7.2   Numerical Integration Based on Interpolation

4. $f(x) = 1$: LHS $= \int_0^1 dx = 1$, RHS $= (1/90)[7 + 32 + 12 + 32 + 7] = 1$.

$f(x) = x$: LHS $= \int_0^1 xdx = 1/2$, RHS $= (1/90)[32(1/4) + 12(1/2) + 32(3/4) + 7] = 1/2$.

$f(x) = x^2$: LHS $= \int_0^1 x^2 dx = 1/3$, RHS $= (1/90)[32(1/4)^2 + 12(1/2)^2 + 32(3/4)^2 + 7] = 1/3$.

$f(x) = x^3$: LHS $= \int_0^1 x^3 dx = 1/4$, RHS $= (1/90)[32(1/4)^3 + 12(1/2)^3 + 32(3/4)^3 + 7] = 1/4$.

$f(x) = x^4$: LHS $= \int_0^1 x^4 dx = 1/5$, RHS $= (1/90)[32(1/4)^4 + 12(1/2)^4 + 32(3/4)^3 + 7] = 1/5$. Since it is exact for $1, x, x^2, x^3$, and x^4, it follows that it is exact for any linear combination of them, namely all polynomials of degree ≤ 4.

8. $f(x) = e^x$: LHS $= \int_0^1 e^x dx = e - 1$, RHS $= A_0 + A_1 e$. $f(x) = \cos(x\pi/2)$: LHS $= \int_0^1 \cos(\pi x/2)dx = \sin(x\pi/2)/(\pi/2)|_0^1 = 2/\pi$, RHS $= A_0$. Solving $A_0 = 2/\pi$, $A_1 = e^{-1}(e - 1 - A_0) = 1 - 1/e - 2/(\pi e)$.

10. a. $\ell_0(x) = -3(x - 2/3)$: LHS $= 1/2$, RHS $= A$. $\ell_1(x) = 3(x - 1/3)$: LHS $= 1/2$, RHS $= B$. So $A = B = 1/2$.

b. Let $x = \lambda(t) = (b - a)t + a$ so $dx = (b - a)dt$. So $\int_a^b f(x)dx = (b - a)\int_{\lambda(0)}^{\lambda(1)} f(\lambda(t))dt = (1/2)(b - a)[f(\lambda(1/3)) + f(\lambda(2/3))] = [(b - a)/2][f((2a + b)/3) + f((a + 2b)/3)]$.

19. $\int_a^b f(x)dx = \sum_{i=1,\text{ odd}}^{n-1} \int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \sum_{i=1,\text{ odd}}^{n-1} (x_{i+1} - x_{i-1})f(x_i) = (2h)\sum_{k=1}^{n/2} f(x_{2k-1})$ since n even.

7.3   Gaussian Quadrature

7. a. By Theorem 5, Section 6.8, we have $q_0 = 1, q_1 = x - a_1, q_n = (x - a_n)q_{n-1} - b_n q_{n-2}$ where $a_n = \lambda x q_{n-1}, q_{n-1}\rangle/\lambda q_{n-1}, q_{n-1}\rangle$ and $b_n = \lambda x q_{n-1}, q_{n-2}\rangle/\lambda q_{n-2}, q_{n-2}\rangle$. Here $\lambda f, g\rangle = \int_0^1 xfgdx$. For this problem, $a_1 = 2/3, a_2 = 8/15$ and $b_2 = 1/18$ so that $q_1 = x - 2/3$ and $q_2 = x^2 - (6/5)x + (3/10)$. The roots of q_2 are $(6 \pm \sqrt{6})/10$.
Using method of undetermined coefficients, we obtain $A_0 + A_1 = 1/2$ and $[(6 - \sqrt{6})/10]A_0 + [(6 + \sqrt{6})/10]A_1 = 1/3$. Solving, we obtain $A_0 = (3 - 2/\sqrt{6})/12$ and $A_1 = (3 + 2/\sqrt{6})/12$. Hence, $\int_0^1 xf(x)dx \approx [(3 - 2/\sqrt{6})/12]f((6 - \sqrt{6})/10) + [(3 + 2/\sqrt{6})/12]f((6 + \sqrt{6})/10)$.

Note: $x_0 = (6 - \sqrt{6})/10 \approx 0.35505\,10259$, $x_1 = (6 + \sqrt{6})/10 \approx 0.84494\,89743$, and $A_0 = (3 - 2/\sqrt{6})/12 \approx 0.18195\,86183$, $A_1 = (3 + 2/\sqrt{6})/12 \approx 0.31804\,13817$. These results agree with those in Table 25.8 in Abramowitz and Stegun [1965, page 921].

