SEC 2.2

8. Write $x = [(-b+\sqrt{b^2-4ac})/(2a)][(b+\sqrt{b^2-4ac})/(b+\sqrt{b^2-4ac})] = (-2c)/(b+\sqrt{b^2-4ac})$. This works if $b > 0$. If $b < 0$ the troublesome root is

$x = [(-b-\sqrt{b^2-4ac})/(2a)][(b-\sqrt{b^2-4ac})/(b-\sqrt{b^2-4ac})] = (-2c)/(b-\sqrt{b^2-4ac})$.

Summary: If $b > 0$ then $x_1 = (-b-\sqrt{b^2-4ac})/(2a)$ and $x_2 = (-2c)/(b+\sqrt{b^2-4ac}) = c/(ax_1)$. If $b < 0$ then $x_1 = (-b+\sqrt{b^2-4ac})/(2a)$ and $x_2 = (-2c)/(b-\sqrt{b^2-4ac}) = c/(ax_1)$. Here $x_1x_2 = c/a$. This is the standard approach. In addition, if there is loss of precision in the calculation of $\sqrt{b^2-4ac}$ it could be done in double precision to reduce loss of precision in this subtraction. An accurate and reliable procedure for solving the quadratic equation is discussed at length in Young and Gregory [1972, pp.77–83].

9. **a.** $\sqrt{x^2+1} - x = (\sqrt{x^2+1} - x)[(\sqrt{x^2+1}+x)/(\sqrt{x^2+1}+x)] = 1/(\sqrt{x^2+1}+x)$.

b. $\log x - \log y = \log(x/y)$.

c. $x^{-3}(\sin x - x) = x^{-3}(-x^3/3! + x^5/5! - x^7/7! + \cdots) = -1/3! + x^2/5! - x^4/7! + \cdots$.

d. $\sqrt{x+2} - \sqrt{x} = 2/(\sqrt{x+2}+\sqrt{x})$.

e. $e^x - e = (1-e) + x + x^2/2 + x^3/3! + x^4/4! + \cdots$.

f. $\log x - 1 = \log(x/10)$.

g. $(\cos x - e^{-x})/\sin x = [(1 - x^2/2! + x^4/4! - \cdots) - (1 - x + x^2/2! - x^3/3! + \cdots)]/(\sin x) = (x - x^2 + x^3/6 - x^5/120 + \cdots)/(x - x^3/3! + x^5/5! - \cdots) = 1 - x + x^2/3 - \cdots$.

h. $\sin x - \tan x = \sin x - (\sin x/\cos x) = \sin x(\cos x - 1)/(\cos x) = \tan x(\cos x - 1)(\cos x + 1)/(\cos x + 1) = \tan x(\cos^2 x - 1)/(\cos x + 1) = (-\tan x \sin^2 x)/(\cos x + 1)$.

15. **a.** $f(0) = 0$ using L'Hospital's rule.

b. Near $x = 2\pi n$, loss of significance since $\cos x \approx 1$.

c. Use $f(x) = (\sin^2 x)/[x(1+\cos x)]$

d. From identity $1 - \cos x = 2\sin^2(x/2)$, use $f(x) = (2/x)\sin^2(x/2)$ or original formula for values when $\cos x \approx -1$.

SEC 2.3

7. Characteristic equation $\lambda^2 - 2\lambda - 1 = 0$ has roots $1 \pm \sqrt{2}$ so general solution of the form $x_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$. No, unstable since $(1+\sqrt{2}) > 1$.

8. Assume $r_n = \lambda^n$ gives characteristic equation $\lambda^2 - \lambda - 1 = 0$ with roots $\lambda_1 = [(1+\sqrt{5})/2]$ and $\lambda_2 = [(1-\sqrt{5})/2]$. General solution: $r_n = A\lambda_1^n + B\lambda_2^n$ where $A = [(1+\sqrt{5})/2]/\sqrt{5}$ and $B = [(1-\sqrt{5})/2]/\sqrt{5}$. Now $r_n/r_{n-1} = (A\lambda_1^n + B\lambda_2^n)/(A\lambda_1^{n-1} + B\lambda_2^{n-1})$ $= \lambda_1[A + B\theta^n]/[A + B\theta^{n+1}] \to \lambda_1$ since $\theta = \lambda_2/\lambda_1 <$. The convergence has linear behavior.

