Highly efficient and accurate spectral approximation of the angular Mathieu equation for any parameter values $q$

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Abstract. The eigenpairs of the angular Mathieu equation under the periodicity condition are accurately approximated by the Jacobi polynomials in a spectral-Galerkin scheme for small and moderate values of the parameter $q$. On the other hand, the periodic Mathieu functions are related with the spheroidal functions of order $\pm 1/2$. It is well-known that for very large values of the bandwidth parameter, spheroidal functions can be accurately approximated by the Hermite or Laguerre functions scaled by the square root of the bandwidth parameter. This led us to employ the Laguerre polynomials in a pseudospectral manner to approximate the periodic Mathieu functions and the corresponding characteristic values for very large values of $q$.

Key Words: Mathieu function; spectral methods; Jacobi polynomials; Laguerre polynomials

AMS Subject Classifications: 33E10; 33D50; 65L60; 65L15

1 Introduction

Mathieu functions were first introduced by Mathieu in 1868 while investigating the vibrating modes of an elliptic membrane [32]. The eigenpairs of the Mathieu equation are needed in many scientific phenomena including wave motion in elliptic coordinates such as acoustic and electromagnetic scattering from an elliptic structures [14, 19, 26, 31], particle in a periodic potential [22] and vibrational spectroscopy of molecules with near resonant frequencies [37, 43]. Theoretical aspects of the Mathieu functions have been studied by many authors, including Stratton [42], McLachlan [33], Sips [41], Meixner & Schäfke [34] and Wang and Zhang [45] (cf. also [44]).

As is seen for many physical and mathematical problems in elliptic geometries the separation of variables process in elliptic coordinates leads to the Mathieu equations. If one wants to solve these problems with large wavenumbers, it is very important to be able to obtain accurate numerical solutions for the angular Mathieu equation for very
large values of $q$ since it is related to the wavenumber parameter present in these equations.

Mathieu functions remain difficult to employ, mainly because of the impossibility of analytically representing them in a simple and handy way [41]. There are numerous studies on the numerical computation of the Mathieu functions and corresponding eigenvalues. Erricolo used Blanch’s algorithm for computing the expansion coefficients of Mathieu functions [16]. Erricolo and Carluccio provided software to compute angular and radial Mathieu functions for complex $q$ values [17]. Shirts presented two algorithms for the computation of eigenvalues and solutions of Mathieu’s differential equation for non-integer orders [39, 40]. Alhargan introduced algorithms for the computation of all Mathieu functions of integer order which can deal with a range of the order $n \ (0—200)$ and the parameter $q \ (0—4n^2)$ [3]. Cöösson and co-workers describe a numerical algorithm which allows a flexible approach to the computation of all the Mathieu functions [12]. Cojocaru in [13] provided Mathieu functions computational toolbox implemented in Matlab. MATSLISE is another software package for the computation of the Mathieu eigenpairs by using the power of high-order piecewise constant perturbation methods [29] and many others [1, 9, 21, 24, 25, 28, 30].

Most of the above algorithms employ the well-known trigonometric series representation

\[
\begin{align*}
\text{ce}_{2n}(\eta, q) &= \sum_{k=0}^{\infty} A_{2k}^{(2n)}(q) \cos(2k\eta), & \text{ce}_{2n+1}(\eta, q) &= \sum_{k=0}^{\infty} A_{2k+1}^{(2n+1)}(q) \cos((2k+1)\eta) \\
\text{se}_{2n+1}(\eta, q) &= \sum_{k=0}^{\infty} B_{2k+1}^{(2n+1)}(q) \sin((2k+1)\eta), & \text{se}_{2n+2}(\eta, q) &= \sum_{k=0}^{\infty} B_{2k+2}^{(2n+2)}(q) \sin((2k+2)\eta)
\end{align*}
\]

(1.1)

for computing the periodic Mathieu functions where $A$ and $B$ are known as the expansion coefficients. There are several ways of computing these expansion coefficients such as continued fractions method [33], the forward and the backward recurrence relations [9, 17] and as the eigenvectors of tri-diagonal matrix-eigenvalue problems [12, 13]. Each has its advantages and disadvantages. However, these algorithms are not suitable for very large values of $q$. Thus, the aim of this study is to construct accurate and efficient spectral algorithms for the computation of the integer order periodic Mathieu functions and the corresponding characteristic values for both small and very large values of the real parameter.

The rest of the paper is organized as follows: Sections 2 and 3 are concerned with the construction of the spectral methods for small and very large values of $q$, respectively. Some numerical results are presented in Section 4. The last section concludes the paper with some remarks.
2 Spectral formulation for small and moderate parameter values

For any given value of the parameter \( q \), the angular Mathieu equation

\[
\frac{d^2 \Phi}{d \eta^2} + (\lambda - 2q \cos 2\eta) \Phi = 0
\]  

