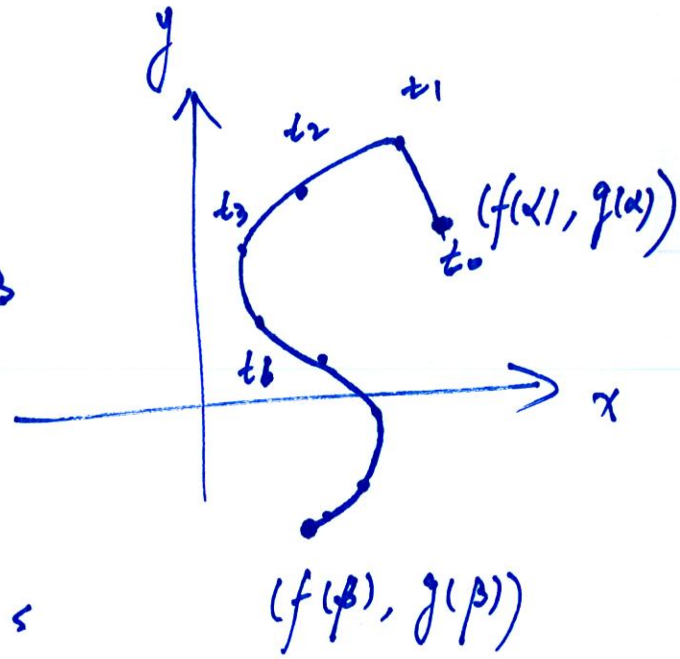
Chap. 10

Recall:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \alpha \le t \le \beta$$

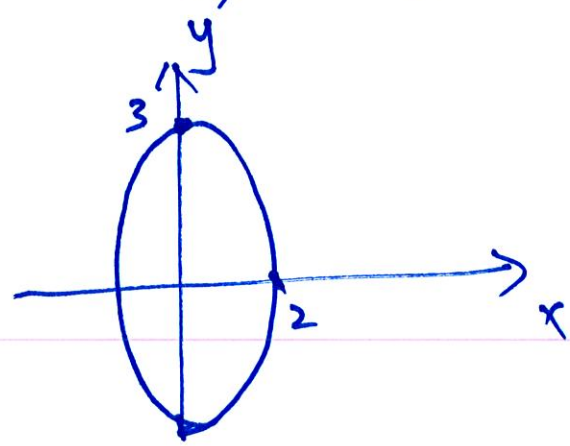


curve by parametric equs

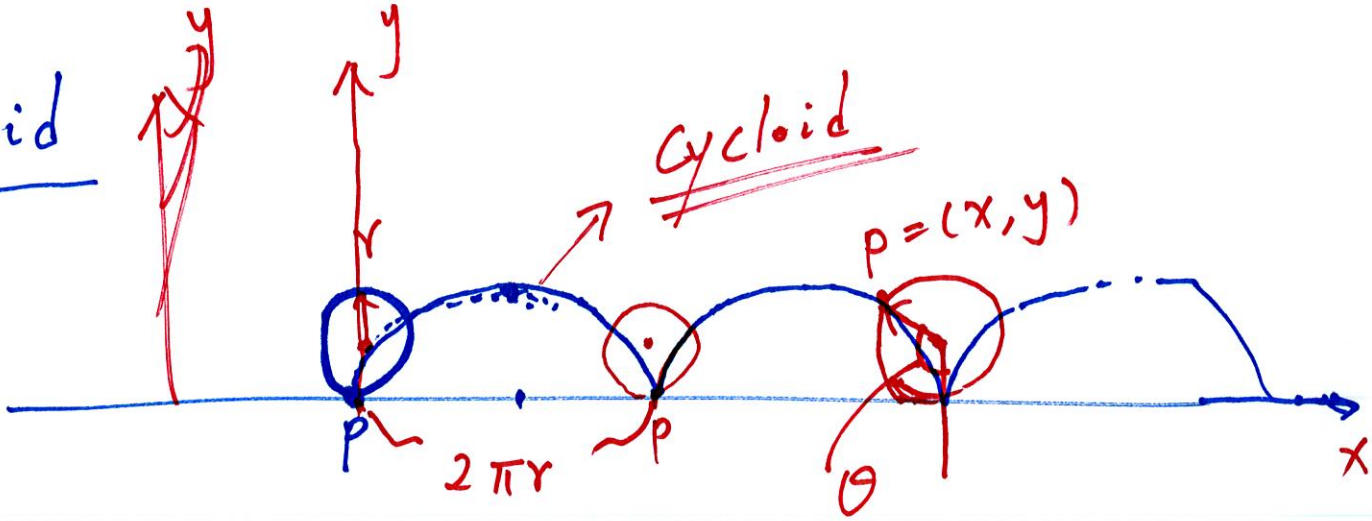
Ex. $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}, \quad 0 \le t \le 2\pi$

