

### 6.10 Interpolation on Higher Dimensions

14. If the nodes are  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ , and  $(1, 0)$ , then the  $4 \times 4$  interpolation matrix is nonsingular. The general case is similar.
15. Let the vertices of the triangle be  $z_1, z_2, z_3$ , and let the sides be the lines  $H_{11}, H_{21}, H_{31}$ . Label everything so that  $z_j \in H_i$  if and only if  $i \neq j$ . Then apply Theorem 3. Another proof is implicit in the text following the corollary.

### 6.12 Trigonometric Interpolation

1. Assume  $0 = g(x_j) = \sum_{k=0}^{N-1} a_k E_k(x_j)$ . Multiply by  $\overline{E_m(x_j)}$  and sum on  $j$  to get  $0 = \sum_{k=0}^{N-1} a_k \sum_{j=0}^{N-1} E_k(x_j) \overline{E_m(x_j)} = \sum_{k=0}^{N-1} a_k N \langle E_k, E_m \rangle_N = N a_m$ . Hence  $a_m = 0$  for  $0 \leq m \leq N - 1$ .
4. If the given equation is true, then  $n \langle f, E_j \rangle_n = \sum_k f(x_k) E_j(x_k) = \sum_k \langle g, E_k \rangle_n E_j(x_k) = \sum_k \langle g, E_k \rangle_n E_k(x_j) = g(x_j)$  by the interpolation theorem.
5. Notice that  $\frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i j k/n} = \frac{1}{n} \sum_{j=0}^{n-1} (e^{2\pi i k/n})^j$ . This geometric series has sum 1 if  $n$  divides  $k$ , for in that case each term is 1. Otherwise, the standard formula  $(r^n - 1)/(r - 1)$  for the sum yields 0 since  $(e^{2\pi i k/n})^n = 1$ . The real part on the left is  $\sum \frac{1}{n} \cos(2\pi i k/n)$ . Since the sum is always real its imaginary part is 0.

### 7.1 Numerical Differentiation and Richardson Extrapolation

7.
  - a. By Taylor series,  $f(x+h) = f + hf' + (h^2/2)f'' + (h^3/6)f''' + (h^4/24)f^{iv} + \dots$ ,  $f(x+2h) = f + 2hf' + (4h^2/2)f'' + (8h^3/6)f''' + (16h^4/24)f^{iv} + \dots$ ,  $f(x+3h) = f + 3hf' + (9h^2/2)f'' + (27h^3/6)f''' + (81h^4/24)f^{iv} + \dots$ . Now  $f(x+h) - f(x+2h) = -hf' - (3h^2/2)f'' - (7h^3/6)f''' - (15h^4/24)f^{iv} - \dots$ . So  $f(x+3h) + 3[f(x+h) - f(x+2h)] = f + h^3 f''' + (36h^4/24)f^{iv} + \dots$  and error term  $-(3h/2)f^{iv}(\xi) = O(h)$ .
  - b. Now  $f(x+2h) - f(x-2h) = 4hf' + (16h^3/6)f''' + (64h^5/120)f^{v} + \dots$  and  $f(x+h) - f(x-h) = 2hf' + (2h^3/6)f''' + (2h^5/120)f^{v} + \dots$ . So  $[f(x+2h) - f(x-2h)] - 2[f(x+h) - f(x-h)] = 2h^3 f''' + (h^5/2)f^{v} + \dots$  and error term  $-(h^2/4)f^{v}(\xi) = O(h^2)$ . Second approximation more accurate.
10. Given  $L = x_n + a_1 n^{-1} + a_2 n^{-2} + \dots$  and replacing  $n$  by  $n^2$ , we have  $L = x_{n^2} + a_1 n^{-2} + a_2 n^{-4} + \dots$ . Taking  $n$  times the latter equation from the former one, we have  $(n-1)L = nx_{n^2} - x_n + (n^{-3} - n^{-2})a_2 + \dots$ . So  $L = (nx_{n^2} - x_n)/(n-1) + [n^{-3}(1-n)/(n-1)]a_2 + \dots$  and  $L = [n/(n-1)]x_{n^2} - [1/(n-1)]x_n + O(n^{-3})$ .
12. Given  $L = \phi(h) + a_1 h + a_3 h^3 + \dots$ . Replacing  $h$  by  $h/2$ , we have  $L = \phi(h/2) + a_1(h/2) + a_3(h/2)^3 + \dots$ . Multiplying the latter equation by 2 and subtracting the former equation, we obtain  $L = 2\phi(h/2) - \phi(h) - (3/4)a_3 h^3 + \dots$ .
13. Given  $L - f(h) = c_6 h^6 + c_9 h^9 + \dots$ . Replacing  $h$  by  $h/2$ , we have  $L - f(h/2) = c_6(h/2)^6 + c_9(h/2)^9 + \dots$ . Multiplying the latter by  $2^6$  and subtracting the former, we obtain  $63L - 64f(h/2) + f(h) = (2^{-3} - 1)c_9 h^9 + \dots$ . Thus,  $L = [64f(h/2) - f(h)]/63 - (1/48)c_9 h^9 + \dots$ .

## 6.14 Computer Problems

2.  $f(x) = \sin(10x)$  has 11 roots in  $[0, \pi]$ , namely  $i\pi/10$  for  $i = 0, 1, 2, \dots, 10$ . Starting with  $t_0 = 0, t_1 = \pi$ , and  $f(t_0) = f(t_1) = 0$ , we obtain  $\ell(f, t_0, t_1; x) = 0$ . Then procedure MAX will evaluate  $f$  at exactly the roots of  $f$ , so that  $d = 0$  and, obviously,  $d < \epsilon$  for any  $\epsilon > 0$ . The algorithm will stop at that point with the linear approximation  $y = 0$  on  $[0, \pi]$ .