No Class on Friday

Recall:

* Basic formula:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

Ex. $$\frac{1}{2+x^2} = \frac{1}{2\left(1+\frac{x^2}{2}\right)} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{x^2}{2}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n \qquad \left|\frac{x^2}{2}\right| < 1$$

$$\Leftrightarrow |x| < \sqrt{2}.$$

* $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{1}{n+1}(x-a)$$

Ex. Find a power series representation for $\tan^{-1} x$.

$$(\tan^{-1} x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$\Rightarrow$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \int (-x^2)^n \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} + C.$$

Take $x=0 \Rightarrow 0 = \tan^{-1} 0 = 0 + C \Rightarrow C = 0$

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

11.10 Taylor and Maclaurin Series.

Assume,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2 + \cdots \quad \text{for } |x-a| < R.$$

Q. How to find $\{c_n\}$?

Take $x = a \Rightarrow \boxed{f(a) = c_0}$

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

Take $x = a \Rightarrow f'(a) = c_1$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

Take $x = a \Rightarrow f''(a) = 2!\, c_2$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \underline{n(n-1)(n-2)\cdots(n-k+1)}\, c_n (x-a)^{n-k}$$

Take $x = a \Rightarrow \boxed{f^{(k)}(a) = k!\, c_k}$

Thm If $f(x)$ has a power series representation at $x = a$ for $|x-a| < R$.

Then $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(Taylor series of $f(x)$)

special case: $a = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called Maclaurin Series

Ex. Find the Maclaurin Series of $f(x) = e^{2x}$.

$f(0) = 1$

$f'(x) = 2e^{2x}$ $\quad$ $f'(0) = 2$

$f''(x) = 2^2 e^{2x}$ $\quad$ $f''(0) = 2^2$

$\vdots$

$f^{(n)}(x) = 2^n e^{2x}$ $\quad$ $f^{(n)}(0) = 2^n$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \qquad R = +\infty.$$

ratio test $\left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \frac{n!}{2^n x^n} \right| = \left| \frac{2}{n+1} x \right| \xrightarrow{n \to \infty} 0$, any x

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R.$$

$$= \sum_{k=0}^{n} c_k (x-a)^k + \sum_{k=n+1}^{\infty} c_k (x-a)^k$$

$$= T_n(x) + R_n(x)$$

$$\downarrow \qquad\qquad \downarrow$$

Taylor expansion of order n — Residue of order n.

Thm If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$.

then $\underline{|R_{n+1}(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}}$ for $|x-a| \le d$.

Ex. Find the Taylor series of $f(x) = e^{2x}$ at $x=1$.

We need $f^{(k)}(1)$: $k=0, 1, 2, \ldots$

$f(1) = e^2$

$f'(x) = 2e^{2x}$ $\qquad$ $f'(1) = 2e^2$

$$f^{(k)}(x) = 2^k e^{2x} \qquad f^{(k)}(1) = 2^k e^2$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!} (x-1)^n \quad \text{for any } x.$$

Ex. Find the Maclaurin series for $f(x) = \sin x$.

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1.$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$\vdots \qquad \vdots$$

$$f^{(2k)}(0) = 0$$

$$f^{(2k+1)}(0) = (-1)^k$$

$$\Rightarrow f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{for any } x.$$