b. Continuing, $a_3 = 18/35$ and $b_3 = 3/50$ so that $q_3 = x^3 - (12/7)x^2 + (6/7)x - (4/35)$. Using a symbolic manipulation package to solve this cubic, we find $x_1 = (4/7) + a$, $x_2 = [(4/7) - (a/2)] + i[b\sqrt{3}/2]$, $x_3 = [(4/7) - (a/2)] - i(b\sqrt{3}/2)$ where $a = (2/49c) + c$, $b = -(2/49c) + c$, $c = \sqrt[3]{(-2/1715) + i(2/245)}$. The numerical values of these roots of the cubic polynomial are $x_0 = 0.21234\,05382, x_1 = 0.59053\,31356, x_2 = 0.91141\,20405$. Using the method of undetermined coefficients, we need to solve the system $1/2 = A_0 + A_1 + A_2$, $1/3 = A_0 x_0 + A_1 x_1 + A_2 x_2$, $1/4 = A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2$, $1/5 = A_0 x_0^3 + A_1 x_1^3 + A_2 x_2^3$, $1/6 = A_0 x_0^4 + A_1 x_1^4 + A_2 x_2^4$, $1/7 = A_0 x_0^5 + A_1 x_1^5 + A_2 x_2^5$. The solution is $A_0 = 0.06982\,69799$, $A_1 = 0.22924\,11064$, $A_2 = 0.20093\,19137$. One can show numerically that this formula is exact for polynomials of degree ≤ 5.
Note: Here we use the values given in Abramowitz-Stegan [1965, Table 25.8, page 921].

9. $\int_{-1}^{1} f(x)dx \approx c[f(x_0) + f(x_1) + f(x_2)]$. $f(x) = 1$: LHS = 2, RHS = $c(3) \Rightarrow c = 2/3$. $f(x) = x$: LHS = 0, RHS = $(2/3)[x_0 + x_1 + x_2]$. $f(x) = x^2$: LHS = 2/3, RHS = $(2/3)[x_0^2 + x_1^2 + x_2^2]$. Let $x_1 = 0$, $x_0 = -x_2 = -1/\sqrt{2}$.

11. $f(x) = 1$: LHS = 2, RHS = 2. $f(x) = x$: LHS = 2, RHS = 2. $f(x) = x^2$: LHS = 8/3, RHS = $2\alpha^2 - 4\alpha + 4$. $f(x) = x^3$: LHS = 4, RHS = $6\alpha^2 - 12\alpha + 8$. Solving, we have $3\alpha^2 - 6\alpha + 2 = 0$ or $\alpha = 1 \pm 1/\sqrt{3}$.

7.4   Romberg Integration

6.   **a.** $a = 1, b = 3, f(x) = 1/x$: $R(0,0) = (1/2)(b-a)[f(a) + f(b)] = 4/3$,
  $R(1,0) = (1/2)R(0,0) + (1/2)(b-a)[f(a + (b-a)/2)] = 7/6$, $R(2,0) = (1/2)R(1,0) + 14(b-a)[f(a + (b-a)/4) + f(a + 3(b-a)/4)] = 67/60$,
  $R(1,1) = (4/3)R(1,0) - (1/3)R(0,0) = 10/9$, $R(2,1) = (4/3)R(2,0) - (1/3)R(1,0) = 11/10$,
  $R(2,2) = (16/15)R(2,1) - (1/15)R(1,1) = 742/678$.
  From the code:

$$R(0,0) = 1.33333$$
$$R(1,0) = 1.16667 \quad R(1,1) = 1.11111$$
$$R(2,0) = 1.11667 \quad R(2,1) = 1.10000 \quad R(2,2) = 1.09926$$

  Exact value $= \ln 3 \approx 1.09861$

   b. $a = 0, b = \pi/2, f(x) = (x/\pi)^2$: $R(0,0) = \pi/16$, $R(1,0) = 3\pi/64$, $R(2,0) = 11\pi/256$, $R(1,1) = \pi/24$, $R(2,1) = \pi/24$, $R(2,2) = \pi/24$.
  From code:

$$R(0,0) = 0.196350$$
$$R(1,0) = 0.147262 \quad R(1,1) = 0.130900$$
$$R(2,1) = 0.134990 \quad R(2,1) = 0.130900 \quad R(2,2) = 0.130900$$

  Exact value $= \pi/24 \approx 0.13090$.

9. $I = T(f,h) + c_1 h + c_2 h^2 = \cdots$ and $I = T(f, h/2) + c_1(h/2) + c_2(h/2)^2 + \cdots$. Combining, we have $I = 2T(f, h/2) - T(f, h) + (1/2 - 1)c_2 h^2 + \cdots$. Let $R(1,1) = 2R(1,0) - R(0,0)$.
In general, $R(n,1) = 2R(n,0) - R(n-1,0)$. Now, $I = R(1,1) + b_2 h^2 + b_3 h^3 + \cdots$ and $I + R(2,1) + b_2(h/2)^2 + b_3(h/2)^3 + \cdots$. Combining, we have $I = (4/3)R(2,1) - (1/3)R(1,1) + (1/2 - 1)b_3 h^3 + \cdots$.
Let $R(2,2) = (4/3)R(2,1) - (1/3)R(1,1)$. In general, $R(n,2) = (4/3)R(n,1) - (1/3)R(n-1,1)$. Now $I = R(2,2) + c_3 h^3 + c_4 h^4 + \cdots$ and $I = R(3,2) + c_3(h/2)^3 + c_4(h/2)^4 + \cdots$.
Combining, we have $I = (8/7)R(3,2) - (1/7)R(2,2) + (1/2 - 1)c_4 h^4 + \cdots$. Let $R(3,3) = (8/7)R(3,2) - (1/7)R(2,2)$. New Eq. (5) is $R(n,m) = R(n, m-1) + (1/2^m - 1)[R(n, m-1) - R(n-1, m-1)]$.