11.7

Goemetric series: $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r|\geq 1 \end{cases}$

$= a(1+r+r^2+\cdots)$

P-series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent} & \text{if } p>1 \\ \text{divergent} & \text{if } p\leq 1. \end{cases}$

Series: $\displaystyle\sum_{n=1}^{\infty} a_n$

Fact $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ divergent.

Integral test: $a_n = f(n)$, $\displaystyle\int_1^{\infty} f(x)\,dx$.

Comparison test, $a_n \leq b_n$

limit test: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$

ratio test: $\displaystyle\lim \left|\frac{a_{n+1}}{a_n}\right| = L$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

alternative series test: $\sum(-1)^{n-1} a_n$, $a_{n+1} \leq a_n$, $\displaystyle\lim_{n\to\infty} a_n = 0$ ...

Power series:

* basic formula:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

* Taylor & Maclaurin Series:

if $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ with $|x-a| < R$

then $a_n = \dfrac{f^{(n)}(a)}{n!}$

R is the radius of the convergence.

Interval of convergence is: three choices

(i) $R-a < x < R+a$ if $\sum a_n (R-a)^n$, $\sum a_n (-R-a)^n$ are divergent.

(ii) $R-a \le x < R+a$

(iii) $R-a \le x \le R+a$.

Ex. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2n}} = a_n$ $\qquad b_n = \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}}$.

limit test: $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum b_n = \sum \frac{1}{\sqrt{n}}$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

Ex. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2n}}$ $= a_n$ (Alternating series)

$\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \le a_n, \ldots$

$$\boxed{\underline{f(n) = a_n} \quad \Rightarrow f(x) = \frac{x}{\sqrt{x^3+2x}} \qquad f'(x) < 0 \text{ for } x \geq c_0}$$

$\Rightarrow$ is convergent

conditionally

Ex. $\sum (-1)^n \frac{1}{n^2+5n+1}$ $\qquad$ $\sum \frac{1}{n^2}$

is absolutely convergent.

Ex. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $\qquad$ $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Since $\sum \frac{1}{n}$ is divergent

$\Rightarrow$ $\sum \sin \frac{1}{n}$ is divergent.

Ex. $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{5^n}$, find the smallest # of terms such that the error of partial sum is ≤ 0.01.

$$S_n = \sum_{k=1}^{n} (-1)^{k} \frac{1}{5^k}$$

$$\Rightarrow |S - S_n| \leq |a_{n+1}| = \frac{1}{5^{n+1}} \leq 0.01$$

$\checkmark$ $n=2$ $\qquad \frac{1}{5^3} = \frac{1}{125} \leq 0.01$

$n \cancel{=} 1$ $\qquad \frac{1}{5^2} = \frac{1}{25} > 0.01$

Ex. (a) $\sum_{n=1}^{\infty} (-1)^n n^3 \left(\frac{1}{3}\right)^n$, (b) $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4n+5}\right)^n$

(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3 \left(\frac{1}{3}\right)^{n+1}}{n^3 \left(\frac{1}{3}\right)^n} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \xrightarrow[n\to\infty]{} \frac{1}{3} < 1$$

(a) is absolutely convergent.

(b): $$\sqrt[n]{\left(\frac{3n+1}{4n+5}\right)^n} = \frac{3n+1}{4n+5} \xrightarrow{n\to\infty} \frac{3}{4} < 1$$

$\sqrt[n]{a_n} =$

(b) is absolutely convergent.

Ex. Find the radius and the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{n+5}\left(\frac{x-2}{3}\right)^n}_{a_n}$

* Use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+6}\left(\frac{x-2}{3}\right)^{n+1}}{\frac{1}{n+5}\left(\frac{x-2}{3}\right)^{n}}\right| = \frac{n+5}{n+6}\,\frac{|x-2|}{3}$$

$$\xrightarrow{n\to\infty} \frac{|x-2|}{3} < 1$$

$\Rightarrow$ Radius of convergence $= \cancel{1}\ 3$.

has to check $\left|\frac{x-2}{3}\right| = 1 \iff \begin{cases} x = 1 \quad -1 \\ x = 3. \quad 5 \end{cases}$

$x=1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+5}(-1)^n = \sum_{n=0}^{\infty}\frac{1}{n+5}$ divergent.

$x=3$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+5}$ is convergent

$1 < x \le 3$ or

Interval of convergence: $(1, 3]$ $(-1, 5]$

Ex. Find the power series at $x = 0$ for

$$f(x) = \frac{5}{x^2 - 3x + 2}$$

$$= \frac{5}{(x-2)(x-1)} = \frac{a}{2-x} + \frac{b}{1-x}$$

$$= \frac{a}{2}\,\frac{1}{1-\frac{x}{2}} + b\,\frac{1}{1-x}$$

$$= \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n + b\sum_{n=0}^{\infty} x^n \qquad \text{for } |x|<1.$$

$\left|\frac{x}{2}\right| < 1$ $\qquad$ $|x| < 1$

$|x| < 2$

Ex. The coefficent of x^6 of the Maclaurin series of $f(x) = \cos \frac{x}{3}$

Maclaurin Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^{n}$

$f'(x) = -\frac{1}{3} \sin \frac{x}{3}$

$f''(x) = -\frac{1}{3^2} \cos \frac{x}{3}$

$\vdots$

$f^{(6)}(x) = -\frac{1}{3^6} \cos \frac{x}{3} \Rightarrow f^{(6)}(0) = -\frac{1}{3^6}$

Answer $= -\frac{1}{3^6 \, 6!}$