MA51400 HW1

SEC. 1.2

8. $e^h = 1+h+h^2/2!+\cdots = 1+O(h) = 1+o(1)$. $(1-h^4)^{-1} = 1+h^4+h^8+\cdots = 1+O(h^4) = 1+o(h^3)$. $\cos(h) = 1-h^2/2!+h^4/4!+\cdots = 1+O(h^2) = 1+o(h)$. $1+\sin(h^3) = 1+h^3-h^9/3!+\cdots = 1+O(h^3) = 1+o(h^2)$.

9. $e^h = 1+h+h^2/2!+h^3/3!+\cdots$ so $[(1+h)-e^h]/h^2 = -1/2-h/6-\cdots \to -1/2$ as $h \to 0$. Hence, $-1/2+O(h^\beta)$ with $\beta=1$ the best value and $-1/2+o(h^\beta)$ with $\beta=0$ the best value.

22. Put $x_n = (1+a\theta^n)/(1+a\theta^{n-1})$. We want to find c and N so that $0<c<1$ and $n \ge N$ implies $|x_{n+1}-1| < c|x_n - 1|$. We have

$$\begin{aligned}\frac{x_{n+1}-1}{x_n-1} &= \frac{[(1+a\theta^{n+1})/(1+a\theta^n)]-1}{[(1+a\theta^n)/(1+a\theta^{n-1})]-1}\\ &= \frac{(1+a\theta^{n+1}-1-a\theta^n)/(1+a\theta^n)}{(1+a\theta^n-1-a\theta^{n-1})/(1+a\theta^{n-1})}\\ &= \frac{a\theta^n(\theta-1)/(1+a\theta^n)}{a\theta^{n-1}(\theta-1)/(1+a\theta^{n-1})}\\ &= \theta\left[\frac{1+a\theta^{n-1}}{1+a\theta^n}\right]\end{aligned}$$

The term $\theta[(1+a\theta^{n-1})/(1+a\theta^n)]$ converges to θ. Select N so that for $n \ge N$, we have $\theta[(1+a\theta^{n-1})/(1+a\theta^n)] < \frac{1}{2}(\theta+1)$. Then we can use $c = \frac{1}{2}(\theta+1) < 1$ for $n \ge N$.

SEC. 1.3

10. The null space of E^r consists of sequences $x = [x_1, x_2, \ldots]$ in which $x_i = 0$ for $i > r$. The null space has dimension r.

11. **a.** Characteristic equation: $x^3-3x^2+4=0$. Roots: $-1, 2$ (double).
    Basis: $[-1,1,-1,1,\ldots,(-1)^n,\ldots]$, $[2,4,8,16,\ldots,2^n,\ldots,\ldots]$, $[1,4,12,32,\ldots,n2^{n-1},\ldots]$.

    b. Characteristic equation: $3x^2-2x+3=0$. Roots: $1 \pm i\sqrt{2}$.
    Basis: $u_n = (1+i\sqrt{2})^n$, $v_n = (1-i\sqrt{2})^n$.

    c. Characteristic equation: $2x^6-9x^5+12x^4-4x^3=0$. Roots: 0 (triple), 1/2 (simple), 2 (double).
    Basis: $x^{(1)} = [1,0,0,\ldots]$, $x^{(2)} = [0,1,0,\ldots]$, $x^{(3)} = [0,0,1,\ldots]$, $x^{(4)} = [1/2,1/4,1/8,\ldots]$, $x^{(5)} = [2,4,8,\ldots]$, $x^{(6)} = [1,4,12,\ldots]$. Here the general term for $x^{(6)}$ is $x_n^{(6)} = n2^{n-1}$.

13. Notice these are not polynomial difference operators. Solve by inspection for first few terms.
    a. $x_{n+1} = n!x_1$ **b.** $x_{n+1} = (1/2)n(n+1)+x_1$ **c.** $x_{n+1} = 2n+x_1$

14. It is obvious that $\Delta = E - I$. If p is a polynomial of degree n, then by Taylor's Theorem $p(x) = \sum_{j=0}^n [p^{(j)}(a)/j!](x-a)^j$. Put $x = E$ and $a = I$ to get $p(E) = \sum_{j=0}^n (1/j!)p^{(j)}(I)\Delta^j$.

27. Characteristic equation: $\lambda^2-2\lambda-2=0$. Roots: $1\pm\sqrt{3}$.
    General solution: $z_n = \alpha(1+\sqrt{3})^n + \beta(1-\sqrt{3})^n$.
    Initial values give $1 = x_1 = \alpha(1+\sqrt{3})+\beta(1-\sqrt{3})$ and $1-\sqrt{3} = x_2 = \alpha(1+\sqrt{3})^2+\beta(1-\sqrt{3})^2$.
    So solution is $\alpha = 0$ and $\beta = 1/(1-\sqrt{3})$.