$\Downarrow$

$$\begin{cases} \dfrac{x}{2} = \cos t \\ \dfrac{y}{3} = \sin t \end{cases} \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

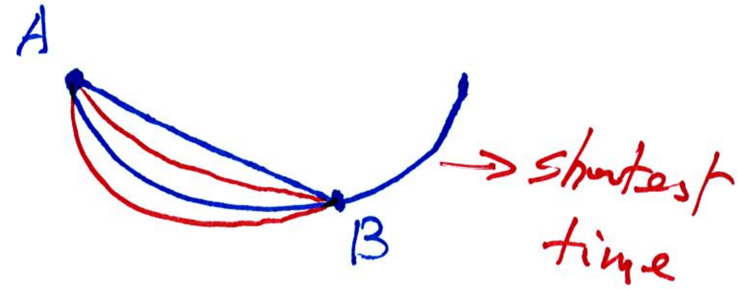


Cycloid

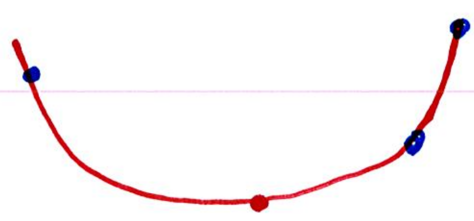


$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases} \qquad 0 \le \theta$$

(i)



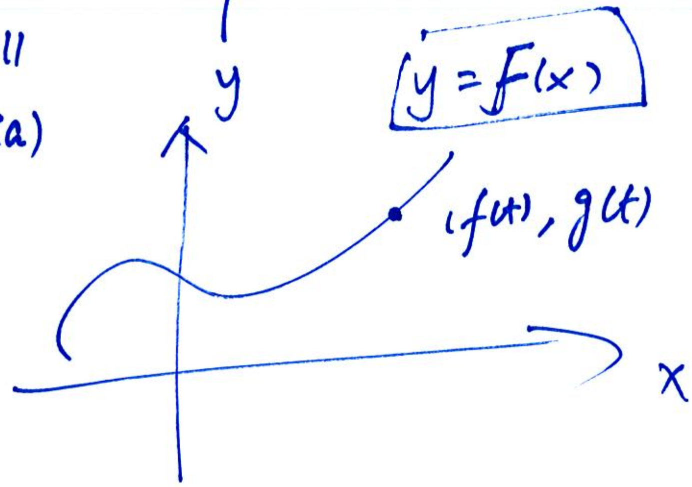
(ii)



10.2 Calculus with parametric curves

$$\left(y = F(x) \right) \qquad a \le x \le b$$

$$\iff \begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad \underset{\substack{\parallel \\ f(a)}}{\alpha} \le t \le \beta = f(b)$$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= g'(t) \big/ f'(t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) =$$

