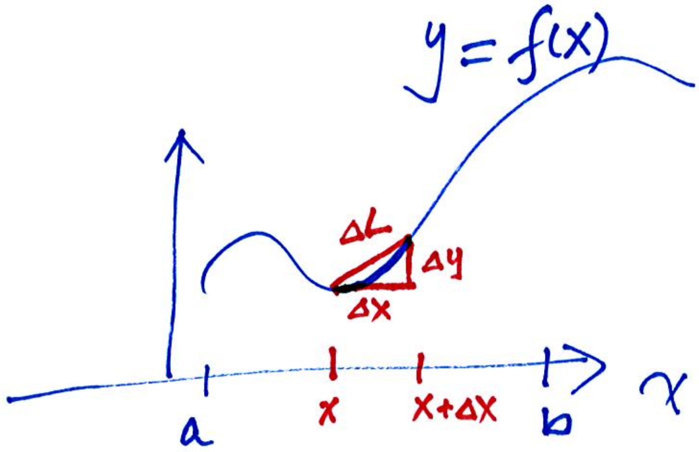
chap 8

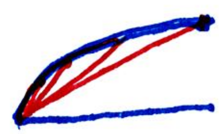
8.1. Arc length



$$\Delta L = \sqrt{\Delta x^2 + \Delta y^2}$$

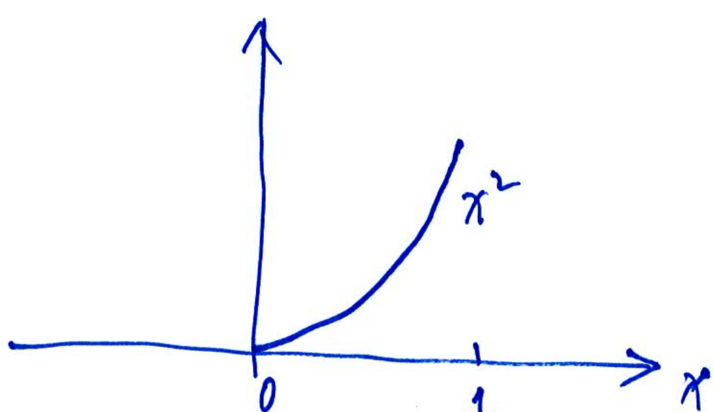
$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$



$$= \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

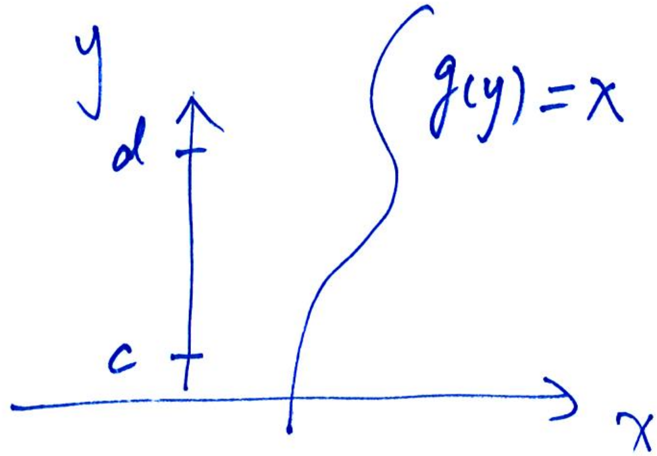
Ex. $y = x^2$



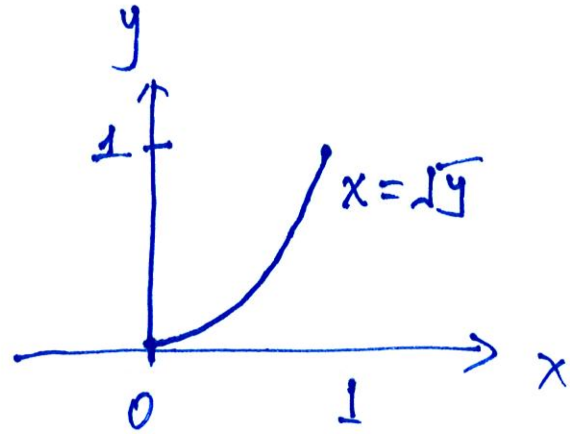
$$L = \int_0^1 \sqrt{1 + (2x)^2}\ dx$$

$$\underline{\underline{u=2x}}\ \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du = \frac{1}{2}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^2 = \cdots$$

$$\mathcal{L} = \int_c^d \sqrt{1+g'(y)^2}\, dy$$



$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$



$$\mathcal{L} = \int_0^1 \sqrt{1+\left(\tfrac{1}{2} y^{-\frac{1}{2}}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1+\frac{1}{4y}}\, dy = \int_0^1 \sqrt{\frac{1+4y}{4y}}\, dy$$

