§7.4 rational function

Ex $\displaystyle\int \frac{\sqrt{x+16}}{x}\,dx$ (rationalize)

$u=\sqrt{x+16}$

$u^2 = x+16$

$2u\,du = dx$

$$\int \frac{u}{u^2-16}\,2u\,du = 2\int \frac{u^2-16+16}{u^2-16}\,du$$

$$= 2\int\left(1 + 16\,\frac{1}{(u+4)(u-4)}\right)du$$

$$= 2\left(u + \int\left(\frac{A}{u+4} + \frac{B}{u-4}\right)du\right) = \cdots$$

$$\frac{A}{u+4} + \frac{B}{u-4} = \frac{A(u-4)+B(u+4)}{(u+4)(u-4)}$$

$u=4 \Rightarrow 8B = 16 \Rightarrow B = 2$

$u=-4 \quad -8A = 16 \Rightarrow A = -2.$

7.5 Please read by yourself.

7.6 Integration by tables

$$\int \frac{x^2}{\sqrt{9-4x^2}}\,dx$$

$\sqrt{a^2-u^2}$

$\sqrt{9-4x^2} = \sqrt{3^2-(2x)^2}$

$$\underline{\underline{u=2x}} \int \frac{1}{4}\frac{u^2}{\sqrt{3^2-u^2}}\frac{1}{2}\,du$$

$$= \frac{1}{8}\int \frac{u^2}{\sqrt{3^2-u^2}}\,du = \frac{1}{8}\left[-\frac{u}{2}\sqrt{3^2-u^2} + \frac{3^2}{2}\sin^{-1}\frac{u}{3} + C\right]$$

$$= \frac{1}{8}\left[-x\sqrt{9-4x^2} + \frac{9}{2}\sin^{-1}\frac{2}{3}x\right] + C.$$

Ex. $\int x^3 \cos 3x\, dx \xrightarrow[x=\frac{1}{3}u]{u=3x} \int \frac{1}{3\cancel{x}\cancel{x}} u^3 \cos u\, du$

$= \frac{1}{81}\left[\int u^3 \cos u\, du\right] = \frac{1}{81}\left[u^3 \sin u - 3\underline{\int u^2 \sin u\, du}\right]$

$= \frac{1}{81}\left[u^3 \sin u - 3\left(-u^2 \cos u + 2\underline{\int u \cos u\, du}\right)\right]$

$= \frac{1}{81}\left\{u^3 \sin u + 3u^2 \cos u - 6\left(u \sin u - \int \sin u\, du\right)\right\}$

$= \frac{1}{81}\left\{u^3 \sin u + 3u^2 \cos u - 6(u \sin u + \cos u + c)\right\}$

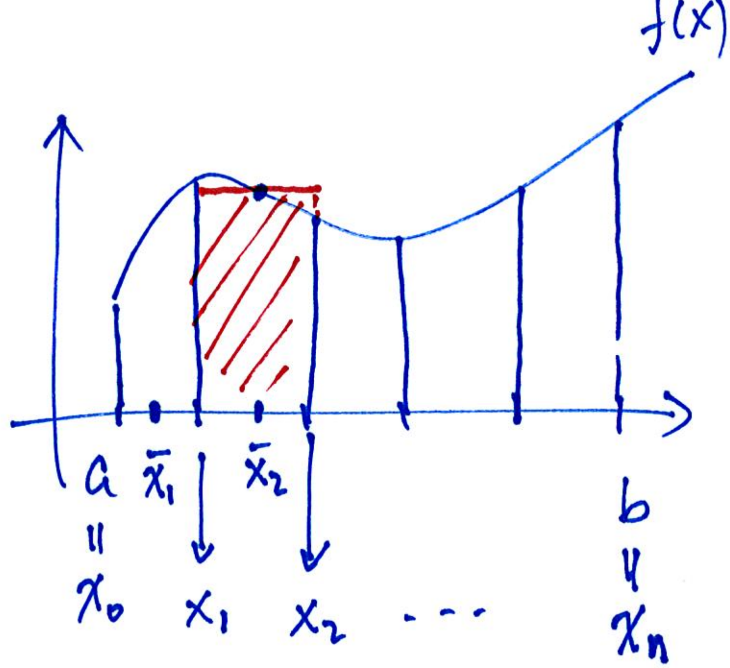
$\underline{\underline{u=3x}}$ ....

$\int u^n \cos u\, dx = u^n \sin u - n\int u^{n-1} \sin u\, du$

$\int u^n \sin u\, dx = -u^n \cos u + n\int u^{n-1} \cos u\, du$

7.7 Approximating integrals

$$\int_a^b f(x)\,dx$$



(I) Midpoint rule

$$\Delta A_k = h f(\bar{x}_k) \qquad [a,b] = [x_0, x_1] + [x_1, x_2] + \cdots [x_{n-1}, x_n]$$

$$= h f\left(\frac{x_k + x_{k-1}}{2}\right) \quad \text{with } x_{k+1} - x_k = \underline{\underline{h}}$$

