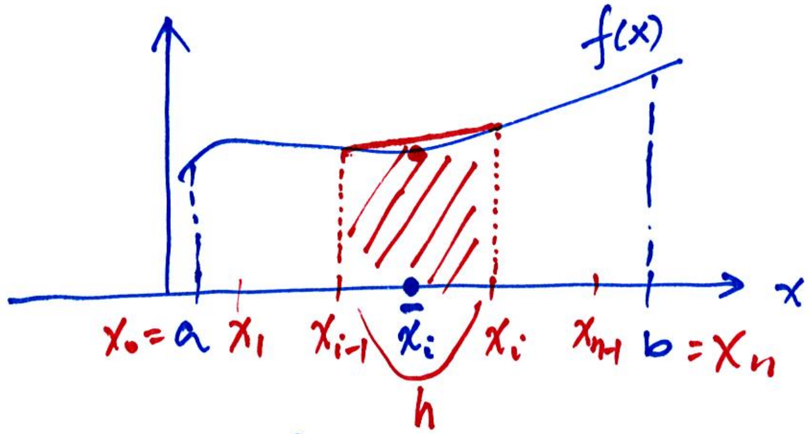
Summary of §7.7:



Midpoint rule:

$$\Delta x = h = \frac{b-a}{n}, \quad x_i = a + ih$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2}$$

$$\int_a^b f(x)\,dx \simeq M_n = h \sum_{k=1}^{n} f(\bar{x}_k) = h\left(f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right)$$

Trapezoid rule:

$$\int_a^b f(x)\,dx \simeq T_n = \frac{h}{2}\left\{f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right\}$$

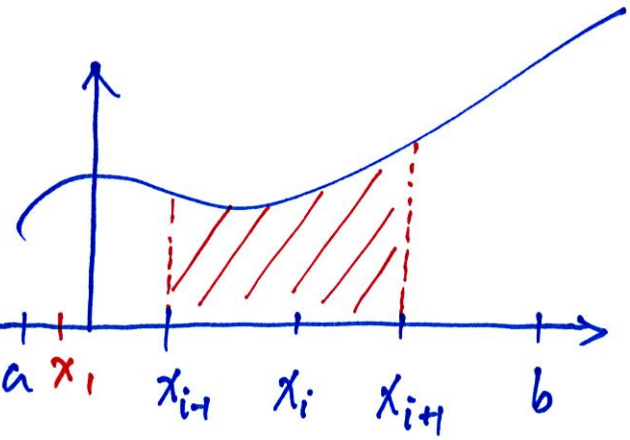
Errors:

$$\left| \int_a^b f(x)\,dx - M_n \ (\text{or } T_n) \right| \leq K\,h^2$$

Simpson's rule: Divide $[a,b]$ by $n=2m$ intervals

$$\Delta x = h = \frac{b-a}{n} \quad (n=2m), \quad x_i = a + ih$$

$$\Delta A = \frac{h}{3}\left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right)$$



$$\int_a^b f(x)\,dx \simeq S_n = \frac{h}{3}\Big\{ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + \underline{\underline{4}}\,f(x_{n-1}) + f(x_n) \Big\}$$

Error:

$$\left| \int_a^b f(x)\,dx - S_n \right| \le K h^{4}$$

$$\int_0^{0.6} e^{-x^2}\,dx \simeq S_6 \overset{h = \frac{0.6}{6} = 0.1}{=\!=\!=} \frac{0.1}{3}\Big\{ e^{-0^2} + 4 \cdot e^{-0.1^2} + 2e^{-0.2^2} + 4e^{-0.3^2} + 2e^{-0.4^2} + 4e^{-0.5^2} + e^{-0.6^2} \Big\} = \cdots$$

§7.8 Improper integrals

Type I: $\int_a^b f(x)\,dx$ with a or/and b is "$\pm\infty$".

$0<p<1 \Rightarrow$ divergent

$\Rightarrow$ $p>1 \Rightarrow$ convergent.

Ex. $\int_1^{+\infty} \frac{1}{x^p}\,dx$ for $p>0$, $p\neq 1$

Def. $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$

If the limit exists $\Rightarrow \int_a^{+\infty} f(x)\,dx$ **convergent**

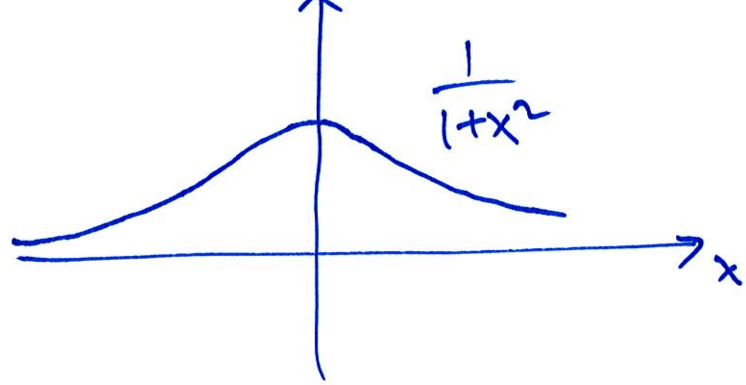
" ~~do~~ does not exists $\Rightarrow$ " **divergent**

