11.8. Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$ — power series about $x = a$

Q. How to find the $\underline{\text{interval}}$ of convergence

radius of convergence R ?

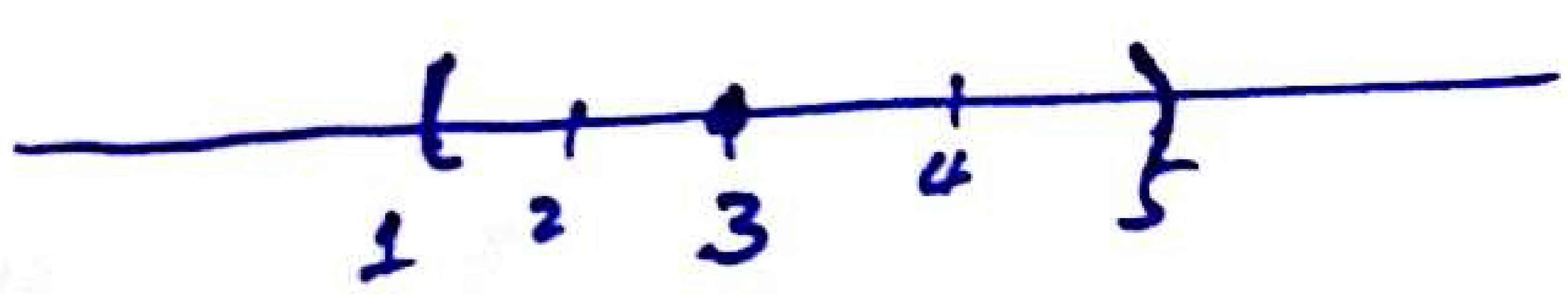
A. Use ratio test.

Ex. $$\underbrace{\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-3)^n}{2^n}}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}}{2^{n+1}} \frac{2^n}{|x-3|^n} = \frac{|x-3|}{2} < 1.$$

(i) we need $\boxed{|x-3| < 2}$ $\Rightarrow$ The radius of convergence $R = 2$.

(ii) what about



$|x-3| = 2$?

$x = 1$ or 5 ?

$x=1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^{2n+1}$ is divergent

$x=5$: $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{2^n}=\sum_{n=0}^{\infty}(-1)^{n+1}$ is divergent.

Interval of convergence $=(1,5)$

Ex. $\displaystyle J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (Bessel function) of order 0

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}\cdot\frac{2^{2n}(n!)^2}{x^{2n}}$$

$$=\frac{x^2}{2^2(n+1)^2}\xrightarrow{n\to\infty}0$$

Interval of convergence: $(-\infty,+\infty)$

radius of convergence: $R=\infty$

Ex. $\displaystyle\sum_{n=0}^{\infty} n!\,(x-2)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x-2|^{n+1}}{n!\,|x-2|^n} = \frac{(n+1)\,|x-2|}{1}$$

$$\xrightarrow{n\to\infty} \begin{cases} \infty & x \neq 2 \\ 0 & x = 2 \end{cases}$$

Interval of convergence: $\{2\}$

radius of convergence: $R = 0$.

11.9

For $\sum_{n=0}^{\infty} x^n$ — interval of convergence $|x|<1$, radius of convergence $R=1$.

It is also a geometric series. $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$ for $|r|<1$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x} \quad \text{for } |x|<1$$

* We can represent $\boxed{\frac{1}{1-x} \text{ as } \sum_{n=0}^{\infty} x^n !}$

Basic formula: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$.

Ex. Represent $\frac{1}{1+x^2}$ as a power series:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2| < 1 \iff |x| < 1$

Ex. $\frac{1}{3+2x} = \frac{1}{3\left(1+\frac{2x}{3}\right)}$

$$= \frac{1}{3}\frac{1}{1-\left(-\frac{2x}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n$$

for $\left|\frac{-2x}{3}\right| < 1 \iff |x| < \frac{3}{2}$.

Thm: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent for $\underline{|x-a| < R}$.

Then $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{1}{n+1}(x-a)^{n+1} + C.$$

for $|x-a| < R$.

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Ex. $$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\int_0^1 J_0(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \int_0^1 x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \frac{1}{2n+1}$$

Q. Compute $\int_0^1 J_0(x)\,dx$ up to three digit of accuracy.

we only need to find the smallest n such that

$$\frac{1}{2^{2(n+1)}((n+1)!)^2(2(n+1)+1)} \leq 0.001$$

Ex. $\underline{\ln(1+x)}$ as a power series

$$f(x) = \ln(1+x) \quad , \quad f'(x) = \frac{1}{1+x}$$

$$= \sum_{n=0}^{\infty} (-x)^n$$

$$\Rightarrow \ln(1+x) = f(x) = \int f'(x)\,dx$$

$$= \sum_{n=0}^{\infty} \int (-x)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} + C .$$

Take $x=0 \Rightarrow \ln 1 = C \Rightarrow C=0$

$$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}$$