Review (12 - 20)

7.3 Trig. substitution

$\int f(x)\,dx$

$f(x)$:	substitution
$\sqrt{a^2-x^2} = a\cos\theta$,	$x = a\sin\theta$
$\sqrt{a^2+x^2} = a\sec\theta$,	$x = a\tan\theta$
$\sqrt{x^2-a^2} = a\tan\theta$	$x = a\sec\theta$

$\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$

$\frac{d}{dx}(\tan\theta) = \sec^2\theta$, $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\sin 2\theta}{2}$

Ex. $\int \sqrt{9x^2+16}\, dx = \int \sqrt{(3x)^2+4^2}\, dx$

$\underline{\underline{3x = 4\tan\theta}}$

$3\,dx = 4\sec^2\theta\, d\theta$

$\int 4\sec\theta \, \frac{4}{3}\sec^2\theta\, d\theta.$

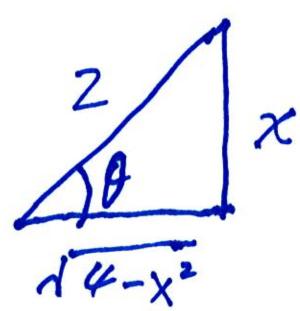
$\theta = \sin^{-1}\frac{x}{2}$

Ex. $\int \sqrt{4-x^2}\, dx \;\underset{dx = 2\cos\theta\, d\theta}{\overset{x = 2\sin\theta}{=\!=}}\; \int 2\cos\theta\, 2\cos\theta\, d\theta$

$= 2\int (1+\cos 2\theta)\, d\theta = 2\theta + \underline{\sin 2\theta} + C$

$= 2\sin^{-1}\frac{x}{2} + 2\,\frac{\sqrt{4-x^2}}{2}\,\frac{x}{2} + C.$

$\sin 2\theta = 2\cos\theta \sin\theta$



* Partial fraction.

$$\int \frac{P(x)}{Q(x)}\,dx \quad , \quad P(x),\ Q(x) \text{ are polynomials.}$$

(i) $\deg(P(x)) \geq \deg(Q(x))$, long division first.

Ex. $$\int \frac{x^2+1}{x+1}\,dx = \int \left(x-1+\frac{2}{x+1}\right)dx$$

$$\begin{array}{r l}
 & x-1 \\
x+1 \,\big) & x^2+1 \\
-) & x^2+x \\
\hline
 & -x+1 \\
-) & -x-1 \\
\hline
 & 2
\end{array}$$

(ii) Factorize $Q(x)$ and then partial fraction

ex. $$\frac{x+2}{x^2-3x+2} = \frac{x+2}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

(iii) To complete the square.

$$\frac{x+3}{x^2+4x+8} = \frac{x+3}{x^2+4x+4+4} = \frac{x+3}{(x+2)^2+2^2}$$

$$\int \frac{x+2+1}{(x+2)^2+2^2}\,dx = \int\left(\frac{x+2}{(x+2)^2+2^2} + \frac{1}{(x+2)^2+2^2}\right)dx$$

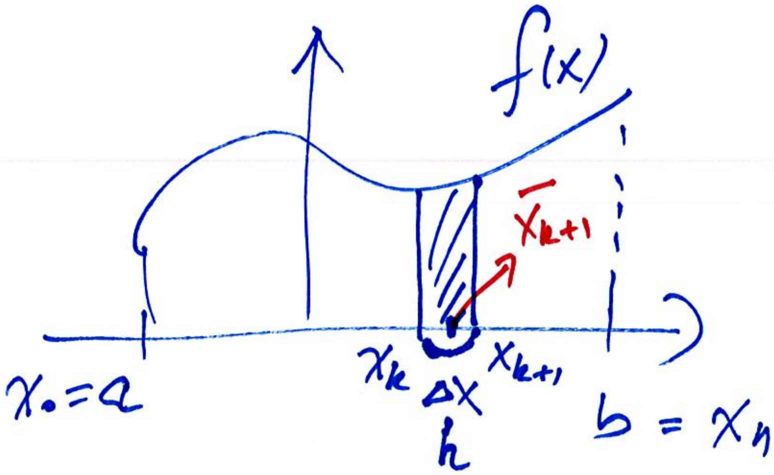
$$\overset{x+2=u}{=\!=\!=} \int\left(\frac{u}{u^2+2^2} + \frac{1}{u^2+2^2}\right)du$$

Ex. $$\int \frac{3x^3+5x^2+9}{(x+1)^2(x-2)(x^2+3x+9)}\,dx$$

$$= \int\left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{Dx+E}{x^2+3x+9}\right]dx$$

§7.7 Approximations.