(2.1)
supplemented with a certain periodic boundary conditions admits two linearly independent families of periodic solutions with period \( \pi \) or \( 2\pi \) for specific values of the separation constant \( \lambda \). Such values of the parameter \( \lambda \) are known as characteristic values or eigenvalues. When the solutions \( \Phi(\eta) \) are even with respect to \( \eta = 0 \) the characteristic values are denoted as \( a_n(q) \) whereas for odd solutions they are represented as \( b_{n+1}(q), n = 0, 1, \ldots \). Periodic eigenfunctions corresponding to the \( a_n(q) \) and \( b_{n+1}(q) \) are denoted by \( ce_n(q; \eta) \) and \( se_{n+1}(q; \eta) \), respectively. The notations \( ce \) and \( se \) are due to Whittaker and Watson [47] and they stand for cosine-elliptic and sine-elliptic, respectively.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Mathieu functions</th>
<th>Eigenvalues</th>
<th>Character</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi(0; q) = \Phi(\pi/2; q) = 0 )</td>
<td>( ce_{2n}(q; \eta) )</td>
<td>( a_{2n}(q) )</td>
<td>even ( \beta ) – periodic</td>
</tr>
<tr>
<td>( \Phi'(0; q) = \Phi'(\pi/2; q) = 0 )</td>
<td>( ce_{2n+1}(q; \eta) )</td>
<td>( a_{2n+1}(q) )</td>
<td>even ( 2\beta ) – periodic</td>
</tr>
<tr>
<td>( \Phi(0; q) = \Phi'(\pi/2; q) = 0 )</td>
<td>( se_{2n+1}(q; \eta) )</td>
<td>( b_{2n+1}(q) )</td>
<td>odd ( 2\beta ) – periodic</td>
</tr>
<tr>
<td>( \Phi(0; q) = \Phi(\pi/2; q) = 0 )</td>
<td>( se_{2n+2}(q; \eta) )</td>
<td>( b_{2n+2}(q) )</td>
<td>odd ( \beta ) – periodic</td>
</tr>
</tbody>
</table>

Before describing the numerical scheme, we first transform the angular Mathieu equation to a more tractable form for each boundary condition set in Table 1. The angular Mathieu equation in (2.1) and Table 1 allow us to write

\[
ce''_{2n} + (a_{2n} - 2q \cos 2\eta)ce_{2n} = 0, \quad ce'_{2n}(0; q) = ce'_{2n}(\pi/2; q) = 0.
\]  

(2.2)

that can be transformed to an equivalent algebraic form

\[
(1 - x^2)y''_n - xy'_n - \frac{1}{2}qxy_n = \mu_n(q)y_n, \quad \mu_n(q) = -\frac{1}{4}a_{2n}(q)
\]  

(2.3)

by the mapping \( x = \cos 2\eta \in (-1, 1) \) where \( y_n(x; q) = ce_{2n}(x; q) \). The connection formula

\[
ce_{2n}(q; \eta) = Cy_n(\cos 2\eta; q)
\]  

(2.4)

reveals that the function \( u \) does not have to satisfy any boundary conditions at all. Next, we consider the system

\[
ce''_{2n+1} + (a_{2n+1} - 2q \cos 2\eta)ce_{2n+1} = 0, \quad ce'_{2n+1}(0; q) = ce'_{2n+1}(\pi/2; q) = 0
\]  

(2.5)

and have found the corresponding algebraic form to be

\[
(1 - x^2)y''_n + (1 - 2x)y'_n - \frac{1}{2}qxy_n = \mu_n(q)y_n, \quad \mu_n(q) = \frac{1}{4}[1 - a_{2n+1}(q)]
\]  

(2.6)
upon use of the transformations \( x = \cos 2\eta \) and \( ce_{2n+1}(x;q) = (1+x)^{1/2}y_n(x;q) \) on both independent and dependent variables, respectively. Returning back to the original variables we see that the solution

\[
ce_{2n+1}(\eta;q) = C(\sqrt{2}\cos \eta)y_n(\cos 2\eta;q)
\]  

(2.7)

satisfies the boundary conditions in (2.5) meaning that we don’t need to impose any boundary conditions on (2.6). Thirdly, the maps \( x = \cos 2\eta \) and \( se_{2n+1}(x;q) = (1-x)^{1/2}y_n(x;q) \) transform the system

\[
se''_{2n+1} + (b_{2n+1} - 2q\cos 2\eta)se_{2n+1} = 0, \quad se_{2n+1}(0;q) = se_{2n+1}(\pi/2;q) = 0
\]  

(2.8)

to the equivalent algebraic form

\[
(1-x^2)y''_n - (1+2x)y'_n - \frac{1}{2}qxy_n = \mu_n(q)y_n, \quad \mu_n(q) = \frac{1}{4}[1-b_{2n+1}(q)]
\]  

(2.9)

without any boundary conditions since the solution

\[
se_{2n+1}(\eta;q) = C(\sqrt{2}\sin \eta)y_n(\cos 2\eta;q)
\]  

(2.10)

readily satisfies the conditions in (2.8). Finally, the system

\[
se''_{2n+2} + (b_{2n+2} - 2q\cos 2\eta)se_{2n+2} = 0, \quad se_{2n+2}(0;q) = se_{2n+2}(\pi/2;q) = 0
\]  

(2.11)

can be converted to the form

\[
(1-x^2)y''_n - 3xy'_n - \frac{1}{2}qxy_n = \mu_n(q)y_n, \quad \mu_n(q) = \frac{1}{4}[4-b_{2n+1}(q)]
\]  

(2.12)

by means of \( x = \cos 2\eta \) and \( se_{2n+2}(x;q) = (1-x^2)^{1/2}y_n(\eta;q) \). Again, the solution in original variable

\[
se_{2n+2}(\eta;q) = C(\sqrt{2}\sin \eta)(\sqrt{2}\cos \eta)y_n(\cos 2\eta;q)
\]  

(2.13)

satisfies the specified boundary conditions.

