

§ 7.4. Rational functions by partial fractions.

$$\begin{aligned} \text{ex. } & \int \frac{x+5}{x^2-x-6} dx \\ &= \int \left(\frac{8/5}{x-3} - \frac{3}{5} \cdot \frac{1}{x+2} \right) dx \\ &= \frac{8}{5} \ln|x-3| - \frac{3}{5} \ln|x+2| + C. \end{aligned}$$

I. If $\frac{P(x)}{Q(x)}$ with $\deg(P(x)) < \deg(Q(x))$,

we say $\frac{P(x)}{Q(x)}$ is proper. Factorize $Q(x)$ first.

$$\begin{aligned} \frac{x+5}{x^2-x-6} &= \frac{x+5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \\ &= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)} \end{aligned}$$

$$\Rightarrow x+5 = A(x+2) + B(x-3)$$

$$x=3 \Rightarrow 5A = 8 \Rightarrow A = 8/5, \quad x=-2 \Rightarrow B = -3/5$$

II. If $\deg(P(x)) \geq \deg(Q(x))$, then we say

$\frac{P(x)}{Q(x)}$ is improper. In this case, do a
long division first.

ex. $\int \frac{x^3 + 5}{x-1} dx$

$$= \int (x^2 + x + 1 + \frac{6}{x-1}) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 6 \ln|x-1| + C$$

$$\begin{array}{r} x^2 + x + 1 \\ x-1 \overline{) x^3 + 0x^2 + 0x + 5} \\ \underline{-(x^3 - x^2)} \\ x^2 + 0x + 5 \\ \underline{-(x^2 - x)} \\ x + 5 \\ \underline{-(x - 1)} \\ 6 \end{array}$$

$$\Rightarrow x^3 + 5 =$$

$$(x-1)(x^2 + x + 1) + 6$$

$$\text{Ex. } \int \frac{x^2 - 5}{x^2 - x - 6} dx = \int 1 + \frac{x+1}{x^2 - x - 6} dx$$

$$= \int \left(1 + \frac{4}{5} \frac{1}{x-3} + \frac{1}{5} \frac{1}{x+2} \right) dx$$

$$= x + \frac{4}{5} \ln|x-3| + \frac{1}{5} \ln|x+2| + C$$

$$x+1 = A(x+2) + B(x-3)$$

$$x=3 \Rightarrow A = \frac{4}{5}; \quad x=-2 \Rightarrow B = \frac{1}{5}$$

$$\begin{aligned} \frac{x+1}{x^2-x-6} &= \frac{(x+1)}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \\ &= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)} \end{aligned}$$

$$\begin{array}{r} 1 \\ \hline x^2 - x - 6 \overline{) x^2 - 0x + 5} \\ -) x^2 - x - 6 \\ \hline x + 1 \end{array}$$

$$\Rightarrow x^2 - 5 = (x^2 - x - 6) + x + 1$$

$$a \ln b = \ln(b^a), \quad \ln a + \ln b = \ln(a \cdot b)$$

$$\ln a - \ln b = \ln \frac{a}{b}$$

$$\text{Ex. } \int \frac{x^2 + 5x + 1}{x^3 - 3x^2 + 2x} dx$$

$$= \int \left(\frac{1}{2} \frac{1}{x} + \frac{15}{2} \frac{1}{x-2} - 7 \frac{1}{x-1} \right) dx$$

$$= \frac{1}{2} \ln |x| + \frac{15}{2} \ln |x-2| - 7 \ln |x-1| + C$$

$$= \ln \sqrt{|x|} + \ln \sqrt{|x-2|}^{15} - 7 \ln |x-1| + C$$

$$= \ln \left\{ \frac{\sqrt{|x|} \cdot \sqrt{|x-2|}^{15}}{|x-1|^7} \right\} + C$$

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-2)(x-1)$$

$$\frac{x^2 + 5x + 1}{x^3 - 3x^2 + 2x} = \frac{x^2 + 5x + 1}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}$$

$$= \frac{A(x-2)(x-1) + B(x-1)x + C(x-2)x}{x(x-2)(x-1)}$$

$$\begin{array}{l} x=0 \Rightarrow 1 = 2A \\ x=1 \Rightarrow 7 = -C \\ x=2 \Rightarrow 4 + 0 + 1 = 2B \end{array} \Rightarrow \begin{cases} A = \frac{1}{2} \\ C = -7 \\ B = \frac{15}{2} \end{cases}$$

$$\text{Ex. } \int \frac{x^2 + 5}{x^3 - x^2 - x + 1} dx$$

$$= \int \left(\frac{1}{2} \frac{1}{x-1} + 3 \frac{1}{(x-1)^2} + \frac{3}{2} \frac{1}{x+1} \right) dx$$

$$= -\frac{1}{2} \ln|x-1| - 3 \frac{1}{x-1} + \frac{3}{2} \ln|x+1| + C$$

$$\begin{array}{r} x^2 - 1 \\ \hline x-1 \overline{) x^3 - x^2 - x + 1} \\ \underline{-x^3 + x^2} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

$$\begin{aligned} \Rightarrow x^3 - x^2 - x + 1 &= (x-1)(x^2 - 1) \\ &= (x-1)^2 (x+1) \end{aligned}$$

$$\begin{aligned} \frac{x^2 + 5}{x^3 - x^2 - x + 1} &= \frac{x^2 + 5}{(x-1)^2 \cdot (x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \\ &= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)} \end{aligned}$$

$$x=1 \Rightarrow 6 = 2B \Rightarrow B=3 \quad x=0 \Rightarrow \boxed{5 = -A + B + C}$$

$$x=-1 \Rightarrow 6 = 4C \quad C = \frac{3}{2} \quad \Rightarrow A = -\frac{1}{2}$$