$$\int_a^b f(x)\,dx$$



split $[a, b]$ into n intervals

$$h = \frac{b-a}{n}, \quad x_k = a + kh, \quad k = 0, 1, \cdots n.$$

~~Mid~~ Midpoint rule: $\bar{x}_k = \dfrac{x_{k-1} + x_k}{2}$

$$\int_a^b f(x)\,dx \simeq M_n = h\left\{ f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right\}.$$

Trapezoid rule

$$\int_a^b f(x)\,dx \simeq T_n = \frac{h}{2}\Big\{ f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \Big\}.$$

Simpson's rule: ($n = 2m$ intervals)

$$\int_a^b f(x)\,dx \simeq S_n = \frac{h}{3}\Big\{f(x_0) + 4f(x_1) + 2f(x_2) + 4\ldots$$
$$+ \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big\}$$

1	4	2	4	2	......	2	4	1
x_0	x_1	x_2	x_3			x_{n-2}	x_{n-1}	x_n

Ex. $\int_1^2 \tan x\,dx$ with 4 intervals by ~~Simple~~ Simpson's rule:

$$h = \frac{2-1}{4} = \frac{1}{4}$$

$$S_4 = \frac{1/4}{3}\Big\{\tan 1 + 4\tan\frac{5}{4} + 2\tan\frac{3}{2} + 4\tan\frac{7}{4} + \tan 2\Big\}.$$

§7.8 Improper integral.

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \boxed{\int_a^t f(x)\,dx}$$

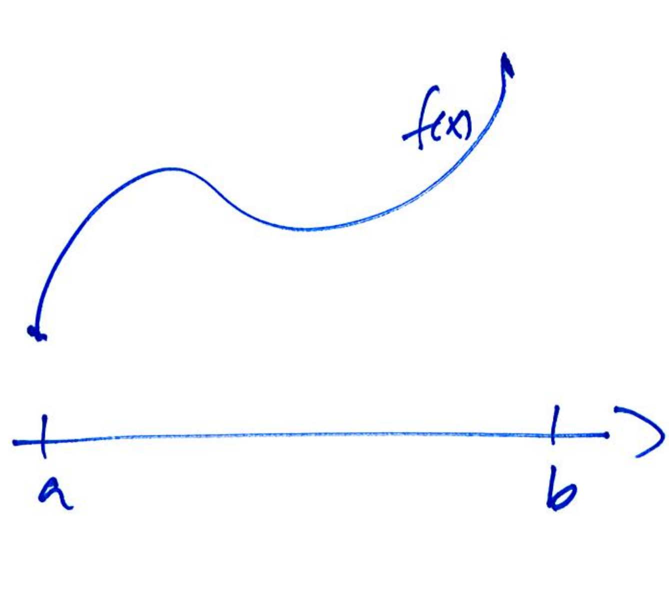
if the limit exists $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent

———— does not exist $\Rightarrow$ 〃 is divergent.

* $\int_1^\infty \frac{1}{x^p}\,dx$ is $\begin{cases} \text{convergent for } p>1 \\ \text{divergent if } p\le 1. \end{cases}$

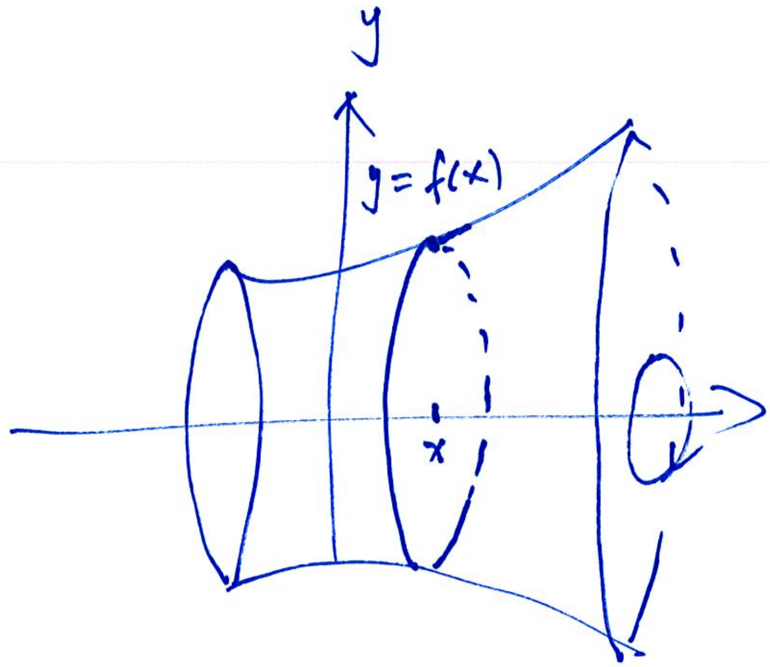
* 8.1 Arc length

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$



8.2 Surface of revolution

about x-axis.