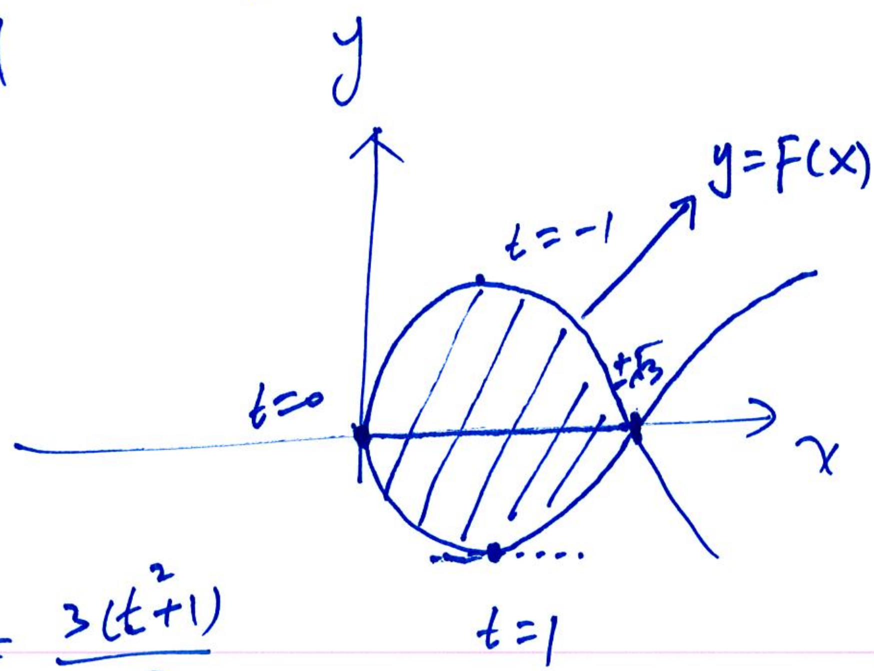
$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(g'(t)/f'(t)\right)}{f'(t)}$$

Ex. $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$

Compute the slope $\overbrace{\text{at}}^{\text{of the tangent line}}$ $(1, -2)$ (with $\underline{t=1}$)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t}$$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{t=1} = 0$$



$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

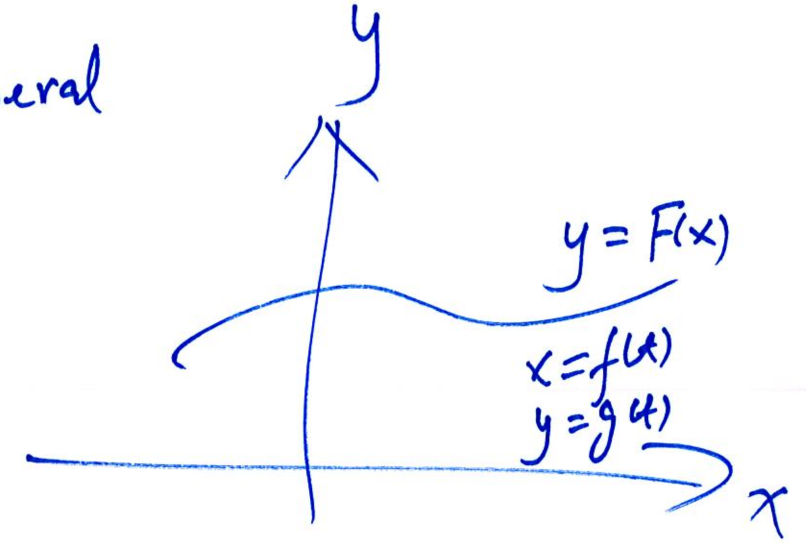
$$= \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

$$y = t^3 - 3t = 0 \Rightarrow t = 0, \pm\sqrt{3}$$

Ex. Compute the area.

$$A = 2\int_0^3 \underset{y}{\underset{\shortparallel}{F(x)}}\,dx \;\underset{dx=2t\,dt}{\overset{x=t^2}{=\!=}}\; 2\int_0^{-\sqrt{3}}(t^3-3t)\,2t\,dt$$

