Review for EXAM I

* equations for spheres

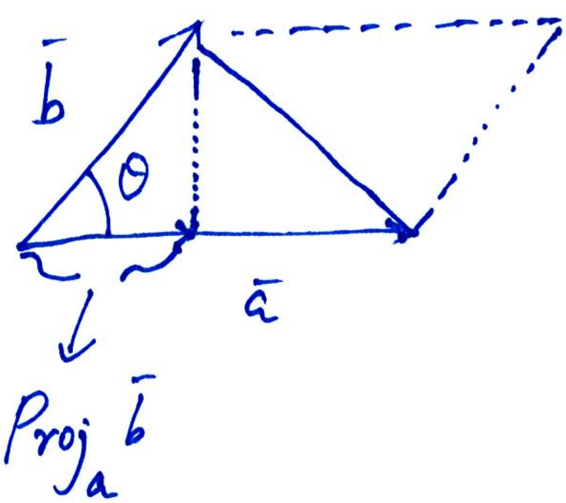
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

sphere with center at (a, b, c) and radius R.

* vectors,

$$\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta$$

$$|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta$$



$$|\text{Proj}_a \bar{b}| = |\bar{b}| \cos\theta$$

$$= \frac{|\bar{a}|\,|\bar{b}| \cos\theta}{|\bar{a}|} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|}$$

* $$\text{Proj}_{\bar{a}} \bar{b} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|} \frac{\bar{a}}{|\bar{a}|}$$

Area of the triangle

$$A = \frac{1}{2} |\bar{a} \times \bar{b}|$$

if $\bar{a} \cdot \bar{b} = 0 \iff \bar{a} \perp \bar{b}$

Ex. Find x s.t $\bar{a} = (1, -x)$, $\bar{b} = (4, x)$ are orthogonal

sol'n: we need $\bar{a} \cdot \bar{b} = 0$

$$\Rightarrow 4 - x^2 = 0 \quad \Rightarrow x = \pm 2.$$

Ex. $\bar{b} = (1, 2, 0)$, $\bar{a} = (3, 1, 2)$

Find $\text{Proj}_{\bar{a}} \bar{b}$:

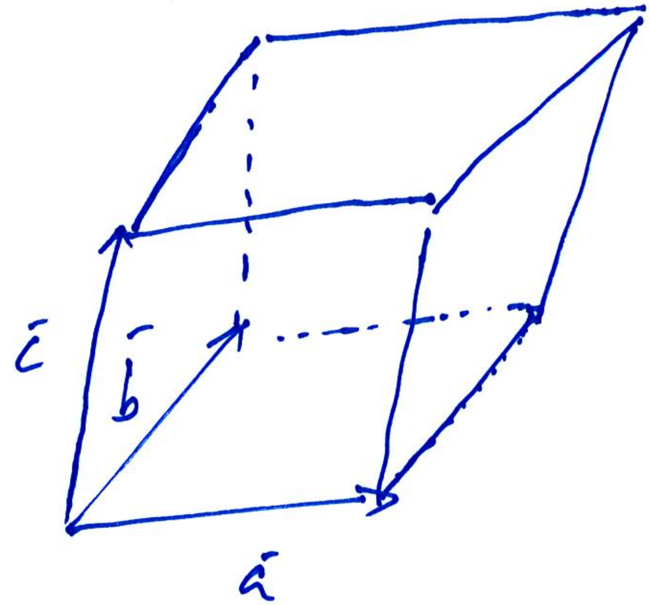
$\bar{a} \cdot \bar{b} = 3 + 2 = 5$

$|\bar{a}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$

$$\text{Proj}_{\bar{a}} \bar{b} = \frac{5}{14} (3, 1, 2)$$

Triple product.

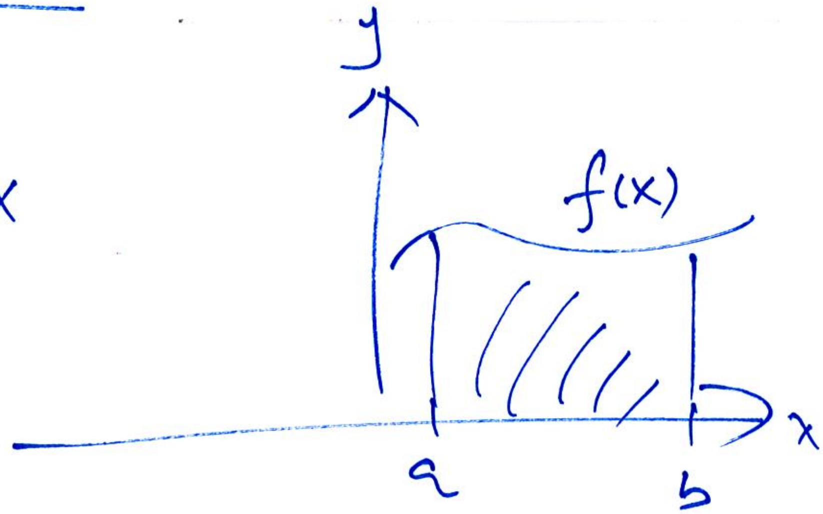
$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