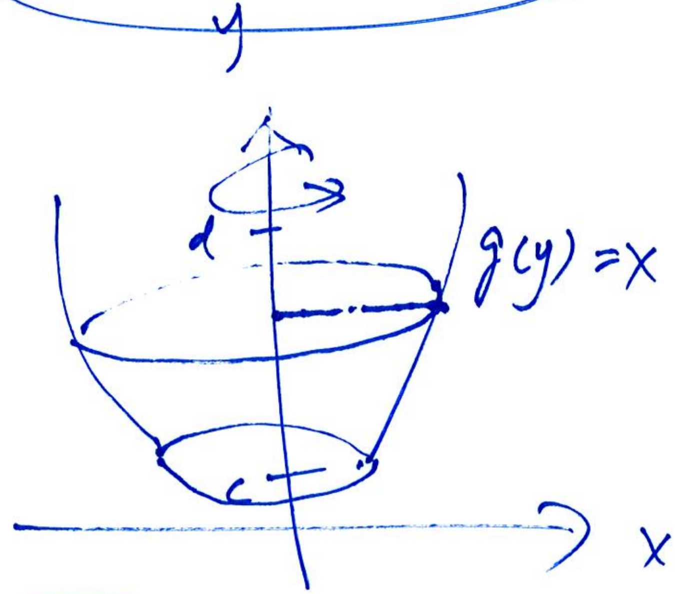


$$S = \int_a^b 2\pi f(x) \underline{\sqrt{1+f'(x)^2}\,dx}$$

$$= \int 2\pi y \, ds$$

$$ds = \sqrt{1+f'(x)^2}\, dx.$$

about y-axis:

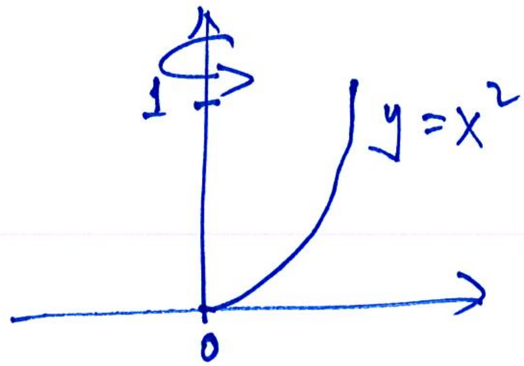


$$S = \int_c^d 2\pi \underline{g(y)} \sqrt{1+g'(y)^2}\, dy$$

$$= \int 2\pi x \, ds$$

$$ds = \sqrt{1+g'(y)^2} \quad ds$$

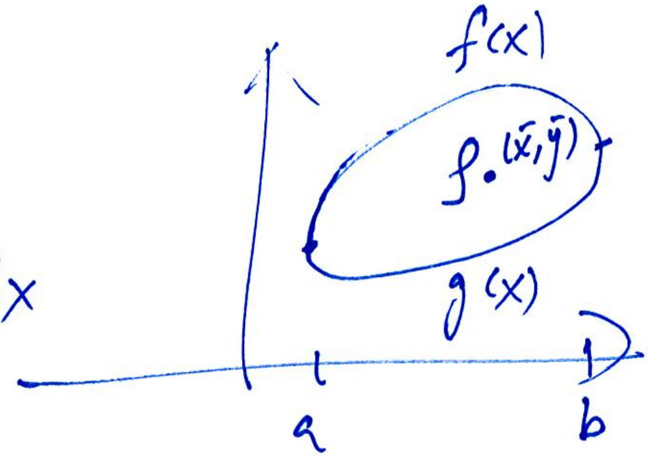
$$S = \int_0^1 2\pi x \, ds$$



$$= \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^1 2\pi x \sqrt{1 + (4x^2)} \, dx \quad \underline{\underline{1+4x^2 = u}} \quad \cdots$$

§ 8.3 Moments & Centroids



moment w.r.t. x:

$$M_x = \rho \int_a^b \frac{1}{2}\left[f^2(x) - g^2(x)\right] dx$$

moment w.r.t to y:

$$M_y = \rho \int_a^b x\left(f(x) - g(x)\right) dx$$

$$m = \rho \int_a^b (f(x) - g(x))\, dx$$

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

§11.1, 11.2.

$\sum a_n$, $\{a_n\}$

How to determine whether the sequence & series are convergent!

(i) If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

(ii) Geometric series

$$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{for } |r| < 1 \\ \text{divergent} & \text{for } r \geq 1 \end{cases}$$