Actually the equations in (2.3), (2.6), (2.9) and (2.12) can be put together to give the equation

\[
L_n^{(\alpha,\beta)} := \left(1-x^2\right)\frac{d^2}{dx^2} + \left[\beta - \alpha - (\alpha + \beta + 2)x\right] \frac{d}{dx} - \frac{1}{2}q x \right) y_n^{(\alpha,\beta)}(x;q) = \mu^{(\alpha,\beta)}(q)y_n^{(\alpha,\beta)}(x;q)
\]  

(2.14)

with

\[
\mu^{(\alpha,\beta)}(q) = \frac{1}{4}[(\alpha + \beta + 1)^2 - \lambda(q)].
\]  

(2.15)

The characteristic values

\[
\lambda(q) = \{a_{2n}(q), a_{2n+1}(q), b_{2n+1}(q), b_{2n+2}(q)\}
\]  

(2.16)
of (2.1) are connected with those of the equation in (2.14) by the relations
\[
\begin{aligned}
-4\mu_n(-\frac{1}{2},-\frac{1}{2}) (q), 1-4\mu_n(-\frac{1}{2},\frac{1}{2}) (q), 1-4\mu_n(\frac{1}{2},-\frac{1}{2}) (q), 4-4\mu_n(\frac{1}{2},\frac{1}{2}) (q)
\end{aligned}
\]  
(2.17)
respectively while the corresponding eigenfunctions
\[
\Phi^{(a,\beta)}(\eta;q) = \{ce_{2n}(\eta;q), ce_{2n+1}(\eta;q), se_{2n+1}(\eta;q), se_{2n+2}(\eta;q)\}
\]  
(2.18)
are related with the solutions of (2.14) by the formula
\[
\Phi^{(a,\beta)}(\eta;q) = C(\sqrt{2}\sin\eta)^{\alpha+\frac{1}{2}}(\sqrt{2}\cos\eta)^{\beta+\frac{1}{2}} y_n^{(a,\beta)}(\cos2\eta;q).
\]  
(2.19)
Notice that, the set of parameter values \((a,\beta) = \{(-\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (\frac{1}{2},\frac{1}{2}) \}\) lead to the eigenpairs \((ce_{2n}, a_{2n}), (ce_{2n+1}, a_{2n+1}), (se_{2n+1}, b_{2n+1})\) and \((se_{2n+2}, b_{2n+2})\), respectively. Here, the constant \(C\) will be chosen in such a way that \(\|\Phi(\eta;q)\|_{L^2(0,\pi)} = \pi\). That is,
\[
\pi = \int_0^{2\pi} \left[ \Phi^{(a,\beta)}(\eta;q) \right]^2 d\eta = 4 \int_0^{\frac{\pi}{2}} \left[ \Phi^{(a,\beta)}(\eta;q) \right]^2 d\eta
\]
(2.20)
leading to
\[
C = \sqrt{\frac{\pi}{2}}
\]  
(2.21)
when the orthonormal eigenfunctions \(\|y_n\|_{L^2((-1,1))} = 1\) of (2.14) is in consideration.