9. As above, $r_n = A\lambda_1^n + B\lambda_2^n = A[(1+\sqrt{5})/2]^n + B[(1-\sqrt{5})/2]^n$. Since the root $|\lambda_1| > 1$, the recurrence relation does not provide a stable means for computing r_n. In this case, $A = 0$ and $B = 1$ so $r_n = [(1-\sqrt{5})/2]^n \to 0$ as $n \to \infty$.

SEC 3.1

4. By theorem, $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0) \le \varepsilon$. So, $-(n+1)\log 2 + \log(b_0 - a_0) \le \log\varepsilon \Rightarrow -(n+1) \le [\log\varepsilon - \log(b_0 - a_0)]/\log 2$.
Hence, $n > [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$.

5. Relative precision $= |r - c_n|/|r| \le \varepsilon$. Since $r \ge a_0 > 0$, we require
$|r - c_n|/a_0 \le \varepsilon$, $2^{-(n+1)}(b_0 - a_0)/a_0 \le \varepsilon$, $-(n+1) \le \log[(\varepsilon a_0)/(b_0 - a_0)]/\log 2$,
$n \ge \log[(b_0 - a_0)/(\varepsilon a_0)]/(\log 2) - 1 = [\log(b_0 - a_0) - \log\varepsilon - \log a_0]/(\log 2) - 1$

7. By Problem 3.1.4, $|r - c_n| \le \varepsilon$ after $n \ge [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$. Here $n \ge 6/(\log 2) - 1 = 18.93$. So in 19 steps, we obtain 10^{-6} absolute accuracy. On `MARC-32`, machine precision 2^{-24} is obtained in $n > (24\log 2)/(\log 2) - 1 = 23$ steps for absolute accuracy to full precision.

By Problem 3.1.5, relative precision $|r - c_n|/|r| \le \varepsilon$ requires $n \ge [\log(b_0 - a_0) - \log\varepsilon - \log a_0)]/(\log 2) - 1$. Here $n \ge [6 - \log 2]/(\log 2) - 1 = [6/\log 2] - 2 = 17.93$. So in 18 steps, we obtain 10^{-6} relative accuracy. On the `MARC-32`, we have $n \ge [24\log 2 - \log 2]/(\log 2) - 1 = 24 - 2 = 22$ steps for relative accuracy to full precision.

SEC 3.2

5. Let $\lim_{n\to\infty} x_n = r$, then $r = 2r - r^2 y \Rightarrow r = 1/y$. So the purpose of the formula is to compute $1/y$. Now $x_{n+1} = 2x_n - x_n^2 y = x_n + (x_n - x_n^2 y) = x_n + x_n^2(1/x_n - y) = x_n + (x_n^{-1} - y)/x_n^{-2}$. So $x_{n+1} = x_n - (x_n^{-1} - y)/(-x_n^{-2})$. Thus, it is Newton's iteration for $f(x) = x^{-1} - y$.

Alternative Solution: $x_{n+1} = 2x_n - x_n^2 y = x_n - f(x_n)/f'(x_n)$. Change y to a and $f(x)$ to y. Solve $x - ax^2 = -y/y'$. Now $(\log y)' = y'/y = 1/[x(ax-1)] = A/x + B/(ax-1)$ implies $A = -1$ and $B = a$. So $(\log y)' = -1/x + a/(ax-1)$ or $\log y = -\log x + \log(ax-1) = \log[(ax-1)/x]$ implies $y = a - 1/x$. Hence, original problem to solve $f(x) = y - 1/x$.

10. $f(x) = x^3 - R$; $f'(x) = 3x^2$. Thus, the Newton's iteration formula for computing $\sqrt[3]{R}$ is: $x_{n+1} = x_n - (x_n^3 - R)/3x_n^2 = (2x_n + R/x_n^2)/3$. For $x > 0$, $f'(x) > 0$ and $f''(x) > 0$. Then by Theorem 2 the Newton iteration will converge from any point > 0. For $x < 0$, Newton method will not always converge since one of the iterates could not be the origin. A quick calculation shows that if we start with the point $x = \sqrt[3]{-R/2}$, then the first iterate will be the point $x_1 = 0$ where $f'(x_1) = 0$ and $x_2 = \infty$. So the method fails in this case. Hence, it converges for all $x > 0$.