Trigonomitic Substitution (II):

$$\sqrt{a^2-x^2} \xrightarrow[dx = a\cos\theta\, d\theta]{x = a\sin\theta} \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\cos\theta$$

$$\sqrt{x^2+a^2} \xrightarrow[dx = a\sec^2\theta\, d\theta]{x = a\tan\theta} \sqrt{a^2(\tan^2\theta+1)}$$

$$= a\sec\theta.$$

$$\sqrt{x^2-a^2} \xrightarrow[dx = a\sec\theta\tan\theta\, d\theta]{x = a\sec\theta} \sqrt{a^2(\sec^2\theta-1)}$$

$$= a\tan\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = \tan^2\theta + 1$$

Ex. $\displaystyle\int \frac{1}{\sqrt{x^2-16}}\,dx \overset{x=4\sec\theta}{=} \int \frac{1}{4\sqrt{\sec^2\theta-1}}\,dx$

$dx = 4\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\cancel{4}\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta$$

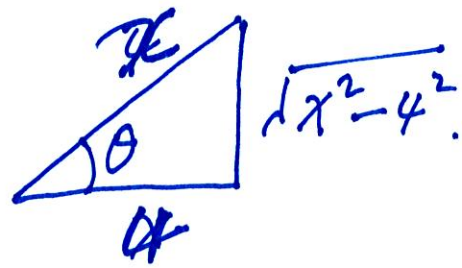
$$= \ln\left|\sec\theta + \tan\theta\right| + C.$$

$$= \boxed{\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-4^2}}{4}\right|} + C$$

$$\left(x = 4\sec\theta \iff \theta = \sec^{-1}\frac{x}{4}\right)$$

$$= \ln\left|\frac{x}{4} + \tan\sec^{-1}\frac{x}{4}\right| + C$$

$$x = 4\,\frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{4}{x}$$



$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

$$\int \frac{1}{\sqrt{x^2 - 16}}\,dx \xrightarrow[dx = 4\sinh\theta\,d\theta]{x = 4\cosh\theta} \int \frac{4\sinh\theta}{4\sqrt{\cosh^2\theta - 1}}\,d\theta.$$

$$= \int d\theta = \theta + C.$$

$$= \boxed{\cosh^{-1}\frac{x}{4}} + C.$$

$$x = \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$\sqrt{x^2-1} = \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$+ \Rightarrow x + \sqrt{x^2-1} = e^\theta$$

$$\Rightarrow \underbrace{\theta}_{=\cosh^{-1}x} = \ln\left|x + \sqrt{x^2-1}\right|.$$

Ex. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{3\sqrt{3}/2} \frac{x^3}{\sqrt{(2x)^2+3^2}^{\,3}}\,dx$

$\underline{\underline{2x = 3\tan\theta}}$

$2\,dx = 3\sec^2\theta\,d\theta$

$$\int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\left[\cancel{3}\sqrt{\tan^2\theta+1}\right]^3}\,\frac{\cancel{3}}{2}\sec^2\theta\,d\theta$$

$\left(\cancel{2\cdot\frac{3\sqrt{3}}{2}}\quad \cancel{2}\cdot\frac{\cancel{3}\sqrt{3}}{\cancel{2}} = \cancel{3}\tan\theta\right)$

$$= \int_0^{\frac{\pi}{3}} \cancel{\frac{27}{16}}\ \frac{3}{16}\,\frac{\tan^3\theta}{\sec^3\theta}\sec^2\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^{\cancel{3}}\theta\cdot\sin\theta}{\cos^2\theta}\,d\theta.$$

$\underline{\underline{u = \cos\theta}}$

$du = -\sin\theta\,d\theta$

$$-\frac{3}{16}\int_1^{\frac{1}{2}} \frac{1-u^2}{u^2}\,du = \frac{3}{16}\int_{\frac{1}{2}}^{1}\left(\frac{1}{u^2}-1\right)du$$

$$= \cdots = 3/32$$

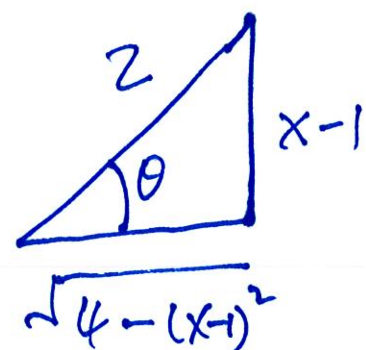
Ex. $\int \frac{x}{\sqrt{3+2x-x^2}}\,dx = \int \frac{x}{\sqrt{2^2-(x-1)^2}}\,dx$

$x-1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ $\quad \int \frac{1+2\sin\theta}{\cancel{2\cos\theta}}\,\cancel{2\cos\theta}\, d\theta$

$= \theta - 2\cos\theta + C$

$\left(\frac{x-1}{2} = \sin\theta\right.$



$= \boxed{\sin^{-1}\frac{x-1}{2} - \cancel{2}\cdot\frac{\sqrt{4-(x-1)^2}}{\cancel{2}} + C}$

$3+2x-x^2 = 3+1-(1-2x+x^2)$

$= 4-(x-1)^2$