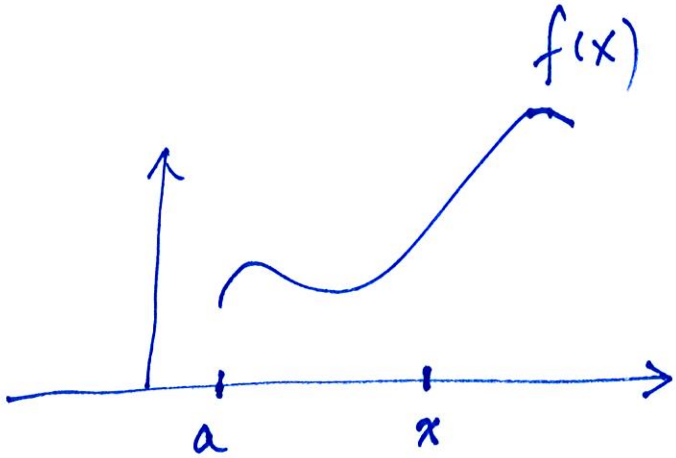
$$\underline{\underline{4y = u^2}} \qquad \int_0^2 \frac{\sqrt{1+u^2}}{\cancel{u}}\, \frac{1}{2}\cancel{u}\, du$$

$$4dy = 2u\,du$$

$$= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du$$

Arc Length function

Arc length of f(x) from a to x is:



$$S(x) = \int_a^x \sqrt{1+f'(t)^2}\,dt$$

Ex. Find the arc length function of $y = x^2 - \frac{1}{8}\ln x$ from $(1, 1)$ ~~and $(3, y(3))$~~

$$y' = 2x - \frac{1}{8x}$$

$$S(x) = \int_1^x \sqrt{1+\left(2t-\frac{1}{8t}\right)^2}\,dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right)dt = \left[t^2 + \frac{1}{8}\ln t\right]_1^x$$

$$= x^2 + \frac{1}{8}\ln x - 1$$

Arc length from $x=1$ to $x=3$:

Answer $= S(3) = 8 - \frac{1}{8}\ln 3$.

$$S(x) = \int_a^x \sqrt{1+f'(t)^2}\,dt \qquad (y=f(x))$$

$$\frac{ds}{dx} = \sqrt{1+f'(x)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

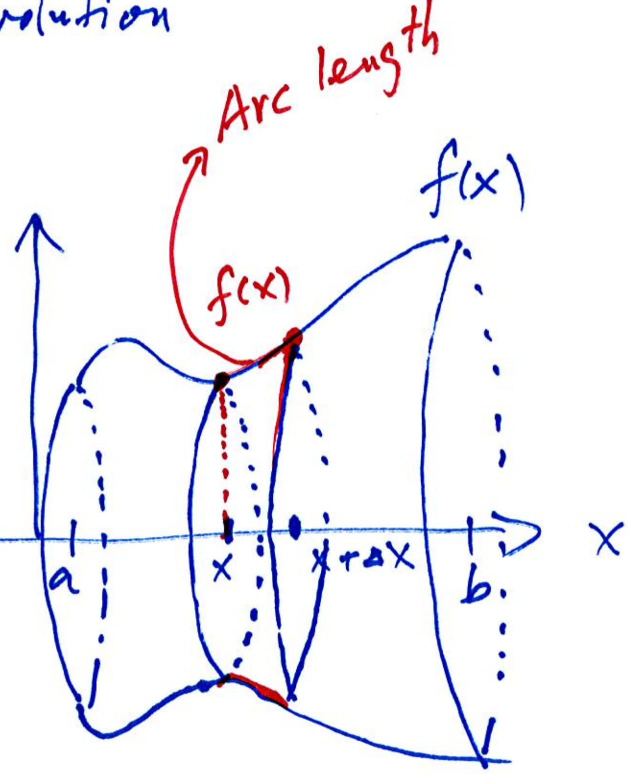
$$\boxed{ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

8.2 Surface area by revolution

revolution by x-axis:

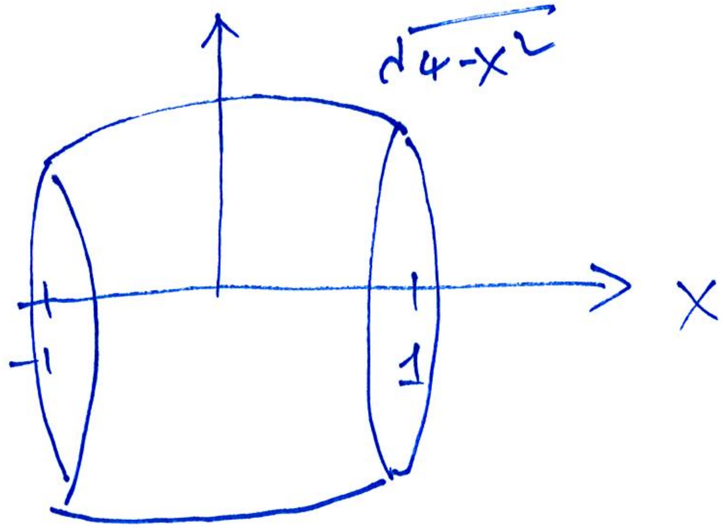
$$\Delta S = 2\pi f(x)\cdot\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta X$$

$$\Rightarrow \boxed{S = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dX}$$



Ex. $y = \sqrt{4-x^2}$ $\quad -1 \le x \le 1$

$y^2 = 4 - x^2$



$$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$$

$$= \int_{-1}^{1} 2\pi \cdot 2\,dx = 8\pi.$$

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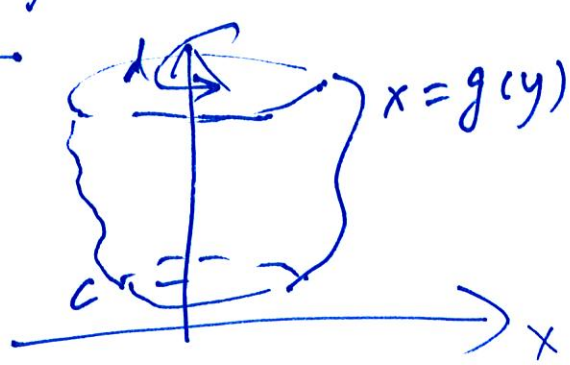
revolution of $x = g(y)$ by y-axis

$$S = \int_c^d 2\pi g(y)\sqrt{1+g'(y)^2}\,dy$$



$$= \int 2\pi x\,ds$$

revolution of $y=f(x)$ about x-axis

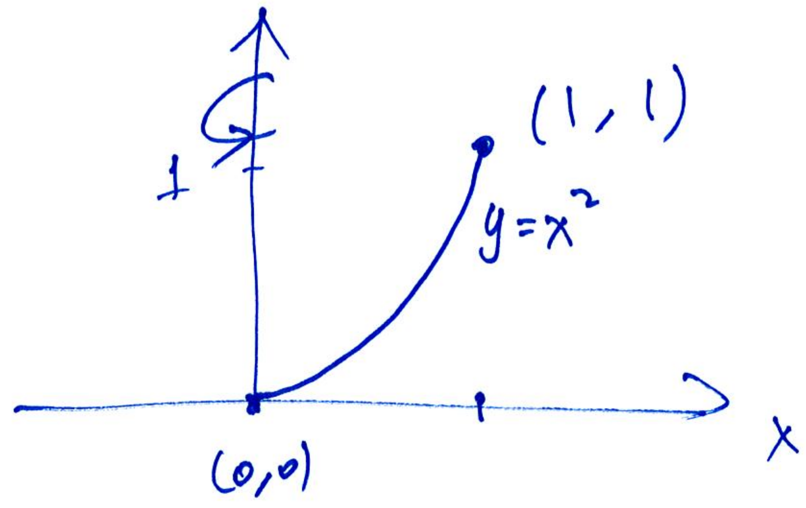
$$S=\int_a^b 2\pi f(x)\,\sqrt{1+f'(x)}\;dx$$

$$=\int 2\pi y\,ds$$

Ex.

$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$



option 1

$$S = \int_0^1 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \pi\int_0^1 \sqrt{1+4y}\, dy \quad \xrightarrow[du = 4dy]{u = 1+4y} \quad \pi\int_1^5 \sqrt{u}\, \frac{1}{4}\, du$$

$$= \frac{\pi}{4} \cdot \frac{1}{\frac{1}{2}+1}\left[u^{3/2}\right]\Big|_1^5 = \cdots$$

option 2

$$S = \int_0^1 2\pi x \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}_{ds}$$

$$= \int_0^1 2\pi x\sqrt{1+4x^2}\, dx = \cdots$$