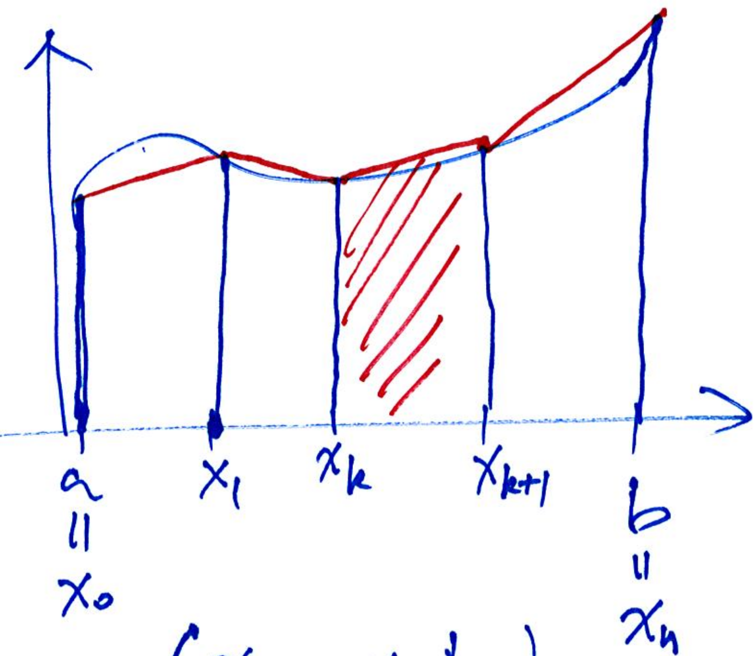
$$\int_a^b f(x)\,dx \simeq M_n = \sum_{k=1}^{n} \Delta A_k = h \sum_{k=1}^{n} f\left(\frac{x_k + x_{k-1}}{2}\right)$$

ex. $\int_0^1 x^3\,dx$ (with $h = \frac{1}{4}$, $n = 4$), $x_k = k \cdot \frac{1}{4}$

$$\simeq \frac{1}{4}\left(f\left(\frac{\frac{1}{4}+0}{2}\right) + f\left(\frac{\frac{2}{4}+\frac{1}{4}}{2}\right) + f\left(\frac{\frac{3}{4}+\frac{2}{4}}{2}\right) + f\left(\frac{1+\frac{3}{4}}{2}\right)\right)$$

$$= \frac{1}{4}\left(\left(\frac{1}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{7}{8}\right)^3\right) = \cdots$$

II. Trapezoid rule



$$\Delta A_k = (x_{k+1} - x_k)\,\frac{f(x_k) + f(x_{k+1})}{2}$$

$$\int_a^b f(x)\,dx \approx \overbrace{\sum_{k=1}^{n} \Delta A_k}^{T_n =} \qquad (x_{k+1} - x_k = h)$$

$$= \frac{h}{2}\left\{ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right\}$$

Ex. $$\int_0^1 x^3\,dx \underset{n=4}{\overset{h=\frac{1}{4}}{\approx}} \frac{\frac{1}{4}}{2}\left\{ 0^3 + 2\cdot\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3 \right\}$$

$$= \ldots$$

How to compute x_k: if we devide $[a, b]$ by n intervals of same length

$$h = \frac{b-a}{n}, \quad x_k = a + k\cdot h = a + k\cdot\frac{b-a}{n}, \quad \underline{\underline{k = 0, 1, \ldots, n}}$$

This means it is comparable to h^2.

<u>Errors</u>

$$\int_a^b f(x)\,dx - T_n \simeq O(h^2)$$

$$\int_a^b f(x)\,dx - M_n \simeq O(h^2)$$

with $h = \frac{b-a}{n}$.

for 6th digits accuracy you need:

$$h^2 \leq 10^{-6}$$

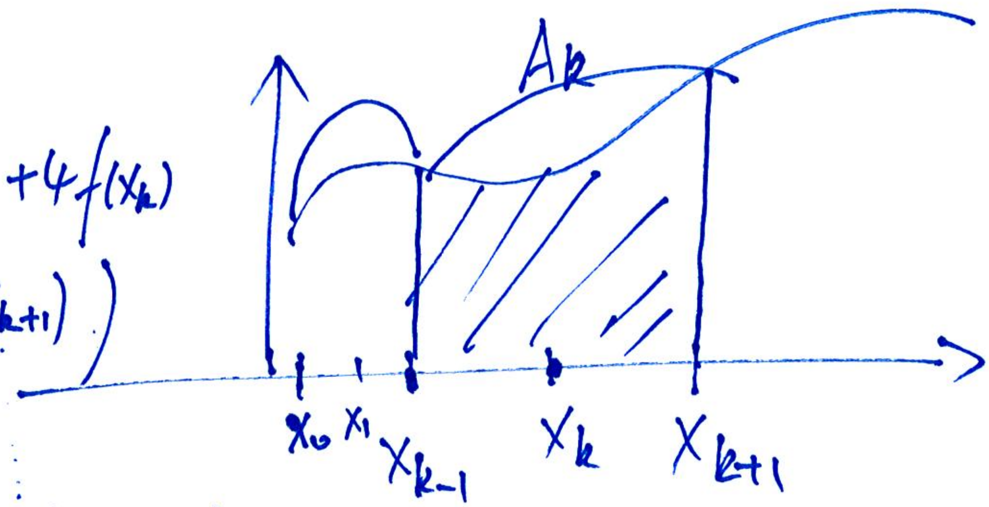
$$\Longleftrightarrow h \leq 10^{-3}$$

$$\Longleftrightarrow \frac{b-a}{n} \leq 10^{-3}$$

$$\Longleftrightarrow n \geq 1000\,(b-a)$$

Simpson rule with $2n$ intervals

$$\Delta A_k = \frac{h}{3}\Big(f(x_{k-1}) + 4f(x_k) + f(x_{k+1})\Big)$$



with $h = x_{k+1} - x_k = x_k - x_{k-1}$

Errors $\simeq O(h^4)$ $\Longrightarrow$ Much better than Midpoint & trapezoid rules!

$$\int_a^b f(x)\,dx \simeq S_n = \sum_{k=1}^{n} \Delta A_k$$

$$= \frac{h}{3}\Big\{ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + \cdots + 4f(x_{2n-1}) + f(x_{2n}) \Big\}$$

with $h = \dfrac{b-a}{2n}$, **Note**: this formula is with $2n$ subintervals!