

SEC 3.4

6. $F(x) = x + f(x)g(x)$, $f(\xi) = 0$, $f'(\xi) \neq 0$. The theory developed in this section shows that the sequence defined by $x_{n+1} = F(x_n)$ will converge *cubically* to ξ if $F'(\xi) = 0$, $F''(\xi) = 0$, $F'''(\xi) \neq 0$. (We want *at least* cubically convergent sequence, so we do not insist on $F'''(\xi) \neq 0$.) $F'(x) = 1 + f'(x)g(x) + f(x)g'(x) \Rightarrow F'(\xi) = 1 + f'(\xi)g(\xi) + f(\xi)g'(\xi) = 1 + f'(\xi)g(\xi)$. We want this to be zero. So $g(\xi) = -1/f'(\xi)$. (Condition 1) $F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \Rightarrow F''(\xi) = f''(\xi)g(\xi) + 2f'(\xi)g'(\xi) + f(\xi)g''(\xi) = f''(\xi)[-1/f'(\xi)] + 2f'(\xi)g'(\xi)$. For this to be zero we require $g'(\xi) = f''(\xi)/\{2[f'(\xi)]^2\}$. (Condition 2) *Note:* If $g(x) = -[f'(x)]^{-1}$ then $g'(x) = [f'(x)]^{-2}[f''(x)]$, so $g'(\xi)$ is off by a factor of 2.
7. Eventually 0.9998477 appears. Let $f(x) = \cos x$. Then $|\cos x - \cos y| = |\sin \xi| |x - y|$ for $x < \xi < y$ since $|\sin \xi| < 1$, $F(x)$ is a contraction and thus has a fixed point.
12. $x = \sqrt{p + \sqrt{p + \sqrt{p + \dots}}}$. Let $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$, $x_3 = \sqrt{p + \sqrt{p + \sqrt{p}}}$, and so on. Observe that $x_2 = \sqrt{p + x_1}$, $x_3 = \sqrt{p + x_2}$, and so on. In general $x_{n+1} = \sqrt{p + x_n}$ (I). Let $f(x) = \sqrt{p + x}$. Equation (I) is the result of using functional iteration on f . If $\lim x_n$ exists, denote it by x . Take limits in Equation (I) to get $x = \sqrt{p + x}$. Hence, $x^2 = p + x$, $x^2 - x - p = 0$, $x = (1 + \sqrt{1 + 4p})/2$. This is the limit of the sequence. For example if $p = 2$, $x = 2$. Try it on your pocket calculator.
13. Use the ideas of Problem 3.4.12. Let $x_1 = 1/p$, $x_2 = 1/(p + (1/p))$, $x_3 = 1/(p + (1/p + (1/p)))$ etc. So $x_2 = 1/(p + x_1)$, $x_3 = 1/(p + x_2)$, and so on. Hence, $x_{n+1} = 1/(p + x_n)$. If $\lim_{n \rightarrow \infty} x_n = x$ then $x = 1/(p + x)$. Hence, $x(p + x) = 1$, $x^2 + px - 1 = 0$, $x = (-p + \sqrt{p^2 + 4})/2$. This illustrates functional iteration with $f(x) = 1/(p + x)$. If $p > 1$, f is a contraction. Use Mean Value Theorem: $|f(x) - f(y)| = |f'(\xi)| |x - y| = |-1/(p + \xi)^2| |x - y|$. Since $p > 1$, all x_n 's will be ≥ 0 , and $1/(p + x)^2 \leq 1/p^2 < 1$. So f is a contraction on $[0, \infty]$. f actually maps $[0, 1]$ into $[0, 1]$, so has a fixed point in $[0, 1]$.
20. a. $|F(x) - F(y)| = |(3 - x^2) - (3 - y^2)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y| \cdot (1/2)$, because $|x| \leq 1/4$, $|y| \leq 1/4$, so $|x + y| \leq |x| + |y| \leq 1/2$. So F is a contraction. But F does not map the interval $[-1/4, 1/4]$ into the interval $[-1/4, 1/4]$. For example $F(0) = 3$.
- b. $|F(x) - F(y)| = |x - y|/2$ so F is a contraction. But F does not map the set $[-2, -1] \cup [1, 2]$ into the set $[-2, -1] \cup [1, 2]$. For example, $F(-1) = -1/2$.
40. $F(x) = x(x^2 + 3R)/(3x^2 + r)$, $F(\sqrt{R}) = \sqrt{R}$, $F'(\sqrt{R}) = 0$, $F''(\sqrt{R}) = 0$, $F'''(\sqrt{R}) \neq 0$.

