6.3 Hermite Interpolation

1.

x	$f(x)$				
0	2	-9	3	7	5
0	2	-6	10	17	
1	-4	4	44		
1	-4	48			
2	44				

So $p(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$.

3. By Theorem 1, there exists a unique polynomial p of degree $\le m$ ($m = 2n+1$) such that $p(x_i) = y_i$ and $p'(x_i) = 0$ for $0 \le i \le n$. By Equation (9), we have $p(x) = \sum_{i=0}^{n} y_i[1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x)$ where $\ell_i(x) = \prod_{j=0,\, j\ne i}^{n}(x - x_j)/(x_i - x_j)$ for $0 \le i \le n$.

4. Let us write $p(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3$. Then $p''(x) = 2c + 6d(x - x_0)$. The four conditions can be written as: $c_{00} = p(x_0) = a$, $c_{02} = p''(x_0) = 2c$, $c_{10} = p(x_1) = a + bh + ch^2 + dh^3$, and $c_{12} = p''(x_1) = 2c + 6dh$ when $h = x_1 - x_0$. So a and c are obtained without restrictions: $a = c_{00}$, $c = c_{02}/2$. d and b can be obtained from last two equations: $\begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} =$ known vector. Det $\begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix} = 6h^2 \ne 0$ iff $h \ne 0 \Rightarrow$ condition: $x_0 \ne x_1$.

6.4 Spline Interpolation

5. $f(1^-) = 1 = f(1^+)$, so f is continuous at $x = 1$. Also $f(2^-) = 3/2 = f(2^+)$, so f is continuous at $x = 2$. $f'(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ 2 - x & x \in [1, 2] \\ 0 & x \in [2, \infty) \end{cases}$.

Thus, $f'(1^-) = 1 = f'(1^+)$ and $f'(2^-) = 0 = f'(2^+)$. Therefore, $f'(x)$ is continuous at $x = 1$, $x = 2$. Hence, f is a quadratic spline function.

7. Enforce the continuity of f at knots: 1,3. At $x = 1$, $a(-1)^2 + 0 = c(-1)^2 \Rightarrow a = c$. At $x = 3$, $c(1)^2 = d(1)^2 + 0 \Rightarrow c = d$. Continuity of f' at knots: At $x = 1$, $2a(-1) + 0 = 2c(-1) \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Continuity of f'' at knots: At $x = 1$, $2a + 0 = 2c \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Thus, in order that f be a cubic spline: $a = c = d$ and b, e any arbitrary values. Next, determine a, b, c, d, e so that f interpolates the table. At $x = 0$, $a(-2)^2 + b(-1)^3 = 26 \Rightarrow 4a - b = 26$. At $x = 1$, $a(-1)^2 + b \cdot 0 = 7 \Rightarrow a = 7$. So $b = 2$ and $c = d = 7$. At $x = 4$, $d(2)^2 + e(1)^3 = 25 \Rightarrow 28 + e = 25 \Rightarrow e = -3$. Then: $a = c = d = 7$, $b = 2$, $e = -3$.

9. Put $q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}z_i h_i$.
Then $q_i(t_i) = -\frac{1}{2}(z_i/h_i)(t_{i+1} - t_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[-(z_i h_i^2)/h_i] + \frac{1}{2}(z_i h_i) + y_i = y_i$ where $h_i = t_{i+1} - t_i$. $q_i'(x) = (z_{i+1}/h_i)(x - t_i) + (z_i/h_i)(t_{i+1} - x)$, $q_i'(t_i) = (z_i/h_i)(t_{i+1} - t_i) = z_i$, $q_i'(t_{i+1}) = (z_{i+1}/h_i)(t_{i+1} - t_i) = z_{i+1}$. $q_{i-1}(x) = \frac{1}{2}(z_i/h_{i-1})(x - t_{i-1})^2 - \frac{1}{2}(z_{i-1}/h_{i-1})(t_i - x)^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1})$ $q_{i-1}(t_i) = \frac{1}{2}(z_i/h_{i-1})(t_i - t_{i-1})^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1}) = \frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1}$.
Continuity Condition $\frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1} = y_i \Rightarrow z_i + z_{i-1} = (2/h_{i-1})(y_i - y_{i-1})$. $q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(x - t_i - h_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) - \frac{1}{2}(z_i h_i) + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) + y_i$.

Here $i = 1, 2, \ldots, n-1$. Q: piecewise quadratic Q, Q' continuous $Q'(t_i) = z_i$ well-defined $q_1(t_2) = q_2(t_2)$ etc. $q_{n-2}(t_{n-1}) = q_{n-1}(t_{n-1})$, i.e., $q_{i-1}(t_i) = q_i(t_i)$ for $i = 2 \ldots n-1$ $z_{i-1} + z_i = (2/h_{i-1})(y_i - y_{i-1})$ $(2 \le i \le n-1)$.
Let $z_1 = 0$ and define inductively $z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$, $i = 2, 3, \ldots, n$. z_i is arbitrary, $z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$ $i = 2, \ldots, n$.
So $z_i = \alpha_i - z_{i-1}$. $z_2 = \alpha_2 - z_1$, $z_3 = \alpha_3 - z_2 = \alpha_3 - (\alpha_2 - z_1) = \alpha_3 - \alpha_2 + z_1$, $z_4 = \alpha_4 - z_3 = \alpha_4 - \alpha_3 + \alpha_2 - z_1$, Etc.. $z_1 = \alpha_1 - \alpha_{i-1} + \alpha_{i-2} \cdots + (-1)^i(\alpha_2 - z_1)$.

