11.7

Goemetric series: $\displaystyle\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

P-series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent} & \text{if } p>1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}$.

<u>Fact</u> $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ divergent.

Integral test: $a_n = f(n)$, $\displaystyle\int_1^{\infty} f(x)\,dx$.

Comparison test: $a_n \leq b_n$

limit test: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$

ratio test: $\displaystyle\lim \left|\frac{a_{n+1}}{a_n}\right| = L$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

alternative series test: $\sum (-1)^{n-1} a_n$, $a_{n+1} \leq a_n$, $\underbrace{\ldots}_{}$ $\lim_{n\to\infty} a_n = 0$

Ex. $\sum_{n=1}^{\infty} \frac{3n+5}{6n-2}$ is divergent.

$$\lim_{n\to\infty} \frac{(3n+5)/n}{(6n-2)/n} = \lim_{n\to\infty} \frac{3+\frac{5}{n}}{6-\frac{2}{n}} = \frac{1}{2} \neq 0.$$

Ex. $\sum_{n=1}^{\infty} \frac{\sqrt{n^5+2n}}{n^4+3n^3+5} = \sum_{n=1}^{\infty} a_n$ converges.

$$\Uparrow$$

$b_n = \frac{\sqrt{n^5}}{n^4} = \frac{1}{n^{3/2}}$, $\sum b_n$ converges

(Use limit test: $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$

then $\sum a_n$, $\sum b_n$ both converge or both divergent.)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^5+2n}\;\; n^{3/2}/n^4}{(n^4+3n^3+5)/n^4} = \lim_{n\to\infty} \frac{\sqrt{1+2\frac{1}{n^4}}}{1+\frac{3}{n}+\frac{5}{n^4}}$$

$$= 1 > 0$$

Ex. $\displaystyle\sum_{n=1}^{\infty} n e^{-2n^2} = \sum_{n=1}^{\infty} a_n$

Use the integral test:

(if $a_n = f(n)$, $\displaystyle\int_1^{\infty} \underline{\underline{f(x)\,dx}}$ exists.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ is convergent)

$f(n) = n e^{-2n^2}$, $\displaystyle\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-2x^2}\,dx$

$$\frac{x^2 = u}{2x\,dx = du} \quad \int_1^{\infty} \frac{1}{2} e^{-2u}\,du = -\frac{1}{4} e^{-2u}\Big|_1^{\infty} = \frac{1}{4} e^{-2}$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} n e^{-2n^2}$ convergent.

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+3}$ is convergent.

not absolutely convergent.

$$a_n = \frac{n^2}{n^3+3} \Rightarrow \lim_{n\to\infty} a_n = 0.$$

Need to check:

$$a_{n+1} \le a_n.$$

(i) $\frac{(n+1)^2}{(n+1)^3+3} \overset{?}{\le} \frac{n^2}{n^3+3}$.

(ii) $f(x) = \frac{x^2}{x^3+3}$, $a_n = f(n)$

we need $f'(x) < 0$:

$$f'(x) = \frac{2x(x^3+3) - x^2(3x^2)}{(x^3+3)^2} = \frac{x(6-x^3)}{(x^3+3)^2} < 0$$

if $x^3 - 6 > 0 \Rightarrow n \ge 2$, $\underline{a_{n+1} \le a_n}$

Ex. $\sum_{n=1}^{\infty} \frac{3}{2+2^n}$, $\sum_{n=1}^{\infty} \frac{3}{2^n}$.

$= \frac{1}{2} \sum_{n=1}^{\infty} 3 \left(\frac{1}{2}\right)^{n-1}$

→ is convergent.

Comparison test: $\frac{3}{2+2^n} \le \frac{3}{2^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{3}{2+2^n}$ convergent

limit test: also works.

Ex. $\sum_{n=1}^{\infty} \frac{3^n}{n!} \Rightarrow$ $\overset{\text{absolutely}}{\text{convergent}}$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0$

11.8 Power Series.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (\text{about } x=0)$$

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n \qquad (\text{about } x=a)$$

Ex. $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n}(x-3)^n}_{a_n}$, for what x, this is convergent?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}|x-3|^{n+1}}{\frac{1}{n}|x-3|^n} = \frac{n}{n+1}|x-3| \xrightarrow{n\to\infty} |x-3|$$

(i) if $|x-3| < 1 \iff -1 < x-3 < 1$
$\iff 2 < x < 4$ convergent

(ii) if $|x-3| > 1 \iff x > 4$ or $x < 2$ divergent.

(iii) $|x-3| = 1 \iff x = 2$ or $x = 4$.

$x = 2$: $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \Rightarrow$ convergent.

$x = 4$: $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ divergent.

To summarize:

$\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converges for $\underline{2 \leq x < 4}$

but diverges for $x < 2$ or $x \geq 4$

The interval of convergence is: $[2, 4)$

radius of convergence: $\underline{1}$.