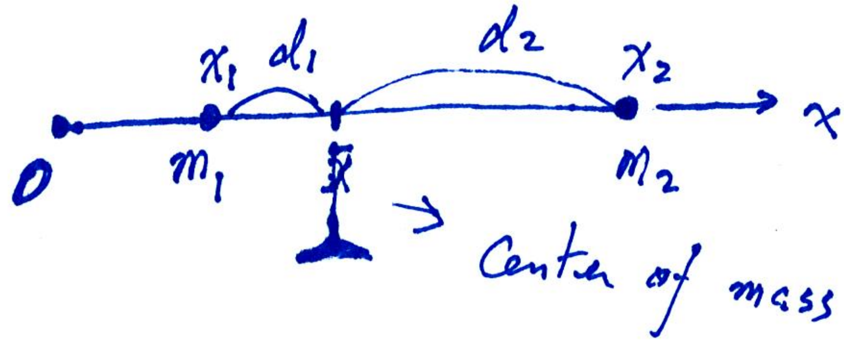
8.3 Applications

* Moments and Center of mass.



$$m_1 d_1 = m_2 d_2$$

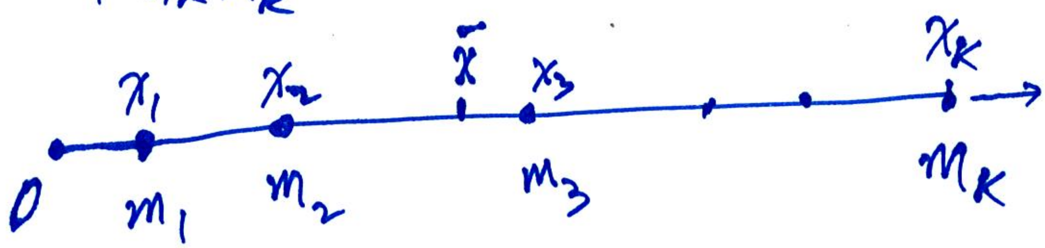
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \begin{array}{l} \rightarrow \text{moment} \\ \rightarrow \text{Total mass} \end{array}$$

$$M = m_1 x_1 + m_2 x_2 + \cdots + m_k x_k$$

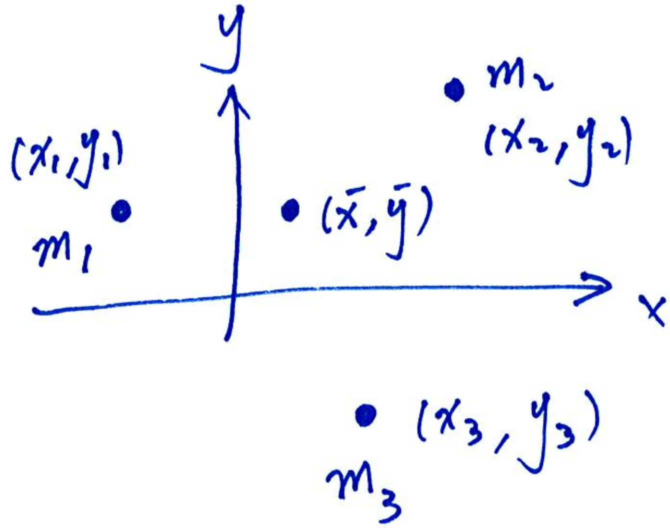
$$m = m_1 + m_2 + \cdots + m_k$$



$$\Rightarrow \bar{x} = \frac{M}{m} = \frac{\sum_{j=1}^{k} m_j x_j}{\sum_{j=1}^{k} m_j}$$

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3$$

(moment w.r.t. y-axis)



$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3$$

(moment w.r.t to x-axis)