So $z_i = \gamma_i - (-1)^i z_1$, $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \gamma_2$, $\gamma_4 = \alpha_3 - \gamma_3$, etc. $\Phi = \sum_{i=2}^{n} z_i 62 = z_2^2 + z_3^2 + z_4^2 + \cdots + z_n^2$ $= (\gamma_2 - z_i)^2 + (\gamma_3 + z_1)^2 + (\gamma_4 - z_1)^2 + \cdots + (\gamma_n - (-1)^n z_1)^2$ $d\Phi/dz_1 = -2(\gamma_2 - z_1) + 2(\gamma_3 + z_1) - 2(\gamma_4 - z_1) - \cdots - 2(-1)^n(\gamma_n - (-1)^n z_1) = 0$ $-\underbrace{\gamma_2 + z_1}_{1} + \underbrace{\gamma_3 + z_1}_{2} - \underbrace{\gamma_4 + z_1}_{3} \cdots - (-1)^n \underbrace{\gamma_n + z_1}_{n-1} = 0$

$(n-1)z_1 - (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n) = 0$ $z_1 = (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_4 + \cdots + (-1)^n \gamma_n)/(n-1)$.

Now $\gamma_2 - \gamma_3 = \alpha_2 - (\alpha_3 - \alpha_2) = 2\alpha_2 - \alpha_3$ and $\gamma_4 - \gamma_5 = \gamma_4 - (\alpha_4 - \gamma_4) = 2\gamma_4 - \alpha_4$. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \alpha_2$, $\gamma_4 = \alpha_4 - \alpha_3 + \alpha_2$, $\gamma_5 = \alpha_5 - \alpha_4 + \alpha_3 - \alpha_2$, etc. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_2 - \alpha_3$, $\gamma_4 = \alpha_2 - \alpha_3 + \alpha_4$, $-\gamma_5 = \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5$, etc. So $[\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n]/(n-1)$ $= [(n-1)\alpha_2 - (n-2)\alpha_3 + (n-3)\alpha_4 - \cdots]/(n-1)$.

Algorithm: For $i = 2 \cdots n$ define $\alpha_i = (2/h_{i-1})(y_i - y_{i-1})$. For $i = 3 \cdots n$ do $\alpha_i - \alpha_{i-1} \to \alpha_i$. For $i = 2 \cdots n$ do $\alpha_2 + (-1)^i \alpha_{i+1} \to \alpha_2$. $z_1 = \alpha_2/(n-1)$.

13. Let $f_1(x) = 1 + x - x^3$, $x \in [0,1]$; $f_2(x) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3$, $x \in [1,2]$; $f_3(x) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3$, $x \in [2,3]$. Since $f_1(0) = 1$, $f_1(1) = 1 = f_2(1)$, $f_2(2) = 0 = f_3(2)$, and $f_3(3) = 10$, $f(x)$ interpolates the table and $f(x)$ is continuous at $x = 1$ and $x = 2$. Also $f_1'(1) = -2 = f_2'(1)$, $f_2'(2) = 4 = f_3'(2)$, $f_1''(1) = -6 = f_2''(1)$, $f_2''(2) = 18 = f_3''(2)$, $f_1''(0) = 0$, and $f_3''(3) = 0$. Hence, $f(x)$ is a natural cubic spline which interpolates the table values.

24. The integral $\int_0^1 S(x)\,dx$ becomes an approximation of the area under the curve $f(t)$ on the interval $[0,1]$ where $R_i = \frac{1}{2}(t_i - t_{i-1})[f(t_i) + f(t_{i-1})]$. This is the composite trapezoid rule with non-uniform segments. Thus, $\int_0^1 S(x)\,dx = \sum_{i=1}^{n} R_i = \frac{1}{2}\sum_{i=1}^{n}(t_i - t_{i-1})[f(t_i) - f(t_{i-1})]$.

30. In the matrix system for the z_i's in the text, the first equation is replaced by $h_0 z_0 + u_1 z_1 + h_1 z_2 = v_1$ and the last equation is replaced by $h_{n-2} z_{n-2} + u_{n-1} z_{n-1} + h_{n-1} z_n = v_{n-1}$. Using Eq. (8) with $i = 0$, we have $-2h_0 z_0 - h_0 z_1 = 6S'(t_0) - b_0$. Using Eq. (9) with $i = n$, we have $h_{n-1} z_{n-1} + 2h_{n-1} z_n = 6S'(t_n) - b_{n-1}$. Use these latter two equations to expand the linear system and solve for $[z_0, z_1, z_2, \ldots, z_{n-2}, z_{n-1}, z_n]^T$.

6.5 The B-Splines: Basic Theory

3. $S(t_m) = \sum_i c_i B_i^2(t_m) = c_{m-2} B_{m-2}^2(t_m) + c_{m-1} B_{m-1}^2(t_m)$
$= c_{m-2} h_m/(h_m + h_{m-1}) + c_{m-1} h_{m-1}/(h_m + h_{m-1}) = (c_{m-2} h_m + c_{m-1} h_{m-1})/(h_m + h_{m-1}) = y_m$.