# §7.4. Rational functions by partial fraction.

ex. $\displaystyle\int \frac{x+5}{x^2-x-6}\,dx$

$$= \int \left( \frac{8/5}{x-3} - \frac{3}{5}\cdot\frac{1}{x+2} \right) dx$$

$$= \tfrac{8}{5} \ln|x-3| - \tfrac{3}{5} \ln|x+2| + C.$$

I. If $\frac{P(x)}{Q(x)}$ with $\deg(P(x)) < \deg(Q(x))$,

we say $\frac{P(x)}{Q(x)}$ is proper. factorize $Q(x)$ first.

$$\frac{x+5}{x^2-x-6} = \frac{x+5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$\Rightarrow x+5 = A(x+2) + B(x-3)$$

$x=3 \Rightarrow 5A = 8 \Rightarrow A = 8/5$, $x=-2 \Rightarrow B = -3/5$

II. If $\deg(P(x)) \geq \deg(Q(x))$, then we say $\frac{P(x)}{Q(x)}$ is improper. In this case, do a long division first.

ex. $$\int \frac{x^3+5}{x-1}\,dx$$

$$= \int \left(x^2+x+1+\frac{6}{x-1}\right)dx = \frac{1}{3}x^3+\frac{1}{2}x^2+x+6\ln|x-1|+C$$

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$$\begin{array}{r l}
 & x^2 + x + 1 \\
x-1 \,\big) & x^3 + 0x^2 + 0x + 5 \\
 & -)\; x^3 - x^2 \\
\hline
 & x^2 + 0x + 5 \\
 & -)\; x^2 - x \\
\hline
 & x + 5 \\
 & -)\; x - 1 \\
\hline
 & 6
\end{array}$$

$\Rightarrow x^3+5 = (x-1)(x^2+x+1)+6.$

Ex. $\displaystyle\int \frac{x^2-5}{x^2-x-6}\,dx = \int 1 + \underline{\frac{x+1}{x^2-x-6}}\,dx$

$\displaystyle = \int \left(1 + \frac{4}{5}\frac{1}{x-3} + \frac{1}{5}\frac{1}{x+2}\right) dx$

$\displaystyle = x + \frac{4}{5}\ln|x-3| + \frac{1}{5}\ln|x+2| + C$

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$x+1 = A(x+2) + B(x-3)$

$x=3 \Rightarrow A = \frac{4}{5}$ ; $x=-2 \Rightarrow B = 1/5$.

$\displaystyle \frac{x+1}{x^2-x-6} = \frac{(x+1)}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$

$\displaystyle = \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$

$$
\begin{array}{r|l}
 & \quad 1 \\
\cline{2-2}
x^2-x-6 & x^2 - 0x - 5 \\
 & -)\ x^2 - x - 6 \\
\hline
 & \quad x+1
\end{array}
$$

$\Rightarrow x^2-5 = (x^2-x-6) + x+1$

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$a\ln b = \ln(b^a)$, $\ln a + \ln b = \ln(a\cdot b)$

$\ln a - \ln b = \ln\frac{a}{b}$

Ex. $$\int \frac{x^2+5x+1}{x^3-3x^2+2x}\,dx$$

$$= \int \left( \frac{1}{2}\,\frac{1}{x} + \frac{15}{2}\,\frac{1}{x-2} - 7\,\frac{1}{x-1} \right) dx$$

$$= \frac{1}{2}\ln|x| + \frac{15}{2}\ln|x-2| - 7\ln|x-1| + C.$$

$$= \ln\sqrt{|x|} + \ln\sqrt{|x-2|}^{15} - \ln|x-1|^{7} + C.$$

$$= \ln\left\{ \frac{\sqrt{|x|}\cdot\sqrt{|x-2|}^{15}}{|x-1|^{7}} \right\} + C$$

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$$x^3-3x^2+2x = x(x^2-3x+2) = x(x-2)(x-1)$$

$$\frac{x^2+5x+1}{x^3-3x^2+2x} = \frac{x^2+5x+1}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$$

$$= \frac{A(x-2)(x-1) + B(x-1)x + C(x-2)x}{x(x-2)(x-1)}$$

$$x=0 \Rightarrow 1 = 2A$$

$$x=1 \Rightarrow 7 = -C$$

$$x=2 \Rightarrow 4+10+1 = 2B$$

$$\Rightarrow \begin{cases} A = \frac{1}{2} \\ C = -7 \\ B = 15/2 \end{cases}$$

Ex. $\displaystyle\int \frac{x^2+5}{x^3-x^2-x+1}\,dx$

$$= \int \left( -\frac{1}{2}\,\frac{1}{x-1} + 3\,\frac{1}{(x-1)^2} + \frac{3}{2}\,\frac{1}{x+1} \right) dx$$

$$= -\frac{1}{2}\ln|x-1| - 3\,\frac{1}{x-1} + \frac{3}{2}\ln|x+1| + C.$$

$$\begin{array}{r|l}
 & x^2 \quad -1 \\
\hline
x-1 & x^3-x^2-x+1 \\
 & -)\;x^3-x^2 \\
\hline
 & -x+1 \\
 & -)\;-x+1 \\
\hline
 & 0
\end{array}$$

$$\Rightarrow x^3-x^2-x+1 = (x-1)(x^2-1) = (x-1)^2(x+1)$$

$$\frac{x^2+5}{x^3-x^2-x+1} = \frac{x^2+5}{(x-1)^2\cdot(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$x=1 \Rightarrow 6 = 2B \Rightarrow B=3$ $\qquad x=0 \Rightarrow \boxed{5 = -A+B+C}$

$x=-1 \Rightarrow 6 = 4C \quad C = 3/2$ $\qquad \Rightarrow A = -1/2$