Review of §11.1 & §11.2

Sequence $\{a_n\}$

series $\sum_{n=1}^{\infty} a_n$

partial sum: $S_n = \sum_{k=1}^{n} a_k$

$\sum_{n=1}^{\infty} a_n$ convergent $\iff \lim_{n\to\infty} S_n = L$.

$= L$

(i) <u>geometric series</u> $\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r|\geq 1 \end{cases}$

(ii) $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ is divergent.

Integral test.

$\cancel{f(x) = a}$

Given a sequence $\{a_n\}$, decreasing to zero

Let $f(n) = a_n$

(i) If $\int_1^\infty f(x)\,dx$ is convergent

$\Rightarrow \sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent

$\Rightarrow \sum_{n=1}^\infty a_n$ is divergent.

Ex. $\cancel{f(x) =}$ $a_n = \frac{1}{n^2}$, $f(x) = \frac{1}{x^2}$

(then $f(n) = \frac{1}{n^2} = a_n$)

$\int_1^\infty \frac{1}{x^2}\,dx = \frac{1}{-2+1} x^{-1} \Big|_1^\infty = 1. \Rightarrow \sum_{n=1}^\infty \frac{1}{n^2}$ is ~~divergent~~ convergent

Ex. $\cancel{f(x)}$ $a_n = \frac{1}{\sqrt{n}}$. $\Rightarrow f(x) = \frac{1}{\sqrt{x}}$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \frac{1}{-\frac{1}{2}+1}\sqrt{x}\Big|_1^\infty = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Ex. (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (if $p>0$, $\frac{1}{n^p} \searrow 0$)

$$f(x) = \frac{1}{x^p}$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{-p+1} x^{-p+1}\Big|_1^\infty = \begin{cases} \frac{1}{p-1}, & p>1 \\ \text{divergent} & p<1 \end{cases}$$

$$\int_1^\infty \frac{1}{x}\,dx = \ln x\Big|_1^\infty = +\infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent only if $p>1$.

Ex. $a_n = \dfrac{\ln n}{n}$ (for $n \geq 3$, $a_n \searrow 0$)

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{x \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \leq 0$$

if $x \geq e$

$$\int_1^\infty \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 \Big|_1^\infty = \infty$$

$$\Rightarrow \sum_{n=1}^\infty \frac{\ln n}{n} \text{ is divergent.}$$

11.4 Comparison test

$$\sum_{n=1}^{\infty} a_n \quad , \quad \sum_{n=1}^{\infty} b_n \qquad \text{with } a_n, b_n \geq 0$$

then if $a_n \leq b_n$, for all $n \geq N$

(i) ~~and~~ if $\sum_{n=1}^{\infty} b_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is also convergent.

(ii) ~~if~~ if $\sum_{n=1}^{\infty} a_n$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} b_n$ is divergent.

Ex. $a_n = \frac{\ln n}{n}$, $b_n = \frac{1}{n}$

so $a_n \geq b_n \quad (n \geq 2)$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

Ex. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{5}{n^3+2n+3}}_{a_n}$ $\qquad b_n = \frac{5}{n^3}$

$a_n \le b_n$ for $n \ge 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^3} = 5 \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent

$\Rightarrow \displaystyle\sum \frac{5}{n^3+2n+3}$ is convergent.

Limit test: Given $\sum a_n$, $\sum b_n$

with $a_n, b_n \ge 0$

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c \ne 0$, then

either $\sum a_n$, $\sum b_n$ both are convergent,

or divergent.

Ex. $\sum \frac{5}{n^3+2n+3}$ set $b_n = \frac{1}{n^3}$

$$\underbrace{\sum \frac{5}{n^3+2n+3}}_{= a_n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5}{n^3+2n+3}}{\frac{1}{n^3}} = 5 \neq 0$$

$\Rightarrow \sum \frac{5}{n^3+2n+3}$ is convergent.

Ex. $\sum_{n=1}^{\infty} \frac{n^2+5n}{\sqrt{n^5+2n^3+1}}$, $b_n = \frac{n^2}{\sqrt{n^5}} = \frac{1}{\sqrt{n}}$

$$\frac{n^2+5n}{\sqrt{n^5+2n^3+1}} = a_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+5n}{\sqrt{n^5+2n^3+1}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\,(n^2+5n) \,/\, \sqrt{n^5}}{\sqrt{n^5+2n^3+1} \,/\, \sqrt{n^5}} = 1 \neq 0$$

Since $\sum \frac{1}{\sqrt{n}}$ is divergent $\Rightarrow \sum \frac{n^2+5n}{\sqrt{n^5+2n^3+1}}$ is divergent