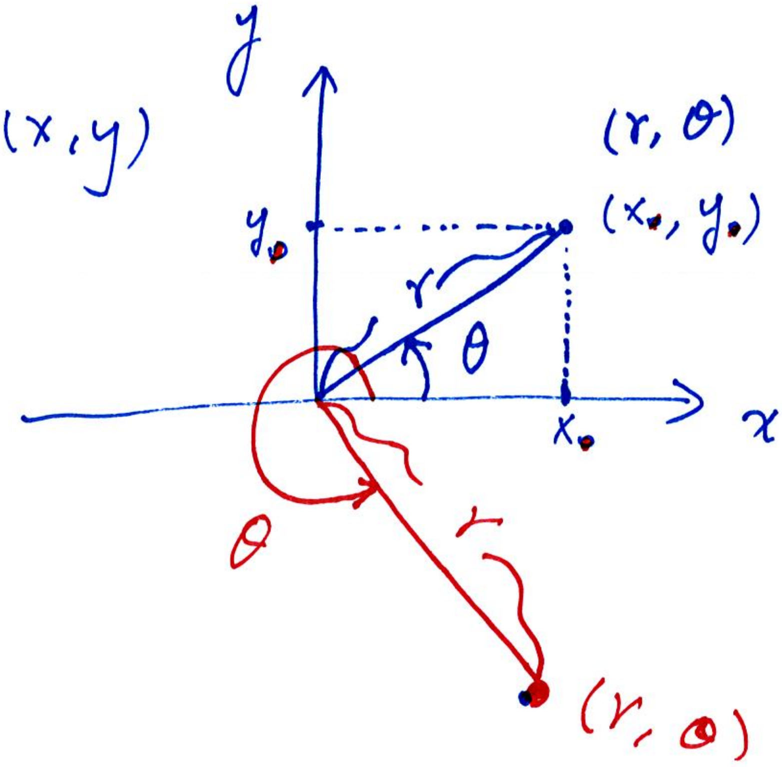
10.3 Polar Coordinates

Cartesian coordinates (x, y)

polar coordinates (r, θ)

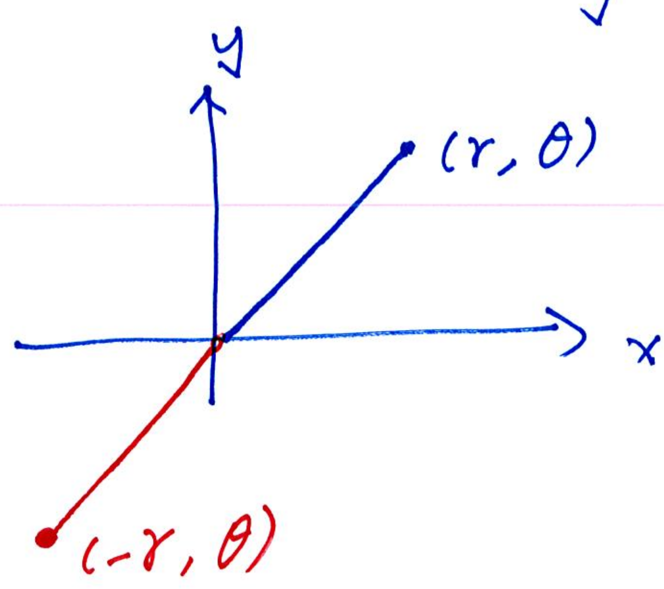


Relation

$$(1) \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(2) \quad \begin{cases} \tan\theta = \dfrac{y}{x} \\ r^2 = x^2+y^2 \end{cases} \Rightarrow \begin{cases} \theta = \tan^{-1}\dfrac{y}{x} \\ r = \sqrt{x^2+y^2} \end{cases}$$

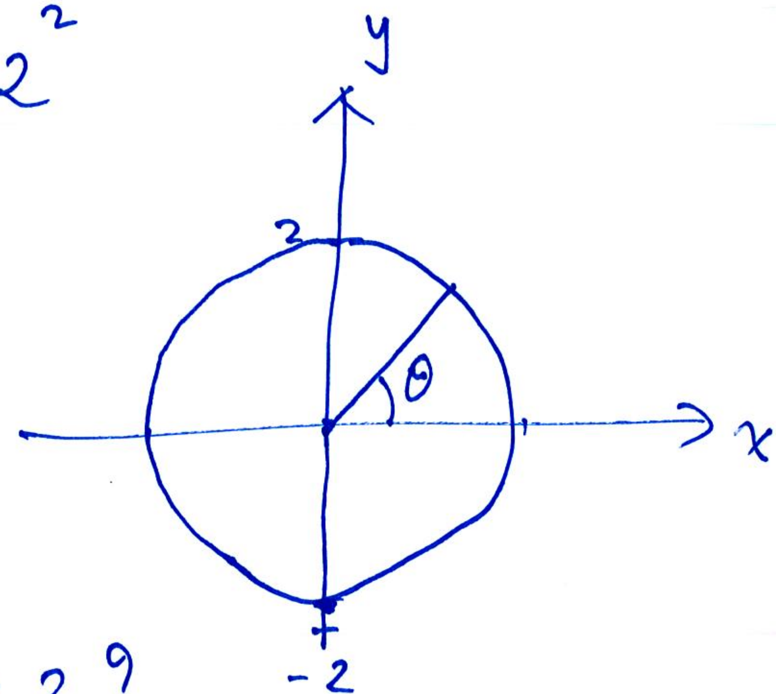
Rmk: $(-r, \theta)$ is defined as in the figure.