Ex. $\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{1}{-p+1} x^{-p+1}\right]_1^t$

$$= \lim_{t\to\infty} \left[\frac{1}{-p+1}\left(t^{-p+1} - 1\right)\right] = \begin{cases} \text{divergent} & -p+1>0 \\ \frac{-1}{1-p} & -p+1<0 \end{cases}$$

Ex. $$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \left(\int_{-\infty}^{0} + \int_{0}^{+\infty}\right) \frac{1}{1+x^2}\,dx$$

$$= 2\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx$$



$$= 2 \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx = 2\cdot\frac{\pi}{2} = \pi.$$

$$\int_{0}^{t} \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_{0}^{t} = \tan^{-1}t = u$$

$$t = \tan u$$

$$\lim_{t\to\infty} \tan^{-1}t = \frac{\pi}{2}$$

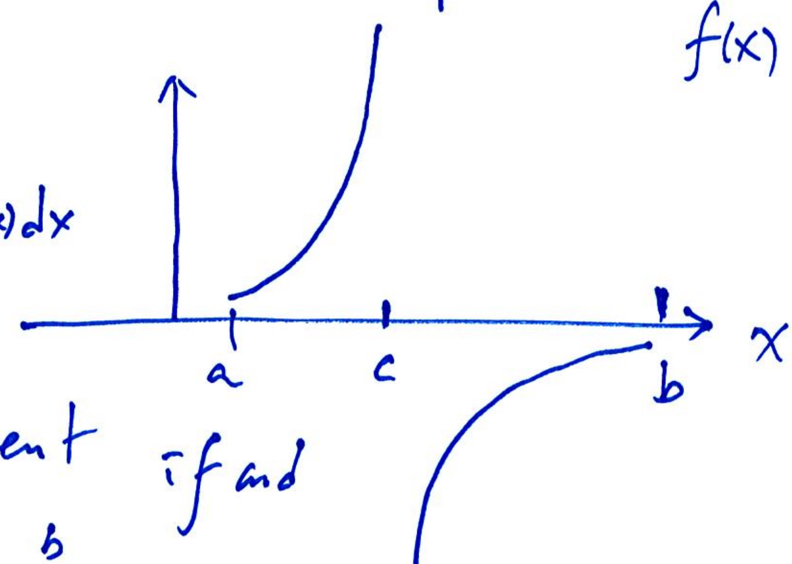
$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{divergent} & 0<p<1 \\ \text{''} & p=1 \\ \frac{1}{p-1} & p>1 \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t = \lim_{t\to\infty} \ln t = +\infty$$

Type II: $f(x)$ discontinuous at some point.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Def. $\int_a^b f(x)\,dx$ is convergent if and only if $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ are convergent.

Ex. $$\int_0^4 \frac{1}{x-2}\,dx$$

Trouble point : $x = 2$.

$$\int_0^4 \frac{1}{x-2}\,dx = \int_0^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx$$

$$\int_0^2 \frac{1}{x-2}\,dx = \Big[\ln|x-2|\Big]_0^2 = +\infty.$$

$\Rightarrow \displaystyle\int_0^4 \frac{1}{x-2}\,dx$ is divergent.

<u>Note</u> $$\int_0^4 \frac{1}{x-2}\,dx = \Big[\ln|x-2|\Big]_0^4$$

$$= \ln 2 - \ln 2 = 0$$

Ex. $\displaystyle\int_1^3 \frac{1}{\sqrt{x-1}}\,dx = 2\sqrt{2}.$

Trouble point: $x = 1.$

$$\int_1^3 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} \int_t^3 \frac{1}{\sqrt{x-1}}\,dx$$

$$= \lim_{t\to 1^+} \left\{ 2\left(\sqrt{2} - \sqrt{t-1}\right) \right\}$$

$$= 2\sqrt{2}.$$

$$\int_t^3 \frac{1}{\sqrt{x-1}}\,dx = \left[\frac{1}{-\frac{1}{2}+1} \sqrt{x-1} \right]_t^3 = 2\left(\sqrt{2} - \sqrt{t-1}\right)$$

Ex. $\displaystyle\int_0^2 \ln x\,dx$

$$\int_0^2 \ln x \, dx$$

Trouble point: $x \le 0$

$$\int_0^2 \ln x \, dx = \lim_{t \to 0^+} \int_t^2 \ln x \, dx = \underline{2\ln 2 - 2}$$

$$\int_t^2 1 \ln x \, dx \xrightarrow[v = x]{u = \ln x} \int u \, dv = uv - \int v \, du$$

$$= x \ln x - \int x \frac{1}{x} dx = \left[x \ln x - x \right]_t^2$$

$$= \underline{2\ln 2 - 2} - (t \ln t - t) \xrightarrow{t \to 0^+} \underline{\underline{2\ln 2 - 2}}$$