11.10 Part C

Recall. Binormial series

k is any real #

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for } |x|<1.$$

with $\displaystyle \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$

Ex. Expand $\dfrac{1}{\sqrt{4-x^2}}$ in Maclaurin Series

$$f(x) = \frac{1}{\sqrt{4-x^2}} = \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{1}{2}\left(1-\left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1+\left(-\left(\frac{x}{2}\right)^2\right)\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\left(\frac{x}{2}\right)^2\right)^n \qquad \left|\left(\frac{x}{2}\right)^2\right| < 1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n}}{2^{2n}} \qquad \Longleftrightarrow |x| < 2$$

Ex. Find the value for $S = \underline{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}\left(\frac{2}{5}\right)^n} = f\left(\frac{2}{5}\right)$

$= \ln\left(1+\frac{2}{5}\right)$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n = \ln(1+x)$$

Q. How to compute $\ln\left(1+\frac{2}{5}\right)$ for up to 2-digits?

Let $S_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \left(\frac{2}{5}\right)^k$

we want $\boxed{|S - S_n| \le 0.01}$

Recall: For $S = \sum_{n=0}^{\infty} (-1)^n a_n$ with $a_n \ge 0$, $a_{n+1} \le a_n$

with $S_n = \sum_{k=0}^{n} (-1)^k a_k$

then $\boxed{|S - S_n| \le a_{n+1}}$

we need $\frac{1}{n+1}\left(\frac{2}{5}\right)^{n+1} \le 0.01$ (find the smallest n)

$\boxed{n=4}$

$\underline{\frac{1}{5} \cdot \frac{2^5}{5^5} \le 0.01}$

Ex. (i) Find the power series representation for

$f(x) = \int e^{-x^2}\,dx$. (ii) Compute $\int_0^1 e^{-x^2}\,dx$ upto 0.001 accuracy.

Recall: If $g(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for $|x-a|<R$

$$\Rightarrow \int g(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{1}{n+1}(x-a)^{n+1} + C$$

$$g'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$\Rightarrow \int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,n!} x^{2n+1} + C.$$

(ii) $\int_0^1 e^{-x^2}\,dx = \Big[\qquad \Big]_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,n!}$

Error $|S - S_n| \le \frac{1}{(2(n+1)+1)\,(n+1)!} \le 0.001$

We need to take $\boxed{n=4}$

Ex. Find the Maclaurin series for

$$f(x) = \frac{1}{x^2 - 5x + 6}$$

Recall: $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

$$f(x) = \frac{1}{(x-2)(x-3)} = \frac{a}{x-2} + \frac{b}{x-3}$$

$$= \frac{-1}{x-2} + \frac{1}{x-3}$$

Recall: $$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n} \quad \text{for } |x| < 1$$

$$f(x) = \frac{1}{2-x} - \frac{1}{3-x} = \frac{1}{2}\,\frac{1}{1-\frac{x}{2}} - \frac{1}{3}\,\frac{1}{1-\frac{x}{3}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n - \frac{1}{3} \sum_{n=0}^{\infty} \cancel{1+\frac{x}{3}} \left(\frac{x}{3}\right)^n \quad \text{for } |x| < 2.$$

$$\left|\frac{x}{2}\right| < 1 \qquad \left|\frac{x}{3}\right| < 1$$

Multiplication & division of power series

Ex. Find the first terms of the power series for $e^x \cos x$ ~~at $x=0$~~.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2}x^2 + \cdots$$

$$e^x \cos x = 1 + x + 0x^2 - \frac{1}{3}x^3 + \cdots$$

$$\begin{array}{rcccccccc}
 & 1 & + x & + \frac{x^2}{2} & + \frac{x^3}{6} & + \cdots \\
 & 1 & & - \frac{1}{2}x^2 & & \\
\hline
 & 1 & + x & + \frac{x^2}{2} & + \frac{x^3}{6} & + \cdots \\
+) & & & - \frac{1}{2}x^2 & - \frac{1}{2}x^3 & \\
\hline
 & 1 & + x & & - \frac{1}{3}x^3 & + \cdots
\end{array}$$