~~In genegral~~ In general

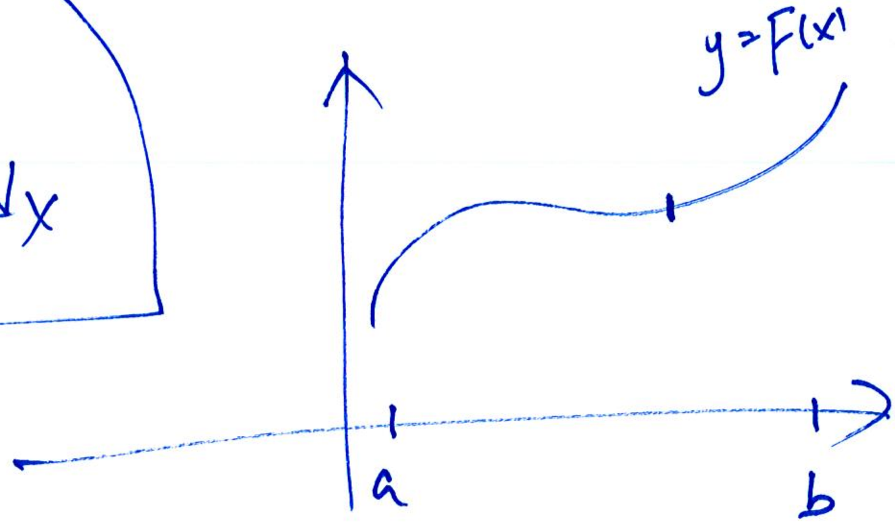


$$A = \int_a^b y\,dx$$

$$\underset{dx=f'(t)\,dt}{\overset{x=f(t)}{=\!=}} \int_\alpha^\beta g(t)\,f'(t)\,dt \quad \text{with} \quad \begin{aligned} a &= f(\alpha) \\ b &= f(\beta). \end{aligned}$$

Arc Length: for $y = F(x)$

$$\boxed{L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$



$\underline{x = f(t)}$

$dx = f'(t)\,dt$

$$\int_\alpha^\beta \sqrt{1+\left(\frac{g'(t)}{f'(t)}\right)^2}\, f'(t)\,dt$$

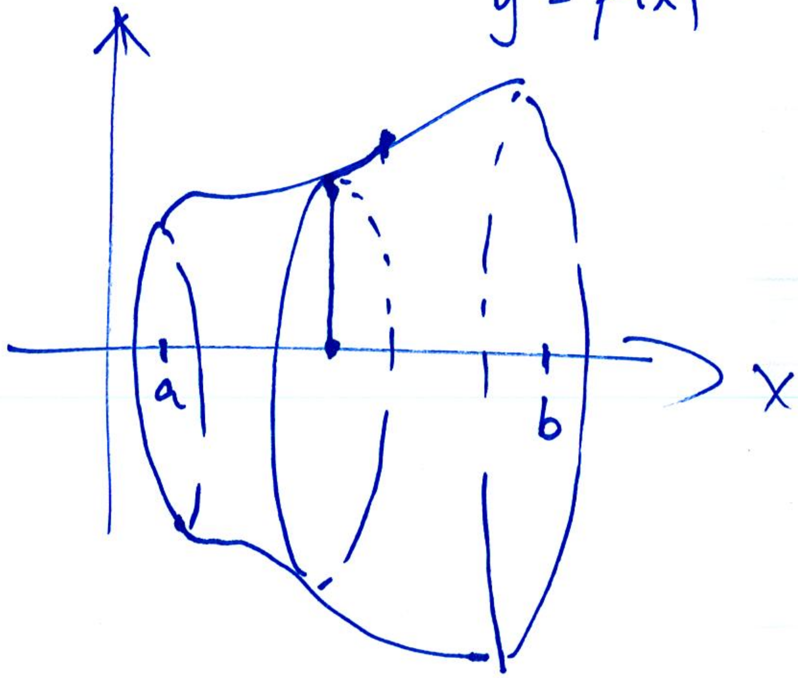
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \alpha \le t \le \beta.$$

$$\boxed{L = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2}\,dt}$$

Ex. $\underline{L = \int_0^{+\sqrt{3}} \sqrt{(2t)^2 + (3t^2-3)^2}\,dt}$.

Surface area

$$A = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



$$\boxed{\underline{\underline{A}} \quad \int_\alpha^\beta 2\pi g(t) \sqrt{(g'(t))^2 + (f'(t))^2}\, dt}$$

$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \alpha \le t \le \beta$

$\underline{\underline{x = f(t)}}$