Now, we will construct the Galerkin spectral formulation of (2.14). Let
\[
\phi_k^{(a,\beta)}(x) = \frac{1}{h_k} P_k^{(a,\beta)}(x), \quad h_k^{(a,\beta)} = \left[ \frac{2^{k+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{k! \Gamma(k+\alpha+\beta+1)} \right]^\frac{1}{2}
\]  
(2.22)
be the normalized Jacobi polynomials of degree \(k\) and order \((a,\beta)\) and
\[
X_N = \text{span} \{ \phi_k^{(a,\beta)}(x) \mid k = 0, 1, \ldots, N \},
\]  
(2.23)
The Spectral-Galerkin method for (2.14) is to find \(y_{n\cdot N}^{(a,\beta)} \in X_N\) such that
\[
(L y_{n\cdot N}^{(a,\beta)}, p_N)_{\omega^{(a,\beta)}(x)} = y_{n\cdot N}^{(a,\beta)}(p_N), \quad \forall p_N \in X_N
\]  
(2.24)
where \(\omega^{(a,\beta)}(x) = (1-x)^a(1+x)^\beta\) is the Jacobi weight function. Now, proposing
\[
y_n^{(a,\beta)}(x;q) \approx y_n^{(a,\beta)}(x;q) = \sum_{k=0}^{N} C_{nk} \phi_k^{(a,\beta)}(x),
\]  
(2.25)
letting \( p_N(x) = \phi_j^{(\alpha, \beta)}(x) \) and keeping in mind that the normalized Jacobi polynomials satisfy the differential equation
\[
\left\{ (1-x^2) \frac{d^2}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x] \frac{d}{dx} + k(k+\alpha+\beta+1) \right\} \phi_k^{(\alpha, \beta)}(x) = 0
\] (2.26)
we see that the Spectral-Galerkin formulation (2.24) of the differential eigenvalue problem (2.14) reduces to the matrix-eigenvalue equation
\[
(S + R)c_n = \mu_n c_n
\] (2.27)
with \( c_n = (c_{n0}, c_{n1}, \ldots, c_{nN})^T, S = [S_{jk}] = -k(k+\alpha+\beta+1) \delta_{jk} \) and
\[
R = [R_{jk}] = -\frac{1}{2}q \int_{-1}^{1} x \phi_k^{(\alpha, \beta)}(x) \phi_j^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx.
\] (2.28)
The three term recurrence relation of the normalized Jacobi polynomials
\[
A_k^{(\alpha, \beta)} \phi_{k+1}^{(\alpha, \beta)}(x) + (B_k^{(\alpha, \beta)} - x) \phi_k^{(\alpha, \beta)}(x) + A_{k-1}^{(\alpha, \beta)} \phi_{k-1}^{(\alpha, \beta)} = 0,
\]
\[
\phi_0^{(\alpha, \beta)} = \frac{1}{h_0^{(\alpha, \beta)}}, \quad \phi_1^{(\alpha, \beta)}(x) = \frac{1}{2h_1^{(\alpha, \beta)}} \left[ (\alpha + \beta + 2)x + \alpha - \beta \right]
\] (2.29)
in which
\[
A_{k-1}^{(\alpha, \beta)} = \frac{2}{2k+\alpha+\beta} \left[ \frac{k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta+1)} \right]^{1/2}, B_k^{(\alpha, \beta)} = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}
\] (2.30)
allows us to write down the entries
\[
R_{jk} = R_{kj} = -\frac{1}{2}q \left\{ \begin{array}{ll}
A_k^{(\alpha, \beta)}, & j = k+1 \\
B_k^{(\alpha, \beta)}, & j = k
\end{array} \right. \] (2.31)
of the symmetric tri-diagonal matrix \( R \) for \( j,k=0,1,\ldots,N \). Notice in (2.31) that the diagonal entries are all zero since we have \( B_k^{(\alpha, \beta)} = 0 \) for all \( k \) when \( (\alpha, \beta) = (\pm \frac{1}{2}, \pm \frac{1}{2}) \). Moreover, the entries of the matrix \( R \) requires special attention when \( (\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2}) \), since in this case the first coefficient \( A_0^{(-1/2, -1/2)} = \sqrt{2}/2 \) while \( A_k^{(-1/2, -1/2)} = 1/2 \) for \( k \geq 1 \).

The matrices have simple structure, more specifically \( S \) is a diagonal and \( R \) is a tri-diagonal matrix with zero main diagonal. Thus, the resulting discrete system is a tri-diagonal matrix. Despite its simple structure, the present formulation yields highly accurate numerical eigenpairs. For small \( q \) values, even the last discrete eigenpair corresponding to \( N \) is correct up to some digits which is promising if we remember the fact that only the two-thirds, often one-half or considerably smaller portion of the computed
Table 2: The number of eigenvalues (ev) having relative errors of order $10^{-14}$ and the number of eigenvectors (ef) having absolute errors of order at most $10^{-13}$ for a given truncation order $N$ and the parameter $q$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$q = 1$</th>
<th>$q = 10^2$</th>
<th>$q = 10^3$</th>
<th>$q = 10^4$</th>
<th>$q = 10^5$</th>
<th>$q = 10^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ev</td>
<td>ef</td>
<td>ev</td>
<td>ef</td>
<td>ev</td>
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<tr>
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<td>94</td>
<td>90</td>
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<td>588</td>
</tr>
</tbody>
</table>

eigenpairs are correct up to some accuracy for a fixed truncation order $N$ [10,46]. Specifically, when $q=1$ and $N=100$, relative errors for all the eigenvalues except the last one are of order $10^{-14}$ and for the same $N$ with $q=10^2$, ninety-four of them have relative errors of order $10^{-14}$.

On the other hand, the number of accurate eigenvectors having the values of the Mathieu functions at some evaluation points is slightly less than that of eigenvalues. The accuracy of the eigenpairs are checked by increasing the truncation order $N$ systematically and observing the stable digits between two consecutive truncation orders $N$ and $N+1$. For the eigenvectors, vector infinity norm is used to measure the errors.

Nevertheless, it can be seen from Table 2 that the number of accurate pairs decreases with a further increment in $q$. Indeed, a dramatic decrease occurs especially for $q \geq 10^5$. Therefore, in this article such $q$ values will be called very large. Actually, the accuracy loss for large values of the parameters is typical for all parameter dependent problems such as the Coffey-Evans [18] and the prolate spheroidal wave equation. Efficient numerical approximation of the latter for very large values of the bandwidth parameter may be found in [4, 27, 35]. An efficient method for very large values of the parameter $q$ will be derived in the next section.

### 3 Spectral formulation for very large parameter values

The main difficulty for very large $q$ values originates from the fact that the eigenfunctions of (2.1) are confined to a small interval around the points $\eta = \pi/2$ and $\eta = 3\pi/2$ when they are considered in the interval $(0,2\pi)$ (see Fig. 1). Therefore, one has to focus on these portions of the interval. One remedy might be the use of mapped Jacobi pseudospectral methods by using a suitable mapping such as the one introduced in [7] which can clusters more points around a desired point. However, here we are in favor of employing another technique which is more efficient and accurate from the numerical point of view.

