11.5 Alternating Series (Recall) ✓

(i) $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ with

$b_n \geq 0$, and $\underline{b_{n+1} \leq b_n}$ for $n \geq N_0$

and $\lim_{n\to\infty} b_n = 0$

then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is convergent. ✓

Ex. $\underline{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}}$

is convergent.

$$\begin{cases} b_n = \frac{1}{\sqrt{n}} \geq 0 \\ b_n \to 0 \\ b_{n+1} \leq b_n \end{cases}$$

(ii) How to estimate $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$?

Define the partial sum

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$$

then if the conditions in (i) are satisfied.

we have

$$\cancel{S = \sum_{n=}} \quad |S - S_n| \le b_{n+1} \quad !$$

Q. How many terms in $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ to get a two-digit accuracy?

We need $b_{n+1} = \frac{1}{\sqrt{n+1}} \le 0.01$

$$\Longleftrightarrow \frac{1}{n+1} \le 0.0001 \quad \Rightarrow n+1 \ge 10000.$$

11.6. Absolute convergence.

$$\sum_{n=1}^{\infty} a_n$$

Def. if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

then we say $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>

Ex. Consider $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$. $\Rightarrow$ is convergent.

$\Rightarrow$ is Not absolutely convergent.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Ex. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ ③ is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \overset{①}{\leq} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent}$$

② is convergent

Sketch of the proof:

(i) $\lim \left|\frac{a_{n+1}}{a_n}\right| = L < 1$

for n big enough, $\left|\frac{a_{n+1}}{a_n}\right| \le r < 1$ $(r \ge L)$

$|a_{n+1}| \le |a_n| r$

$|a_{n+2}| \le |a_{n+1}| r \le |a_n| r^2$

$\vdots$

$|a_{n+k}| \le |a_n| r^k$

$\sum_{k=1}^{\infty} |a_n| r^k$ is convergent for $|r| < 1$

$\Rightarrow \sum |a_n|$ is convergent

(ii) If $\lim \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

for big enough n $\left|\frac{a_{n+1}}{a_n}\right| \geq r > 1$

$$|a_{n+1}| \geq |a_n| r \geq |a_n|$$

$\lim |a_n| \neq 0 \Rightarrow \sum a_n$ is divergent.

(iii) (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \longrightarrow 1$

but $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent.

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$, then $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow 1$

but it is divergent.

Def. If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, we say $\sum_{n=1}^{\infty} a_n$ is <u>conditionally</u> convergent

<u>Thm.</u> If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
then $\sum_{n=1}^{\infty} a_n$ has to be convergent.

proof. $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$$0 \le \underline{a_n} + \underline{|a_n|} \le 2|a_n|$$

by comparison test, $\Rightarrow \sum (a_n + |a_n|)$ is convergent.

$\Rightarrow \sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is convergent.

Ratio test: for $\sum_{n=1}^{\infty} a_n$

(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

(iii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, nothing can be concluded.

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{2^n}$ with $a_n = (-1)^{n-1}\frac{n^4}{2^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^4/2^{n+1}}{n^4/2^n} = \frac{(n+1)^4}{2n^4} \xrightarrow{n\to\infty} \frac{1}{2}$$

$\Rightarrow \sum (-1)^{n-1}\frac{n^4}{2^n}$ is absolutely convergent.

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^n}{n^4}$ is divergent!

Ex. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is absolutely convergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)\, n^n}{(n+1)\,(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\longrightarrow \frac{1}{e} < 1 \quad \left(\lim \left(1+\frac{1}{n}\right)^n = e\right)$$

The root test.

(i) $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1 \Rightarrow \sum a_n$ is absolutely convergent.

(ii) $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1 \Rightarrow \sum a_n$ is divergent

(iii) $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow$ Nothing can be concluded.

Ex. $\sum_{n=1}^{\infty} \left(\frac{2n-1}{5n+3} \right)^n$ is absolutely convergent.

root test:

$$\sqrt[n]{\left(\frac{2n-1}{5n+3}\right)^n} = \frac{2n-1}{5n+3} \xrightarrow{n \to \infty} \frac{2}{5} < 1$$