Polar curves

Ex. what is the curve $r = 2$?

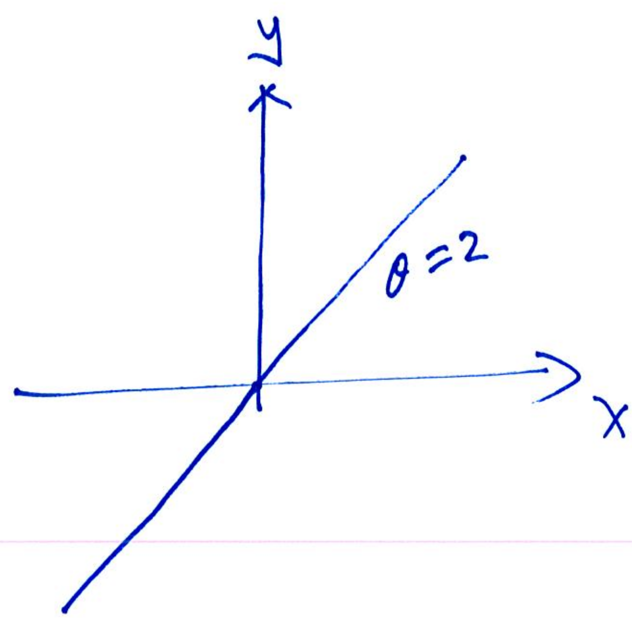
$r = 2 \iff x^2 + y^2 = 2^2$



Ex. How about $\theta = 2$?

$$\boxed{\tan\theta = \frac{y}{x}}$$

$y = (\tan 2)\ x$



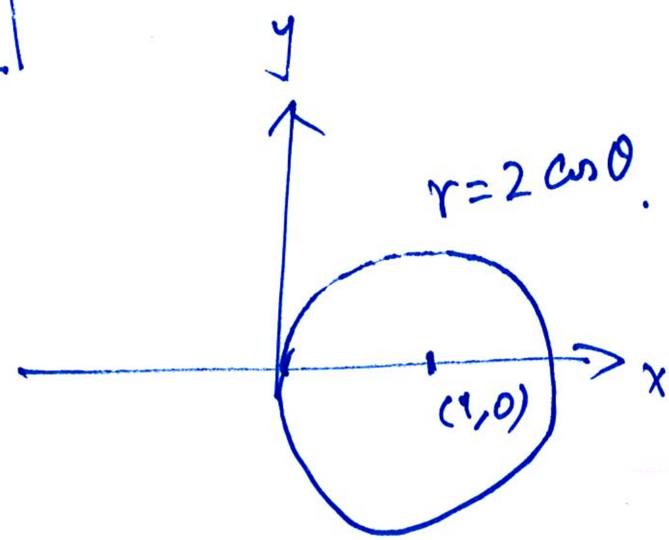
Ex. $r = 2\cos\theta$

$x = r\cos\theta \Longrightarrow$ $\cancel{\cos\theta =}$ $\boxed{x = r\cdot\frac{r}{2}}$

$y \cancel{= r\sin\theta}$

$$\boxed{x^2+y^2 = r^2 = 2x}$$

$$\Rightarrow (x-1)^2+y^2 = 1^2$$



Ex. $\boxed{r=\cos 2\theta}$

$$= \cos^2\theta - \sin^2\theta$$

$$= \frac{x^2}{r^2} - \frac{y^2}{r^2}$$

$$\sqrt{x^2+y^2} = r = \cos 2\tan^{-1}\frac{y}{x}$$

<u>Rmk</u> $r=f(\theta)$ $\qquad$ $y=f(x)$

slope of tangent line for $\underline{r = f(\theta)}$

$$= \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Rightarrow \begin{cases} x = r\cos\theta \\ \quad = f(\theta)\cos\theta \\ y = r\sin\theta \\ \quad = f(\theta)\sin\theta \end{cases}$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{f'(\theta)\sin\theta + r\cos\theta}{f'(\theta)\cos\theta - r\sin\theta}$$

Ex. Find the slope of the tangent line of

$r = \cos 2\theta$ at $(r,\theta) = (0, \frac{\pi}{4})$

$$\frac{dy}{dx} = \left.\frac{-2\sin 2\theta \sin\theta + r\cos\theta}{-2\sin 2\theta\cos\theta - r\sin\theta}\right|_{\substack{r=0 \\ \theta=\frac{\pi}{4}}} = \frac{-2\,\frac{\sqrt{2}}{2}}{-2\cdot\frac{\sqrt{2}}{2}} = 1$$

Ex. Find the slope of the tangent line of $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\left\{ \begin{array}{l} x = r\cos\theta \\ \quad = f(\theta)\cos\theta \\ y = r\sin\theta \\ \quad = f(\theta)\sin\theta \end{array} \right.$$

$$= \left. \frac{\cancel{\cos^2\theta +} \cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} \right|_{\theta = \frac{\pi}{3}}$$

$$= \frac{\frac{1}{2}\frac{\sqrt{3}}{2} + \left(1 + \frac{\sqrt{3}}{2}\right)\frac{1}{2}}{\left(\frac{1}{2}\right)^2 - \left(1 + \frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}} = -1.$$