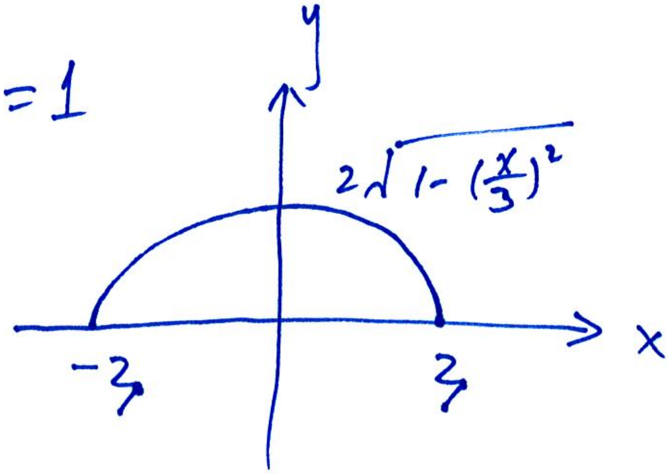
Total mass: $m = m_1 + m_2 + m_3$

$$\Rightarrow \quad \bar{x} = \frac{M_y}{m}, \qquad \bar{y} = \frac{M_x}{m}$$

For K masses:

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m} \quad \text{with} \quad \begin{cases} M_y = \sum_{j=1}^{K} m_j x_j \\ M_x = \sum_{j=1}^{K} m_j y_j \\ m = \sum_{j=1}^{K} m_j \end{cases}$$

Ex. $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, $\rho = 1$



$$y^2 = 4\left(1 - \left(\frac{x}{3}\right)^2\right)$$

$$y = 2\sqrt{1 - \left(\frac{x}{3}\right)^2}$$

by symmetry : $\boxed{\bar{x} = 0}$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{8}{3\pi}.$$

$$M_x = \frac{1}{2}\int_{-3}^{3} 4\left(1 - \left(\frac{x}{3}\right)^2\right)dx = 2\left[6 - \frac{1}{9}\,\frac{1}{3}x^3\Big|_{-3}^{3}\right]$$

$$= 2(6-2) = 8$$

$$m = \int_{-3}^{3} 2\sqrt{1-\left(\frac{x}{3}\right)^2}\,dx = 4\int_0^3 \sqrt{1-\left(\frac{x}{3}\right)^2}\,dx$$

$$\underline{\underline{\frac{x}{3} = \sin\theta}}$$

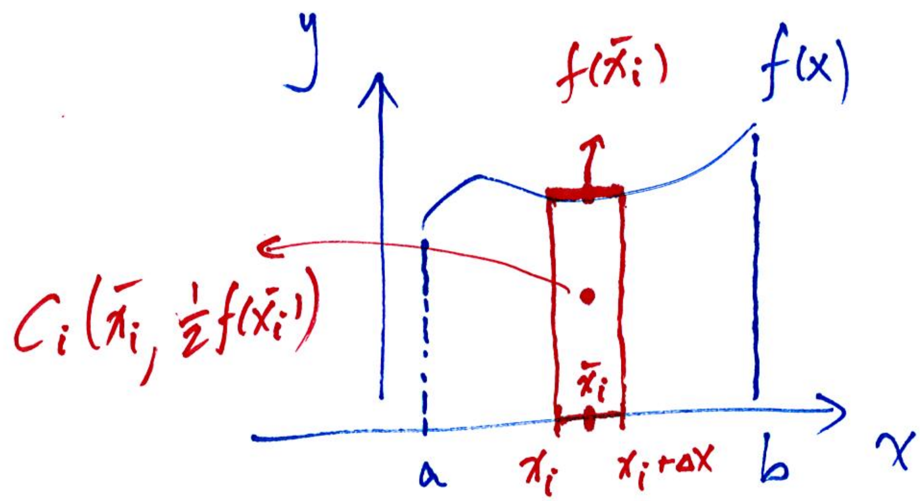
$$dx = 3\cos\theta\,d\theta$$

$$4\int_0^{\frac{\pi}{2}} \cos\theta\; 3\cos\theta\,d\theta = 6\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= 6\left(\frac{\pi}{2} + 0\right) = 3\pi$$

ρ : density

$m_i = \Delta x f(\bar{x}_i) \rho$



$M_y^i = m_i \bar{x}_i = \rho \bar{x}_i f(\bar{x}_i) \Delta x$

$$\lim \sum_{i=1}^{\infty} M_y^i = \boxed{\int_a^b \rho x f(x)\,dx = M_y}$$

$M_x^i = m_i y_i = m_i \frac{1}{2} f(\bar{x}_i) = \rho \frac{1}{2} f(\bar{x}_i)^2 \Delta x$

$$\lim \sum_{i=1}^{\infty} M_x^i = \boxed{\frac{1}{2}\int_a^b \rho f^2(x)\,dx = M_x}$$