SEC 3.5

1. $p(4) = 946$

$$\begin{array}{r} 3 \quad -7 \quad -5 \quad 1 \quad -8 \quad 2 \\ 4 \quad \underline{12 \quad 20 \quad 60 \quad 244 \quad 944} \\ 3 \quad 5 \quad 15 \quad 61 \quad 236 \quad 946 \end{array}$$

4.

$$\begin{array}{r} a_k : \quad 3 \quad -7 \quad -5 \quad \quad 1 \quad \quad -8 \quad \quad 2 \\ u = 3 \quad \quad 9 \quad 6 \quad \quad 12 \quad \quad 45 \quad \quad 123 \\ v = 1 \quad \quad \underline{3} \quad \quad \quad 2 \quad \quad 4 \quad \quad 15 \\ b_k : \quad 3 \quad 2 \quad 4 \quad \quad 15 \quad \quad 41 = b_1 \quad 140 = b_0 \\ u = 3 \quad \quad 9 \quad 33 \quad \quad 120 \quad \quad 438 \\ v = 1 \quad \quad \underline{3} \quad \quad 11 \quad \quad 40 \\ c_k : \quad 3 \quad 11 \quad 40 = c_2 \quad 146 = c_1 \quad 519 = c_0 \end{array}$$

So $J = c_0c_2 - c_1^2 = -556$, $\delta u = (c_1b_1 - c_2b_2)/J = -0.69425$ and $\delta v = (c_1b_0 - c_0b_1)/J = 1.50899$.
Hence, $u = u + \delta u = 2.30576$ and $v = v + \delta v = 2.50899$.

10.

$$\begin{array}{r} 9 \quad -7 \quad 1 \quad -2 \quad 5 \\ 6 \quad \underline{54 \quad 282 \quad 1698 \quad 10176} \\ 9 \quad 47 \quad 283 \quad 1696 \quad 10181 \\ 6 \quad \underline{54 \quad 606 \quad 5334} \\ 9 \quad 101 \quad 889 \quad 7030 \end{array}$$

So $p(6) = 10181$ and $p'(6) = 7030$. Using $z_0 = 6$, we have $z_1 = z_0 - p(z_0)/p'(z_0) = 6 - 10181/7030 = 4.55178$.

16. $\alpha_0 = a_0 + \alpha_1x$, $\alpha_1 = a_1 + x\alpha_2$, $\alpha_2 = a_2 + x\alpha_3, \dots$. Thus $\alpha_0 = a_0 + x(a_1 + x(a_2 + x(a_3 + \dots))) = p(x)$,
 $\beta_0 = \alpha_1 + x\beta_1$, $\beta_1 = \alpha_2 + x\beta_2$, $\beta_2 = \alpha_3 + x\beta_3, \dots$. Thus $\beta_0 = a_1 + x(a_2 + x(a_3 + \dots)) + x(\alpha_2 + x(\alpha_3 + \dots)) = a_1 + 2a_2x + 3a_3x^2 + \dots = p'(x)$, $\gamma_0 = \beta_1 + x\gamma_1$, $\gamma_1 = \beta_2 + x\gamma_2$, $\gamma_2 = \beta_3 + x\gamma_3 \dots$. Thus $\gamma_0 = a_2 + x(a_3 + x(a_4 + \dots)) + x(\alpha_3 + x(\alpha_4 + \dots)) + x(\beta_2 + x(\beta_3 + \dots)) = a_2 + 3a_3x + 6a_4x^2 + \dots$. Therefore, $2\gamma_0 = p''(x)$.