11.2. Series

sequence: $\{a_n\}_{n=1}^{\infty}$

Series $\sum_{n=1}^{\infty} a_n$

Partial sum (of series) $S_n = \sum_{k=1}^{n} a_k$

$\{S_n\}_{n=1}^{\infty}$

Def. If $\{S_n\}_{n=1}^{\infty}$ converges, we say the series $\sum_{n=1}^{\infty} a_n$ converges and

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n}$$

Otherwise, it diverges.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left\{ \frac{1}{k} - \frac{1}{k+1} \right\}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \xrightarrow{\lim_{n\to\infty}} 1$$

$$\begin{array}{rl} & 1 + r + r^2 + \cdots r^{n-1} \\ +) & \quad -r - r^2 + \cdots - r^{n-1} - r^n \\ \hline & 1 \qquad\qquad\qquad\qquad -r^n \end{array}$$

Ex. $a_n = a r^{n-1}$

$$\sum_{n=1}^{\infty} a_n = a\left(\underline{1 + r + r^2 + \cdots + r^n + \cdots}\right)$$

$$S_n = \sum_{k=1}^{n} a r^{k-1} = a\left(\underline{1 + r + \cdots + r^{n-1}}\right)\frac{(1-r)}{1-r}$$

$$= \frac{a}{1-r}\left(1 - r^n\right) \xrightarrow[|r|<1]{n\to\infty} \frac{a}{1-r}$$

$$\boxed{\Rightarrow \sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}}$$

Ex. $\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

Ex. $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \cdots$

$$= \sum_{n=1}^{\infty} 1 \cdot \left(\frac{1}{3}\right)^{n-1} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Ex. $\sum_{n=1}^{\infty} 3^{2n}(5)^{-n} = \sum_{n=1}^{\infty} (3^2)^n \left(\frac{1}{5}\right)^n$

$$= \sum_{n=1}^{\infty} \left(\frac{9}{5}\right)^n \quad \text{divergent.}$$

<u>Thm</u> (i) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\boxed{\lim_{n\to\infty} a_n = 0}$

(ii) If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3n^3+2n+5}$ divergent.

$$\lim_{n\to\infty} \frac{n^3/n^3}{(3n^3+2n+5)/n^3} = \lim_{n\to\infty} \frac{1}{3+\frac{2}{n^2}+\frac{5}{n^3}}$$

$$= \frac{1}{3}$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{n^6+2n+1}$ (divergent)

$$\lim_{n\to\infty} \frac{n^5/n^5}{(n^6+2n+1)/n^5} = \lim_{n\to\infty} \frac{1}{n+\frac{2}{n^4}+\frac{1}{n^5}} = 0$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\sim \int_1^{\infty} \frac{1}{x}\,dx = +\infty$$

Ex. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)} + 2^{2n}5^{1-n}\right) = ?$

$$= \sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)}\right) + \sum_{n=1}^{\infty} 2^{2n}5^{1-n}$$

$$\sum_{n=1}^{\infty}\frac{5}{n(n+1)} = 5\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 5.$$

$$\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty}(2^2)^n\left(\frac{1}{5}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4(4)^{n-1}\left(\frac{1}{5}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1} = \frac{4}{1-\frac{4}{5}} = 20$$

Hence $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)} + 2^{2n}5^{1-n}\right) = 5 + 20 = 25$