11.10 Taylor and Maclaurin Series

Q. How to represent $f(x)$ in power series about $x = a$?

Thm. If $f(x)$ has a power series resporentation at $x = a$.

$$(*) \quad f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n \quad , \quad |x-a| < R$$

(radius of convergence)

Then $\boxed{C_n = \dfrac{f^{(n)}(a)}{n!}}$

__Rmk__ $(*)$ is called the Taylor series of $f(x)$ and if $a = 0$, it is also called Maclaurin Series.

Ex. Expand $f(x) = \cos x$ in power series at $x = 0$.

$k=0 \quad f(x) = \cos x \qquad\qquad f(0) = 1.$

$(k=0) \quad f'(x) = -\sin x \qquad\qquad f'(0) = 0$

$k=1 \quad f''(x) = -\cos x \qquad\qquad f''(0) = -1,$

0

1

0

-1

$\vdots$

$$\begin{cases} f^{(2k)}(x) = (-1)^k \cos x \\ f^{(2k+1)}(x) = (-1)^{k+1} \sin x \end{cases}$$

$$f^{(2k)}(0) = (-1)^k$$

$$f^{(2k+1)}(0) = 0$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x-0)^{2k}$$

for any x.

Q. What is the power series representation of $f(x) = \cos x$ at $x = \frac{\pi}{4}$?

$$f^{(2k)}(x) = (-1)^k \cos x \qquad f^{(2k)}\left(\tfrac{\pi}{4}\right) = (-1)^k \frac{\sqrt{2}}{2}$$

$$f^{(2k+1)}(x) = (-1)^{k+1} \sin x \qquad f^{(2k+1)}\left(\tfrac{\pi}{4}\right) = (-1)^{k+1} \frac{\sqrt{2}}{2}$$

$$\cos x = \sum_{n=0}^{\infty} C_n \left(x - \frac{\pi}{4}\right)^n$$

$$= \sum_{k=0}^{\infty} C_{2k} \left(x - \frac{\pi}{4}\right)^{2k} + \sum_{k=0}^{\infty} C_{2k+1} \left(x - \frac{\pi}{4}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \frac{\sqrt{2}}{2}}{(2k)!} \left(x - \frac{\pi}{4}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \frac{\sqrt{2}}{2}}{(2k+1)!} \left(x - \frac{\pi}{4}\right)^{2k+1}$$

Ex. What is the power series representation of $f(x) = \sin x$ at $x = \frac{\pi}{4}$?

$$\sin x = -(\cos x)'$$

$$= \frac{\sqrt{2}}{2}\left\{ \underbrace{\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} \left(x-\frac{\pi}{4}\right)^{2k-1}} + \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!}\left(x-\frac{\pi}{4}\right)^{2k}} \right\}$$

Ex. What is the power series represention of $f(x) = x^2 \cos x$ at $x = 0$?

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\Rightarrow \boxed{x^2 \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+2}}$$

Q. $f(x) = (x+1)^2 \cos x$ at $x = 0$?

$$= (x^2+2x+1)\cos x = \underline{x^2 \cos x} + \underline{2x \cos x} + \underline{\cos x}$$

Ex. $f(x) = (1+x)^k$ k is any real #.

in Maclaurin series ?

$f(x) = (1+x)^k$ $f(0) = 1$.

$f'(x) = k(1+x)^{k-1}$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $f''(0) = k(k-1)$

$\vdots$

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$, $f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for } |x| < 1.$$

where $\binom{k}{n} := \dfrac{k(k-1)\cdots(k-n+1)}{n!}$ is the binormial coefficient.

Ex. Find the power series representation of

$$f(x) = \frac{1}{\sqrt{2+x}} \quad \text{at} \quad x = 0.$$

$$f(x) = \frac{1}{\sqrt{2+x}} = (2+x)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}\sqrt{1+\frac{x}{2}}} = \frac{1}{\sqrt{2}}\underline{\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{x}{2}\right)^n$$

$$= \frac{1}{\sqrt{2}} \left\{ 1 - \frac{1}{2}\left(\frac{x}{2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{2}\right)^2 + \cdots \right\}.$$