Consider the general equation

$$\left[-\frac{d^2}{d\theta^2} + \tan\theta \frac{d}{d\theta} + \frac{\nu^2}{\cos^2\theta} + c^2 \sin^2\theta\right] \Psi_{\nu n}(\theta;c) = \lambda_{\nu}(\nu;c) \Psi_{\nu n}(\theta;c)$$  (3.1)
Figure 1: \( se_1(\eta,q) \) for specified values of \( q \) on \([0,2\pi]\).

where \( c \) and \( \nu > -1 \) is a real parameter. The maps \( t = \sin \theta \) and \( \Psi_{vn}(\theta;c) = (1-t^2)^{\nu} \psi_{vn}(t;c) \) takes the equation to the algebraic form

\[
(1-t^2)\psi''_{vn} - 2(\nu+1)t\psi'_{vn} - c^2 t^2 \psi_{vn} = \mu_n(v;c)\psi_{vn}
\]

(3.2)

where \( \mu_n(v;c) = v(v+1) - \lambda_n(v;c) \). Stratton [42] and later Chu and Stratton [11] defined the spheroidal functions as the solutions of the last equation that remains finite at the singular points \( t = \pm 1 \). For integral orders \( \nu = m = 0,1,2,... \) the functions \( \psi_{vn}(t;c) \) are related to the spheroidal wave functions that arise from separation of the Helmholtz equation in spheroidal coordinates. For half-integer values \( \nu = \pm \frac{1}{2} \) they are related to the angular Mathieu functions that arise from separation of the Helmholtz equation in elliptic coordinates [38]. Thus, we may make use of (3.1) to approximate the eigenpairs of the angular Mathieu equation for very large values of \( q \) whose details will be explained below.

Note that, the solutions \( \psi_{vn} \) of (3.2) that remain finite at the end points \( t = \pm 1 \) suggest the use of boundary conditions \( \Psi_{vn}(\pm \pi/2;c) = 0 \) for (3.1). This makes equation (3.1) into a reflection symmetric system so that the even and odd states can be separated. Actually the transformations \( x = -\cos 2\theta \) and \( \Psi_{v2n}(x;c) = (1-x)^{\nu/2}y_{vn}(x) \) lead to the equation

\[
(1-x^2)y''_{vn} + \left[ -\left( v + \frac{1}{2} \right) - \left( v + \frac{3}{2} \right) \eta \right] y'_{vn} + \frac{c^2}{8} xy_{vn} = \frac{1}{4} \left[ v(v+1) + c^2 - \lambda_{2n}(v;c) \right] y_{vn}
\]

(3.3)

where \( x \in (-1,1) \). On returning back to the original variables \( \Psi_{v2n}(\theta;c) = \cos^\nu \theta y_{vn}(-\cos2\theta) \), we see that the even indexed eigenfunctions are even functions of \( \theta \). For the odd states, first we use the map \( \Psi_{v2n+1}(\theta;c) = \sin \theta \phi_{v2n+1}(\theta;c) = \) where \( \phi_{v2n+1} \) is necessarily an even function of \( \theta \). This suggest use of the same transformations employed for the even states.
leading to the equation

\[(1-x^2) y''_{vn} + \left[ -\left( v - \frac{1}{2} \right) - \left( v + \frac{5}{2} \right) \eta \right] y'_{vn} - \frac{c^2}{8} xy_{vn} = \frac{1}{4} \left( (v+1)(v+2) + \frac{c^2}{2} - \lambda_{2n+1}(v;c) \right) y_{vn} \]

(3.4)

where the odd state eigenfunctions \( \Psi_{v2n+1}(\theta;c) = \sin \theta \cos \theta y_{vn}(-\cos 2\theta) \) are indeed an odd function of \( \theta \). Therefore, instead of (3.2), we have two equations to handle the even and odd states of (3.1) separately. Actually the last two equations can be unified as

\[ \left\{ (1-x^2) \frac{d^2}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x] \frac{d}{dx} - \frac{1}{8} c^2 x \right\} y^{(a,\beta)}_{n}(x;q) = \mu^{(a,\beta)}(v;c) y^{(a,\beta)}_{n}(x;q) \]

(3.5)

in which

\[ \mu^{(a,\beta)}(v;c) = \frac{1}{4} \left[ \left( \alpha + \beta + \frac{1}{2} \right) \left( \alpha + \beta + \frac{3}{2} \right) - \lambda(v;c) \right] \]

(3.6)

where the parameters \((\alpha,\beta) = (v,-\frac{1}{2})\) and \((\alpha,\beta) = (v,\frac{1}{2})\) yield the even and odd states, respectively.

