Chap 11 Infinite sequences & series

§11.1 Sequences

Ex. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$, $\left\{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \cdots\right\}$, $\{a_n\}_{n=1}^{\infty}$ with

$a_n = \frac{n}{2n+1}$

Ex. Fibonacci sequence

$a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$, $n = 1, 2, \cdots$

$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$

Def. A sequence $\{a_n\}$ has a limit L if

$\lim_{n\to\infty} a_n = L$ ~~$(-L \neq \pm\infty)$~~

We say $\{a_n\}$ is convergent if $\lim_{n\to\infty} a_n = L$ $(\neq \pm\infty)$

Otherwise, it is divergent

Ex. $\left\{(-1)^n\right\}_{n=1}^{\infty} = \left\{-1, 1, -1, 1, -1, \cdots\right\}$

<u>divergent</u>

<u>Thm</u> $\lim_{x\to\infty} f(x) = L$ and let $a_n = f(n)$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$.

Ex. $f(x) = \frac{1}{x^r}$

$$\lim_{x\to\infty} \frac{1}{x^r} = \begin{cases} 0 & r > 0 \\ 1 & r = 0 \\ +\infty & r < 0 \end{cases}$$

$$\Rightarrow \boxed{\lim_{x\to\infty} \frac{1}{n^r} = 0 \quad \text{if} \quad r > 0}$$

<u>Thm</u> If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

Ex. $a_n = \frac{(-1)^n}{n} \to 0$: $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$

Ex. $\displaystyle\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

$$\lim_{n\to\infty} \frac{n/n}{(2n+1)/n} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$$

$$= \frac{1}{2+\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{2}.$$

~~Ex.~~ $\cancel{a_n = \{ \quad a_n =}$

Ex. $\displaystyle\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

$$= \cos(0) = 1.$$

Ex. $\displaystyle 0 \le a_n = \frac{n!}{n^n} = \frac{1}{n}\cdot\left[\frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n}\right] \le \frac{1}{n} \to 0$

$$\lim_{n\to\infty} a_n = 0.$$

Ex. $a_n = r^n$ (geometric sequence)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r|<1 \\ 1 & r=1 \\ \infty & |r|>1 \\ \text{divergent.} & r=-1 \end{cases}$$

Def. $\{a_n\}$ is an increasing sequence if $a_{n+1} \geq a_n$, $n \geq 1$

$\{a_n\}$ is a decreasing sequence if $a_{n+1} \leq a_n$, $n \geq 1$

$\{a_n\}$ is bounded above if $a_n \leq M$, $n \geq 1$

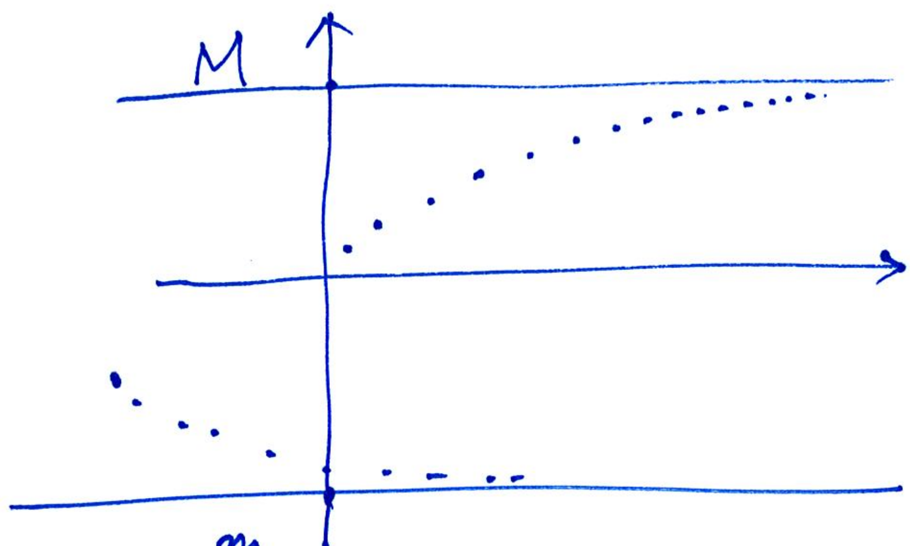
$\{a_n\}$ is bounded below if $a_n \geq m$, $n \geq 1$

if $\{a_n\}$ is bounded above & below, we say $\{a_n\}$ is a bounded sequence.

We say $\{a_n\}$ is monotonic sequence if $\{a_n\}$ is increasing sequence or decreasing sequence.

Thm If $\{a_n\}$ is a bounded monotonic sequence, then $\{a_n\}$ is convergent,

i.e. there exist $L\ (\neq +\infty)$, $\lim_{n\to\infty} a_n = L$.



Ex. $a_1 = 3$, $\boxed{a_{n+1} = \frac{1}{2}(a_n + 2)}$, $n \geq 1$.

$\left\{3, \frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \cdots\right\}$

(i) a_n is decresing: (by induction)

assume ~~a_{k+1}~~ $a_{k-1} \geq a_k$, $k = 2, \cdots, n$

we need to show $a_n \geq a_{n+1}$

$a_{n+1} = \frac{1}{2}(a_n + 2) \leq \frac{1}{2}(a_{n-1} + 2) = a_n$ ✓

We need to show $a_n \geq 2$ (by induction)

we assume

$$a_k \geq 2 \quad k = 1, 2, \cdots n$$

then $a_{n+1} = \frac{1}{2}(a_n + 2) \geq \frac{1}{2}(2+2) = 2$ ✓

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 2) = \frac{1}{2}(L+1)$$

$$\Rightarrow \boxed{L = 2}$$

Ex. $a_1 = 5$, $\underline{a_{n+1} = \frac{1}{2}(a_n + 7)}$ $\nearrow 7$

$$L = \lim a_{n+1} = \lim \frac{1}{2}(a_n + 7) = \frac{1}{2}(L+7)$$

$$\Rightarrow L = 7.$$