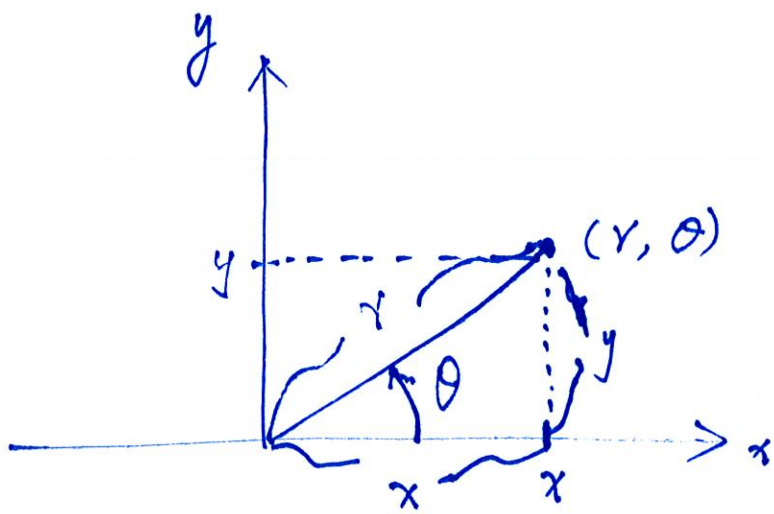
Review of Polar coordinates

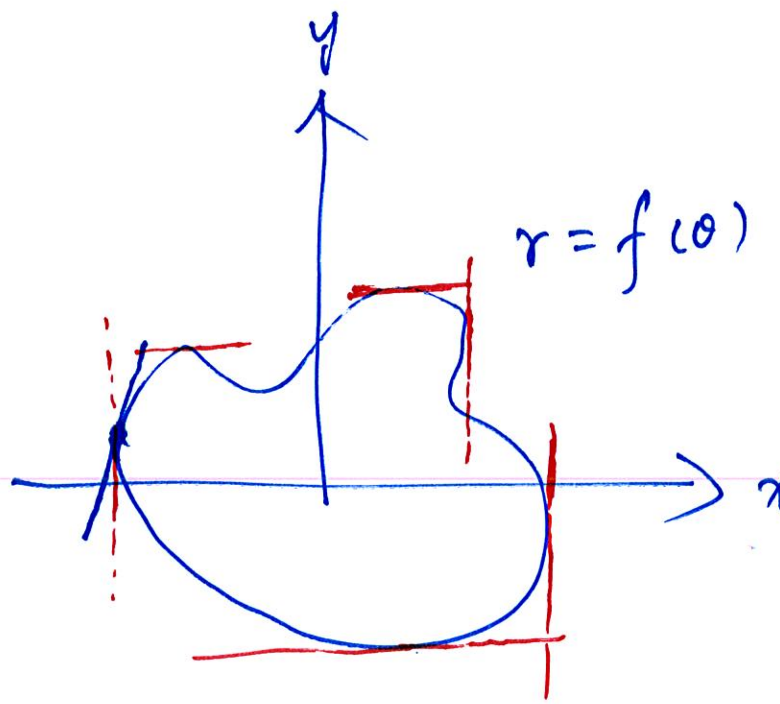
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$\iff \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

How to Compute $\frac{dy}{dx}$ given $\boxed{r = f(\theta)}$?

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$



$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\sin\theta\right)$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\cos\theta\right)$$

Ex. Given $r = 1 + \cos\theta$ $\quad 0 \le \theta < 2\pi$.