Now, taking \( c^2 = 4q \) and comparing (3.5)-(3.6) with (2.14)-(2.15) we obtain the connection relations

\[ a_{2n}(q) = \frac{1}{4} - 2q + \lambda_{2n} \left( -\frac{1}{2} i \sqrt{4q} \right), \quad b_{2n+1}(q) = \frac{1}{4} - 2q + \lambda_{2n} \left( \frac{1}{2} i \sqrt{4q} \right) \]

\[ a_{2n+1}(q) = \frac{1}{4} - 2q + \lambda_{2n+1} \left( \frac{1}{2} i \sqrt{4q} \right), \quad b_{2n+2}(q) = \frac{1}{4} - 2q + \lambda_{2n+1} \left( \frac{1}{2} i \sqrt{4q} \right) \]

(3.7)

among the eigenvalues of the Mathieu and the generalized spheroidal equation in (3.1). Meanwhile, the well-known relations [33]

\[ a_{2n}(-q) = a_{2n}(q), \quad a_{2n+1}(-q) = b_{2n+1}(q), \quad b_{2n+2}(-q) = b_{2n+2}(q) \]

(3.8)

among the eigenvalues of the Mathieu equation, together with (3.7) lead to the interesting relations

\[ \lambda_{2n} \left( -\frac{1}{2} i c \right) - \lambda_{2n} \left( -\frac{1}{2} i c \right) = c^2, \quad \lambda_{2n+1} \left( -\frac{1}{2} i c \right) - \lambda_{2n} \left( \frac{1}{2} i c \right) = c^2 \]

\[ \lambda_{2n} \left( \frac{1}{2} i c \right) - \lambda_{2n+1} \left( -\frac{1}{2} i c \right) = c^2, \quad \lambda_{2n+1} \left( \frac{1}{2} i c \right) - \lambda_{2n+1} \left( \frac{1}{2} i c \right) = c^2 \]

(3.9)

among the eigenvalues of (3.1) when \( v = \pm 1/2 \).

For the spheroidal wave equation, that is when \( v = m = 0,1,2,... \) in equation (3.1), we have presented highly accurate and efficient method for very large values of the bandwidth parameter \( c \) [4]. It was based on the idea that when \( c \) tends to infinity, the prolate spheroidal wave functions can accurately be approximated by the Hermite functions.
scaled by the square root of the parameter $c$ [6, 15, 20, 27, 36]. Clearly, they can also be approximated by the Laguerre functions of order $\pm \frac{1}{2}$ due to the connections

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2), \quad H_{2n+1}(x) = (-1)^n 2^{2n+1} n! L_n^{(\frac{1}{2})}(x^2)$$

(3.10)

between the Hermite and Laguerre polynomials. The same idea can be used with $\nu = \pm \frac{1}{2}$ to approximate the eigenpairs of the Mathieu equation as well. To this end, we transform the equation in (3.1) to another one over the positive half line. This can be accomplished by the maps

$$x = [a \arctanh(\sin\theta)]^2, \quad x \in (0, \infty)$$

(3.11)

and

$$\Psi_{\nu n}(x; c) = x^a \phi_{\nu n}(x; c)$$

(3.12)

applying in respective order. The first map leads to the operational equivalence

$$cos^2 \theta \left( -\frac{d^2}{d\theta^2} + \tan\theta \frac{d}{d\theta} \right) \equiv -4a^2 \left( x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} \right)$$

(3.13)

for the differential part of (3.1) which is not the case for the operator $d^2/d\eta^2$ present in the angular Mathieu equation in (2.1). That is to say, in this case it is not possible to find a transformation from the interval $\eta \in (0, \pi/2)$ to the half line $x \in (0, \infty)$ leading to such an operator with linear and constant polynomial coefficients. That is why we make use of the more general equation in (3.1) and finally take $\nu = \pm \frac{1}{2}$ since in these two cases the solutions of (3.1) are related with the angular Mathieu functions. Application of the above transformations lead to the equation

$$x \phi''_{\nu n} + (\gamma + 1) \phi'_{\nu n} + Q(x) \phi_{\nu n} = \mu^{(\gamma)}_n(v; c) r(x) \phi_{\nu n} \quad \mu^{(\gamma)}_n(v; c) = -\frac{1}{4a^2} \lambda_n(v, c)$$

(3.14)

where

$$Q(x) = -\frac{1}{4a^2} \left[ v^2 + c^2 \text{sech}^2 \left( \sqrt{\frac{x}{a}} \right) \tan^2 \left( \sqrt{\frac{x}{a}} \right) \right], \quad (3.15)$$

$$r(x) = \text{sech}^2 \left( \sqrt{\frac{x}{a}} \right)$$

(3.16)

and $\gamma = 2a - \frac{1}{2}$. Clearly, $\gamma = -\frac{1}{2}$ leads to the even-states whereas $\gamma = \frac{1}{2}$ yield odd-states of (3.1) which can be seen on returning back to the original variables via (3.11)-(3.12). The eigenvalues of (3.1) are related with those of (3.14) by the formula

$$\lambda_{2n}(v, c) = -4a^2 \mu^{(-1/2)}_n(v; c), \quad \lambda_{2n+1}(v, c) = -4a^2 \mu^{(1/2)}_n(v; c)$$

(3.17)

from which we can compute the characteristic values of the Mathieu equation employing the connection formula (3.7) with $\nu = \pm \frac{1}{2}$ and $c = \sqrt{4q}$.