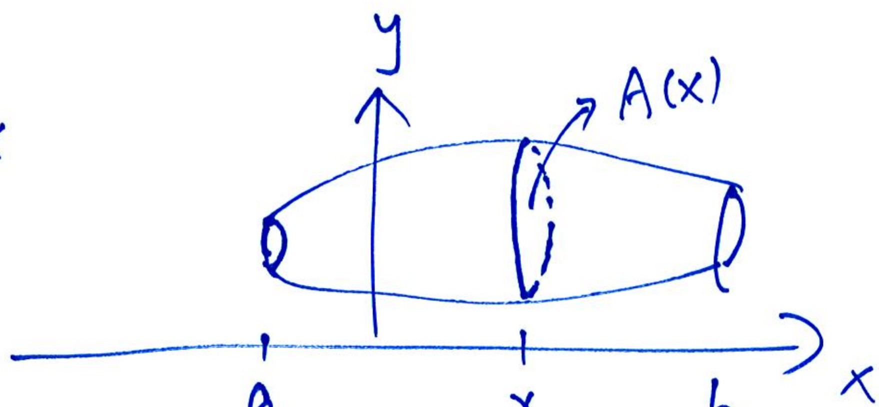
$$V = |\bar{a}\cdot(\bar{b}\times\bar{c})|$$

