MATH/CSE 514 MIDTERM EXAM

March. 6, 2013

Name: **Score:**

1. (5 points) How would you compute $f(x) = \sqrt{1+x} - \sqrt{1-x}$ when x is small? why?

$$f(x) = \sqrt{1+x} - \sqrt{1-x} = \frac{(1+x) - (1-x)}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2x}{\sqrt{1+x} + \sqrt{1-x}}$$

To avoid subtracting nearly equal #s: $\sqrt{1+x}$, $\sqrt{1-x}$ when x is small.

2. (5 points) Give an argument on why the Newton's method converges quadratically by interpreting it as a fixed point iteration.

Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

let $F(x) = x - \frac{f(x)}{f'(x)}$

then Newton's method is a fixed iteration:

$$x_{n+1} = F(x_n)$$

Since $F'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$

If $f(r) = 0$ and $f'(r) \neq 0 \Rightarrow F'(r) = 0$

$$F''(x) = \frac{(f'f'' + ff''')(f')^2 - ff''2f'\cdot f''}{(f'(x))^4}$$

$\Rightarrow F''(r) = \frac{f''(r)}{f'(r)} \neq 0$ if $f''(r) \neq 0$

So Newton's method converges at least quadratically.

3. (10 points) Show that the sequence x_n defined by

$$x_{n+1} = \frac{2x_n + 3}{x_n + 2}$$

converges for any $x_0 \in [-1, \infty)$, and determine its convergence rate. What is $\lim_{n\to\infty} x_n$?

if $x_n \to r \Rightarrow r = \dfrac{2r+3}{r+2}$

$\Rightarrow r = \pm\sqrt{3}$ but since $x_0 \in [-1, \infty)$

$\lim_{n\to\infty} x_n = \sqrt{3}$

let $f(x) = \dfrac{2X+3}{X+2} \Rightarrow f'(x) = \dfrac{1}{(X+2)^2}$

Hence $1 > f'(x) > 0$ for $x \in (-1, \infty)$. for $x_0 \in [-1, +\infty)$ we have $x_1 \geq 0$,

So $f(x)$ is a contractive mapping from $[0, +\infty)$ to $[0, +\infty)$ and has a fixed point.

Since $f'(\sqrt{3}) \neq 0$, the convergence rate is linear.

4. (10 points) Find the polynomial p_4 (of degree less or equal than 4) in Newton's form using the divided difference table such that

$$p_4(0) = 1,\ p_4(1) = 0,\ p_4'(1) = 2,\ p_4''(1) = 2,\ p_4(2) = 3.$$

x	$f(x)$				
0	1	-1	3	-2	1
1	0	2	1	0	
1	0	2	1		
1	0	3			
2	3				

Hence

$$P_4(x) = 1 - x + 3x(x-1) - 2x(x-1)^2 + x(x-1)^3$$

5. (10 points) Let $p_n(f;x)$ be the polynomial of degree $\leq n$ interpolating $f(x) = e^{2x}$ at $x_i = cos\frac{2i+1}{2n+2}\pi$, $i = 0, 1, \cdots, n$. Derive an upper bound for $\|f - p_n\|_{L^\infty(-1,1)}$, and determine the smallest n such that the above error is less than $\frac{1}{2}10^{-2}$.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$f^{n+1}(x) = 2^{n+1} e^{2x} \Rightarrow \max_{x \in [-1,1]} |f^{n+1}(x)| \leq e^2 2^{n+1}$$

$$\max_{x \in [-1,1]} \left| \prod_{i=0}^{n} (x - x_i) \right| \leq \frac{1}{2^n} \quad \text{(Chebyshev points)}$$

$$\Rightarrow |f(x) - p_n(x)| \leq \frac{2e^2}{(n+1)!} \leq \frac{1}{2} 10^{-2}$$

we need $(n+1)! \geq 400 e^2$

the smallest is $n = 6$.

6. (10 points) Given $x_0 < x_1 < \cdots < x_n$. Determine a procedure for constructing a quadratic spline $S(x)$ such that

$$S(x_i) = f(x_i),\ i = 0, 1, \cdots, n;\ S'(x_n) = 0.$$

Let $z_i = S'(x_i)$ $\qquad h_i = x_{i+1} - x_i$

$$\Rightarrow \quad S_i'(x) = \frac{z_i}{h_i}(x_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - x_i)$$

Integrate once $\Rightarrow$

$$S_i(x) = -\frac{z_i}{2h_i}(x_{i+1} - x)^2 + \frac{z_{i+1}}{2h_i}(x - x_i)^2 + C_i$$

From $S_i(x_i) = f(x_i) \Rightarrow C_i = y_i + \frac{h_i}{2} z_i$

From $S_i(x_{i+1}) = f(x_{i+1}) \Rightarrow y_{i+1} = \frac{z_{i+1}}{2} h_i + C_i$

$$= \frac{z_{i+1} + z_i}{2} h_i + y_i, \quad i = 0, 1, \ldots, n-1$$

Algorithm: $z_n = 0$

for $i = n-1, n-2, \ldots, 0$ do

$$z_i = -z_{i+1} + \frac{2}{h_i}\left(f(x_{i+1}) - f(x_i)\right)$$

end do