Here, $a$ is a scaling or an optimization parameter whose optimum value is usually determined by trial and error. However, fortunately, for this problem as is explained
above its optimum value is fixed as \( a_{opt} = \sqrt{c} = (4q)^{1/4} \). To approximate the eigenpairs of (3.14), we utilize the Laguerre functions in a pseudospectral picture since it reduces to the differential equation for the Laguerre functions when \( q(x) \equiv 0 \) and \( r(x) = 1 \). Basically, the idea is to collect the grid points, that are the roots of the order \( \gamma = \pm 1/2 \) Laguerre polynomials, to the small interval where the eigenfunctions are away from zero by means of the scaling factor \( a_{opt} = \sqrt{c} \). Now, consider the weighted interpolation of the solution of (3.14)

\[
\phi_{vn}^N(x,c) = \sum_{k=0}^{N} \ell_k(x) \phi_{vn}(x_k,c)
\]

in which

\[
\ell_k(x) = \frac{\hat{L}^{(\gamma)}_{N+1}(x)}{(x-x_k)\partial_x L^{(\gamma)}_{N+1}(x_k)}, \quad \hat{L}^{(\gamma)}_{N+1}(x) = \frac{1}{h_{N+1}} e^{-x/2} L^{(\gamma)}_{N+1}(x), \quad h_{N+1}^2 = \frac{\Gamma(N+\gamma+2)}{(N+1)!}
\]

are the set of \( N \) th degree Lagrange polynomials and \( x_k \) stand for the \( N+1 \) real and distinct roots of \( L^{(\gamma)}_{N+1}(x) \) which are computed by using the well-known Golub-Welsch algorithm [5,23]. Proposing the interpolant as an approximate solution to (3.14) and requiring the satisfaction of (3.14) at the collocation points \( x_k \), we obtain a discrete representation

\[
\hat{B}\hat{u}_n = \mu_n^{(\gamma)}(v,c)\hat{u}_n
\]

where \( \hat{u}_n = [\phi_{vn}(x_0,c) \phi_{vn}(x_1,c) \ldots \phi_{vn}(x_N,c)]^T \) contains the values of an eigenfunction at the nodal points. Then approximate eigenvalues \( \lambda_{2n}(v,c) \) and \( \lambda_{2n+1}(v,c) \) of (3.14) may be determined from (3.20) with \( \gamma = -\frac{1}{2} \) and \( \gamma = \frac{1}{2} \), respectively. After some algebra one may express the entries of the matrix \( \hat{B} \) as

\[
\hat{B}_{mn} = \cosh^2\left(\sqrt{\frac{m}{m}} / \alpha\right) \begin{cases} 
-2x_m \frac{\partial_x \hat{L}^{(\gamma)}_{N+1}(x_m)}{(x_m-x_n)^2} & \text{if } m \neq n \\
1 - \gamma^2 \frac{2N+\gamma+3}{6} + \frac{x_n}{12} + Q(x_n) & \text{if } m = n
\end{cases}
\]

where \( Q(x_n) \) are the values of the function \( Q(x) \) in (3.15) at the nodal points \( x_n \). This matrix can be factored as \( \hat{B} = SBS^{-1} \) where \( S = \text{diag}(s_1, s_2, \ldots, s_N) \) is diagonal matrix with

\[
s_m = \sqrt{x_m} \cosh\left(\sqrt{\frac{x_m}{x_n}} / \alpha\right) \partial_x \hat{L}^{(\gamma)}_{N+1}(x_m), \quad m = 0,1,\ldots,N
\]

and \( B \) is symmetric matrix with entries

\[
B_{mn} = \begin{cases} 
-2\sqrt{x_m x_n} \cosh\left(\sqrt{\frac{x_m}{x_n}} / \alpha\right) \cosh\left(\sqrt{\frac{x_n}{x_m}} / \alpha\right) & \text{if } m \neq n \\
\cosh^2\left(\sqrt{\frac{x_n}{x_m}} / \alpha\right) \left[1 - \gamma^2 \frac{2N+\gamma+3}{6} + \frac{x_n}{12} + Q(x_n)\right] & \text{if } m = n
\end{cases}
\]
Therefore we may replace the unsymmetric system in (3.20) with the symmetric one

\[ Bu_n = \mu_n^{(\gamma)}(\nu; c) u_n \]  

(3.24)

since the similar matrices share the same eigenvalue set. Clearly, an eigenvector \( \hat{u}_n \) of (3.20) is connected with that of (3.24) by

\[ \hat{u}_n = S u_n. \]  

(3.25)

On the other hand, for an orthonormal eigenfunction of (3.14) we may write

\[ 1 = \int_{0}^{\infty} \Phi_{\nu n}^2(x; c) \text{sech}^2 \left( \sqrt{\alpha} / \alpha \right) x^2 e^{-x} dx = \int_{0}^{\infty} \Psi_{\nu n}^2(x; c) \text{sech}^2 \left( \sqrt{\alpha} / \alpha \right) x^{-1/2} dx \]

(3.26)

where we have used (3.12). However, applying the change of variable \( \sin \theta = \tanh(\sqrt{\alpha} / \alpha) \) in (3.11), to an eigenfunction of (3.1) we obtain

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi_{\nu n}(\theta; c) \cos \theta d\theta = 2 \int_{0}^{\frac{\pi}{2}} \Psi_{\nu n}(\theta; c) \cos \theta d\theta \\
= \frac{1}{\alpha} \int_{0}^{\infty} \Psi_{\nu n}(x; c) \text{sech}^2 \left( \sqrt{\alpha} / \alpha \right) x^{-1/2} dx = \frac{1}{\alpha}.
\]

(3.27)