* Areas & volumes

$$\underline{\text{Area}} = \int_a^b f(x)\,dx$$



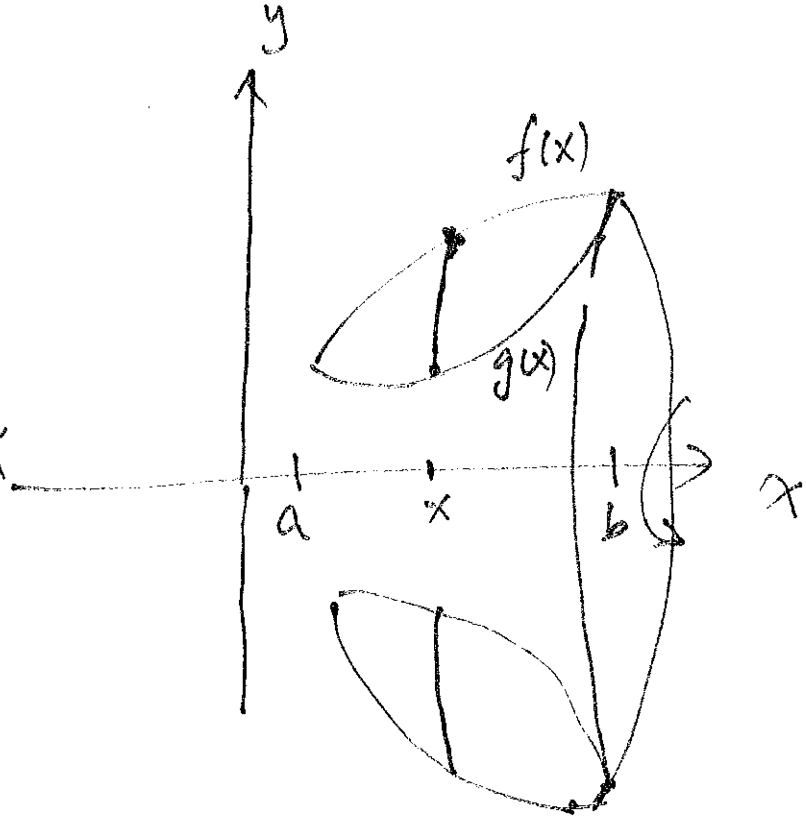
$$\text{Volume} = \int_a^b A(x)\,dx$$



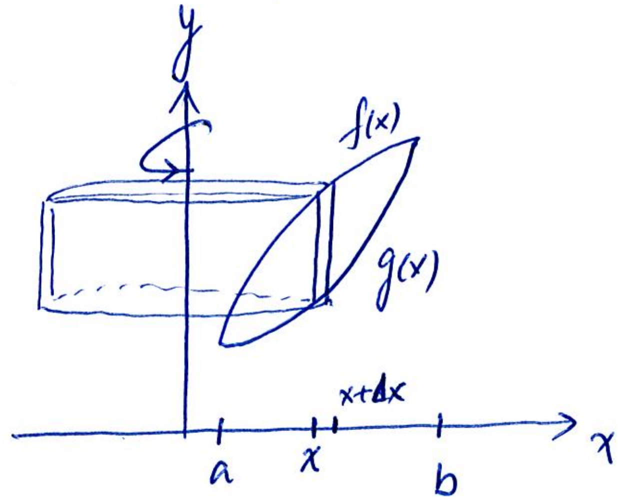
Method of disk / washer

$$A(x) = \pi\left(f^2(x) - g^2(x)\right)$$

$$V = \int_a^b \pi\left(f^2(x) - g^2(x)\right)dx$$



Method of Cylindrical shells : rotate by y-axis.



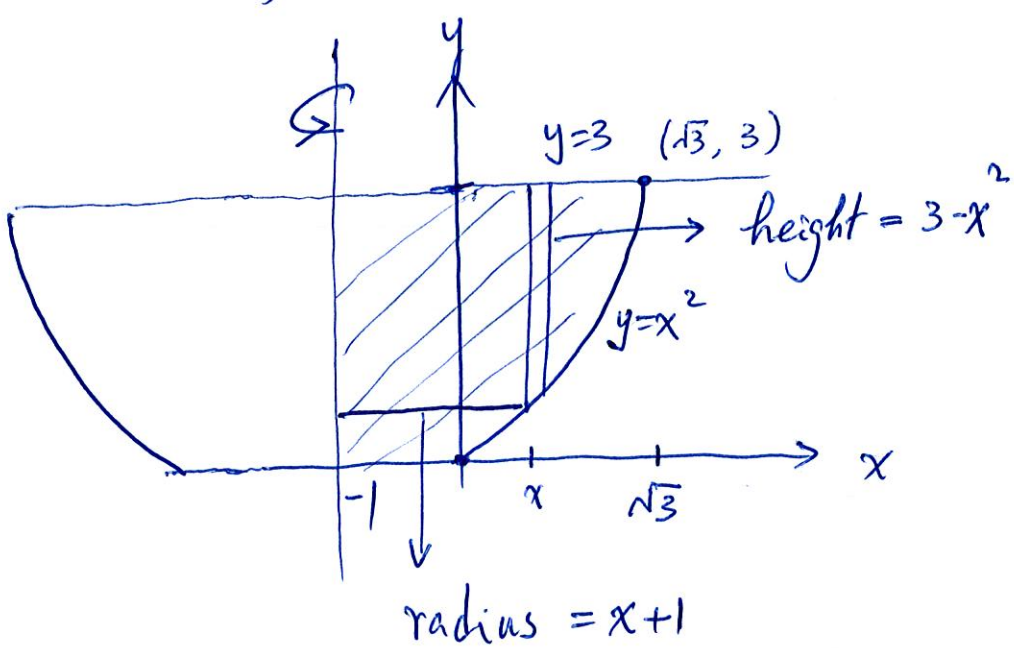
$\Delta V = 2\pi \cdot \text{radius} \cdot \text{height}\ \Delta x$

$= 2\pi x\,(f(x) - g(x))\,\Delta x$

$$V = \int_a^b 2\pi x\,(f(x) - g(x))\,dx$$

ex. Consider the area bounded by $y=0$, $y=3$ and $y=x^2$, rotate by $x=-1$. Find the volume.

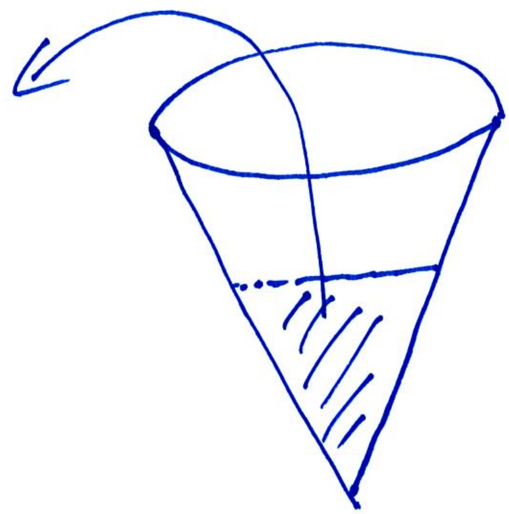
$$\boxed{V = \int_0^{\sqrt{3}} 2\pi(x+1)(3-x^2)\,dx}$$



Work = Force · distance.