Total mass:

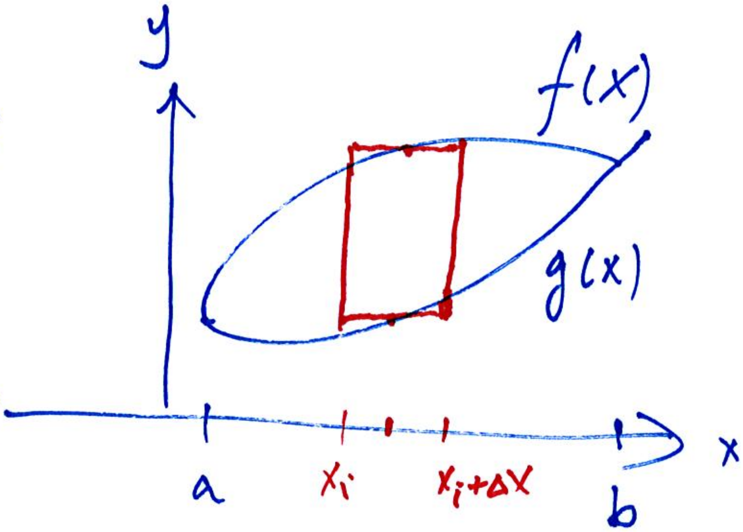
$$m = \rho \cdot \text{Area} = \rho \int_a^b f(x)\,dx$$

Center of mass $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

$$M_y = \rho \int_a^b x\,(f(x) - g(x))\,dx$$

$$M_x = \rho \int_a^b \frac{1}{2}\left\{f^2(x) - g^2(x)\right\} dx$$

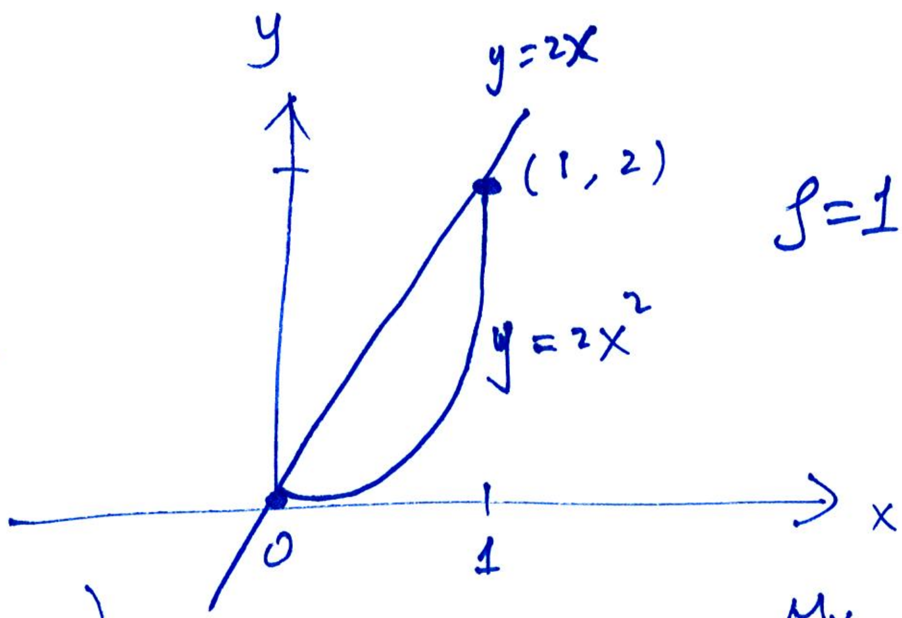


$$m = \rho \int_a^b (f(x) - g(x))\,dx$$

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

Ex. $2x = 2x^2$

$\Rightarrow x = 0, 1$



$\rho = 1$.

$$M_y = \int_0^1 x(2x - 2x^2)\,dx$$

$= \cdots$

$$M_x = \int_0^1 \frac{1}{2}\left((2x)^2 - (2x^2)^2\right) dx = \cdots$$

$$m = \int_0^1 (2x - 2x^2)\,dx = \cdots$$

$\Rightarrow$ $\bar{x} = \frac{M_y}{m}$

$\bar{y} = \frac{M_x}{m}$.