Find the points where the tangent lines are horizontal ($\frac{dy}{dx} = 0$) or vertical ($\frac{dy}{dx} = \infty$)

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left((1+\cos\theta)\sin\theta\right)$$

$$= -\sin^2\theta + (1+\cos\theta)\cos\theta$$

$$= \cos^2\theta - \sin^2\theta + \cos\theta$$

$$= \underline{\cos 2\theta + \cos\theta = 0} \quad \text{when}$$

$\cancel{\theta = \pi}$ $\quad \cancel{\theta = \frac{5\pi}{3}}$ $\quad \cancel{\theta = \frac{7\pi}{3}}$

$$= 2\cos^2\theta - 1 + \cos\theta = 0$$

let $x = \cos\theta$

$$\Rightarrow 2x^2 + x - 1 = 0 \quad \Rightarrow x = \frac{-1 \pm \sqrt{1+8}}{4}$$

$$= \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$$

$\cos\theta = -1$ or $\frac{1}{2}$ $\Rightarrow$ $\theta = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$

horizontal tangent line

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left((1+\cos\theta)\cos\theta\right)$$

$$= -\sin\theta\cos\theta + (1+\cos\theta)(-\sin\theta)$$

$$\cancel{= -2\sin 2\theta - \sin\theta = 0}$$

$$= -\sin\theta(1+2\cos\theta) = 0 \Rightarrow \sin\theta = 0 \text{ or } 1+2\cos\theta = 0 \Rightarrow \begin{cases} \theta = 0, \pi. \\ \text{or} \\ \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \end{cases}$$

vertical tangent line

10.5 Conic Sections.

Parabola $\qquad y = ax^2$ ✓

ellipse $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

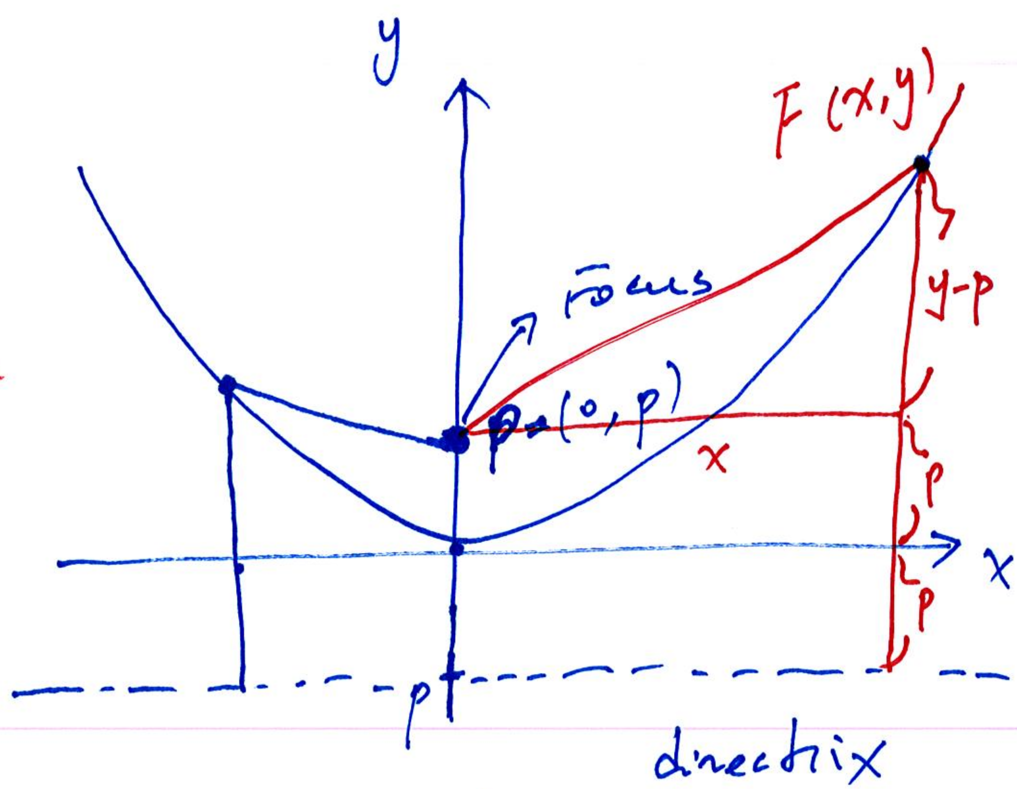
hyperbola $\qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad (a \geq b)$