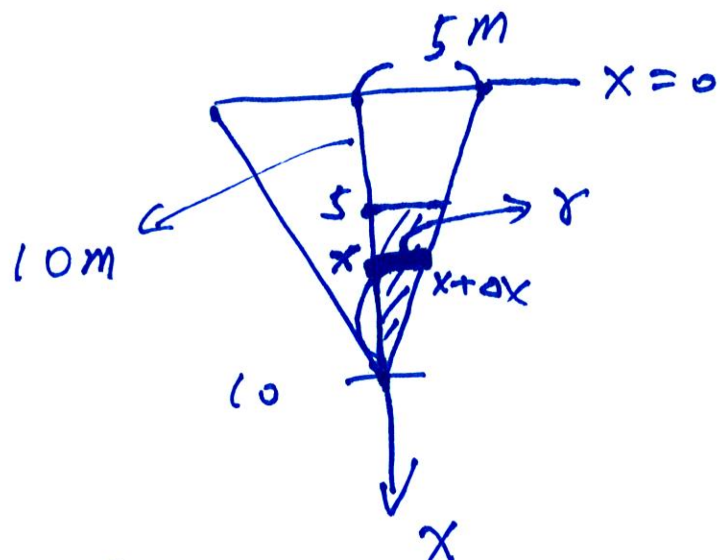
$$W = \int_a^b f(x)\,dx$$

Find the total work



$$\frac{10-x}{r} = \frac{10}{5}$$

$$\Rightarrow r = \frac{1}{2}(10-x)$$



$$\Delta V = \pi r^2 \Delta x$$

$$\Delta \text{Weight} = 1000 \cdot g\, \pi r^2 \Delta x$$

$$\Delta W = x \cdot 1000 g\, \pi r^2 \Delta x$$

$$W = \int_5^{10} x\, 1000 g\, \pi \frac{1}{4}(10-x)^2\, dx.$$

* Integrals

Integration by parts:

$$\int u\, dv = uv - \int v\, du$$

$u = f(x)$
$v = g(x) \Rightarrow$

$$\int_a^b f(x)\left(g'(x)\,dx\right) = f(x)\,g(x)\Big|_a^b - \int_a^b g(x)\left(f'(x)\,dx\right)$$

Ex. $$\int_1^e x \ln x \, dx = (\ln x)\tfrac{1}{2}x^2\Big|_1^e - \int_1^e \tfrac{1}{2}x^2 \cdot \tfrac{1}{x}\, dx$$

$$\begin{pmatrix} u = \ln x, & dv = x\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{2}x^2 \end{pmatrix}$$

$$= \tfrac{1}{2}e^2 - \tfrac{1}{4}x^2\Big|_1^e = \cdots$$

Ex. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx \;\overset{x=\sin\theta}{\underset{dx=\cos\theta\,d\theta}{=\!=\!=}}\; \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \underline{\cos^2\theta\,d\theta} = \int_0^{\frac{\pi}{2}} \underline{\tfrac{1}{2}(1+\cos 2\theta)}\,d\theta$$

$$= \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}} = \cdots$$

Ex. $\displaystyle\int \cos^5\theta\,\sqrt{\sin\theta}\,d\theta = \int \cos^4\theta\,\sqrt{\sin\theta}\;\underline{\cos\theta\,d\theta}$

$$\overset{u=\sin\theta}{\underset{du=\cos\theta\,d\theta}{=\!=\!=}} \int (1-u^2)^2\sqrt{u}\,du$$

$$= \int (1-2u^2+u^4)\sqrt{u}\,du = \cdots$$

(used $\sin^2 x + \cos^2 x = 1$)

Ex. $\int \tan^2 x \underline{\sec^2 x \, dx}$

$\underline{u = \tan x}$

$du = \sec^2 x \, dx$ $\quad \int u^2 \, du = \ldots$

* derivatives of / integration $\left\{ \begin{array}{l} \sin \\ \cos \\ \tan \\ \cot, \\ a^x \\ x^a \\ \ln x \end{array} \right.$

* Values of $\left\{ \begin{array}{l} \sin\theta \\ \cos\theta \\ \tan\theta \\ \cot\theta \end{array} \right.$ at $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

use

$\sec^2 x = \tan^2 x + 1$

$\frac{d}{dx}\tan x = \sec^2 x$

$\sin^2 x + \cos^2 x = 1$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1 - \sin 2\theta)$