$|\overrightarrow{FP}| = \sqrt{x^2 + (y-p)^2}$

$= y + p$

$\Rightarrow x^2 + (y-p)^2 = (y+p)^2$

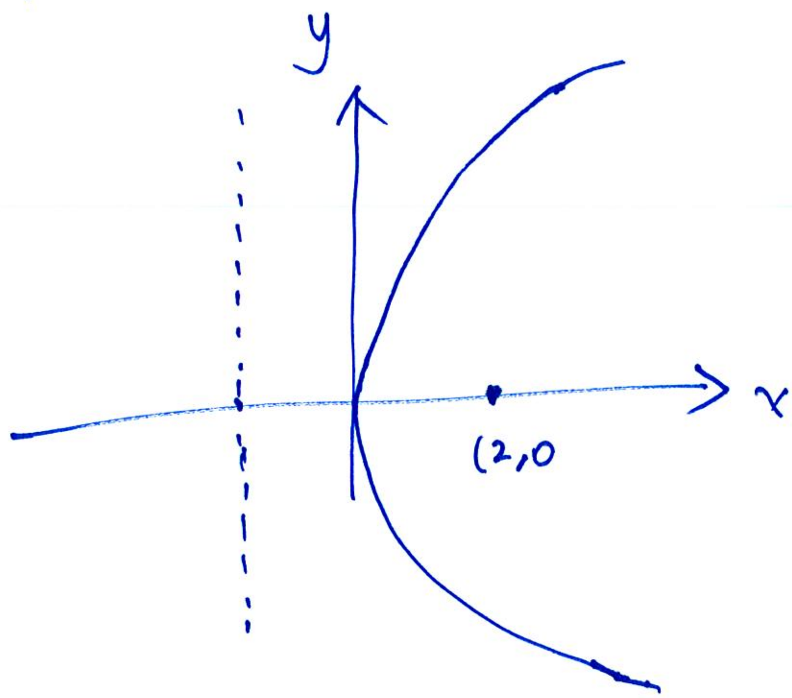
$\Rightarrow \boxed{x^2 = 4py}$



Thm. $x^2 = 4py$ is a parabola with focus at $(0, p)$

Ex. Draw the curve $y^2 = 8x = 4 \cdot 2 x$

Find its focus point.

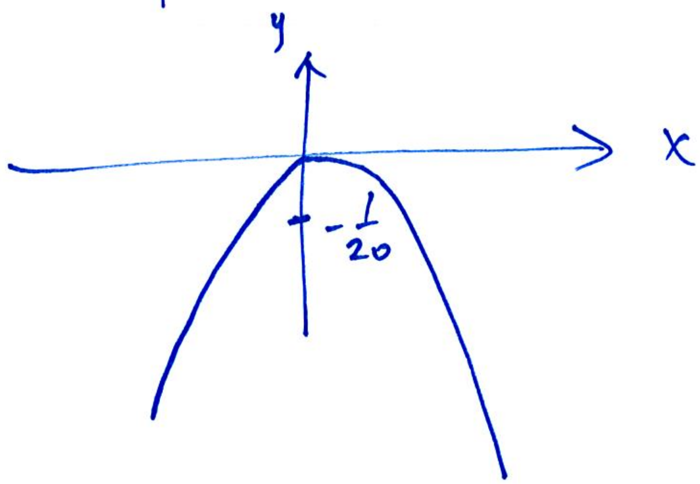


$y = -5x^2$

$\Leftrightarrow x^2 = -\frac{1}{5} y$

$4p = -\frac{1}{5}$

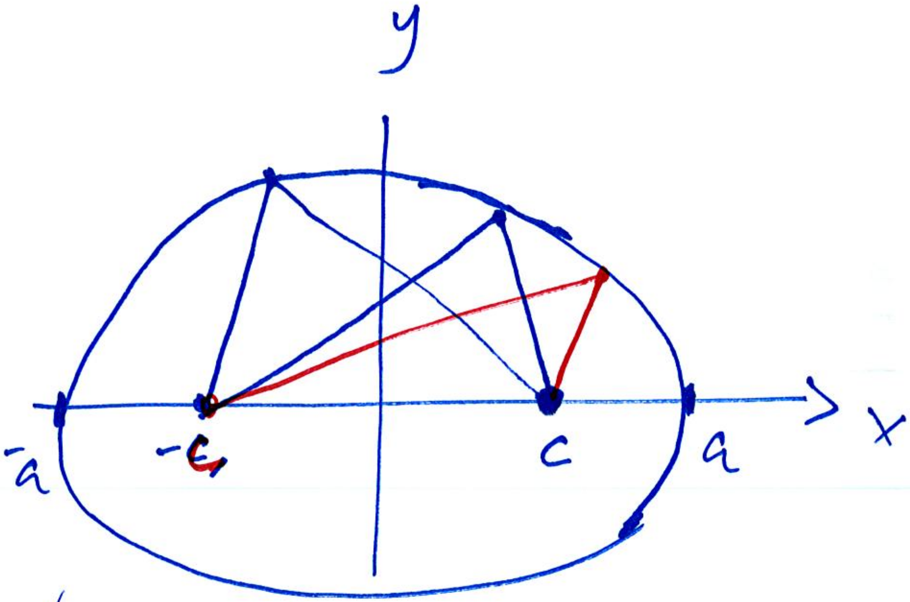
$\Rightarrow p = -\frac{1}{20}$



Ellipse

with Foci at

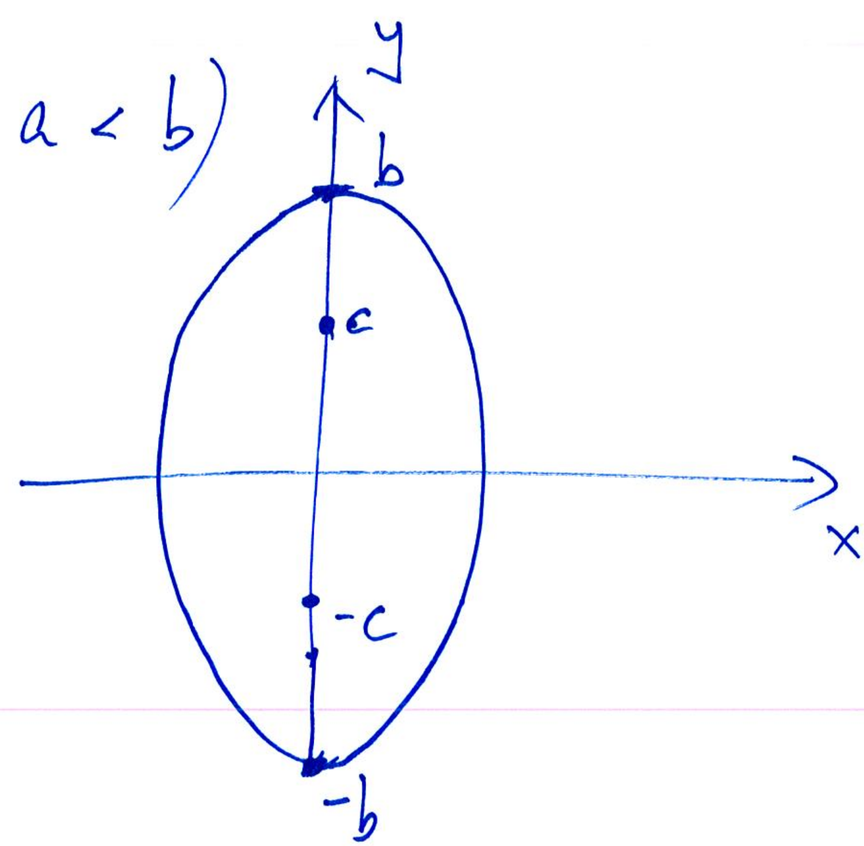
$(c,0), (-c,0)$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a \geq b)$$

with vertices at $(\pm a, 0)$, $c^2 = a^2 - b^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a < b)$$



the Foci are at

$(0, \pm c)$

the vertices are at

$(0, \pm b)$