This means that \( \Psi_{\nu n}(\theta; c) = \sqrt{\alpha} \Psi_{\nu n}(x; c) \) so that the \( L_2^{\cos \theta}(-\pi/2, \pi/2) \) norm of the eigenfunction \( \Psi_{\nu n}(\theta; c) \) in the original variable \( \theta \) is unity. Thus, we may write

\[ \Psi_{\nu n}(\theta_m; c) = \sqrt{\alpha} \Psi_{\nu n}(x_m; c) = \sqrt{\alpha} x_m^{1/2} \cos \left( \sqrt{x_m / \alpha} \right) \partial_x L_{N-1}^{(\gamma)}(x_m) u_{nm} \]

(3.28)

where \( \theta_m = \sqrt{x_m / \alpha} \) and \( u_{nm} \) is the \( m \)-th entry of the eigenvector \( u_n \) of (3.24). Actually an eigenvector of (3.24) depends on the parameters \( \nu \) and \( \gamma \), too. Hence, it is better to adopt the notation \( u_n^{(\nu, \gamma)} \). Finally, using the differential-difference relation

\[ x \partial_x \hat{L}_n^{(\gamma)}(x) = n \hat{L}_n^{(\gamma)}(x) - \sqrt{n(n+\gamma)} \hat{L}_{n-1}^{(\gamma)}(x) \]

(3.29)

of the normalized Laguerre functions in (3.19) with \( n = N+1 \) and \( x = x_\nu \), we may represent the angular Mathieu functions in terms of the spheroidal functions of orders \( \nu = \pm 1/2 \) as

\[ \Phi_n^{(\nu, \gamma)}(\theta_m; q) = \sqrt{\pi / 2} \Psi_{\nu n}(\theta_m; \sqrt{4q}) \]

\[ = -\sqrt{\pi} \alpha (N+1)(N+\gamma+1) / 2 x_m^{1/2} \cos \left( \sqrt{x_m / \alpha} \right) \hat{L}_N^{(\gamma)}(x_m) u_{nm} \]

(3.30)

where the parameters \( (\nu, \gamma) = \{(-\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\} \) lead to the Mathieu functions \( \Phi^{(\nu, \gamma)} = \{c_2n, c_{2n+1}, s_{2n+1}, s_{2n+2}\} \), respectively, for very large values of the parameter \( q \). Here, the values \( \hat{L}_N^{(\gamma)}(x_m) \) of the \( N \)-th normalized Laguerre function of order \( \gamma \) are computed in a stable way by using the recurrence

\[ \sqrt{(n+1)(n+\gamma+1)} \hat{L}_{n+1}^{(\gamma)}(x) = (2n+\gamma+1-x) \hat{L}_n^{(\gamma)}(x) - \sqrt{n(n+\gamma)} \hat{L}_{n-1}^{(\gamma)}(x) \]

(3.31)
where
\[
\hat{L}_0^{(\gamma)}(x) = e^{-x/2}/\sqrt{\Gamma(\gamma+1)}, \quad \hat{L}_1^{(\gamma)}(x) = (\gamma+1-x)e^{-x/2}/\sqrt{\Gamma(\gamma+2)}.
\] (3.32)

The matrix in (3.23) only necessitates the knowledge of the zeros of the \((N+1)\)-st degree Laguerre polynomial of order \(\gamma = \pm 1/2\). It is easy to construct and symmetric with quite reasonable condition number. In fact, numerical experiments reveal that as \(q\) gets larger, the condition number of the matrix \(B\) of size \((N+1) \times (N+1)\) converges to \(4N+1\) and \((4N+3)/3\) when \(\gamma = -1/2\) and \(\gamma = 1/2\), respectively.

Table 3: The number of eigenvalues having relative errors of order \(10^{-14}\) for a given truncation order \(N\) and the parameter \(q\).

<table>
<thead>
<tr>
<th>(q = 10^5)</th>
<th>(q = 10^6)</th>
<th>(q = 10^{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>Jacobi</td>
<td>Laguerre</td>
</tr>
<tr>
<td>100</td>
<td>6</td>
<td>55</td>
</tr>
<tr>
<td>300</td>
<td>134</td>
<td>120</td>
</tr>
<tr>
<td>600</td>
<td>535</td>
<td>160</td>
</tr>
</tbody>
</table>

Now, in Table 3 by comparing with the Jacobi spectral method we see that the Laguerre pseudospectral formulation leads to much better results for large \(q\) values, especially as \(q\) gets larger and larger.

### 4 Numerical results

In this section we present some numerical tables and plots of eigenfunctions for small, moderate and very large values of the parameter \(q\).

It is clear from Table 4 that the algorithms mentioned in this study can accurately approximate the eigenvalues when the parameter \(q\) is small. Here we executed our algorithm in Fortran quadruple precision arithmetic to see if the results agree with those of [1] up to the accuracy quoted. This does not mean that the other algorithms can not attain such accuracy for small \(q\). In contrast, there would have been algorithms cited in this study which can also attain the same accuracy if they were implemented in quadruple precision arithmetic, too.

Then in Table 5 we present the discrete values of the function \(ce_9(\eta,100)\) at the specified angles. First two columns include the results obtained from the present Jacobi-Galerkin method with the truncation orders \(N=25\) and \(N=26\). The last column includes the results from [8] which truncates the second series in (1.1) at \(N = 20\). Although it seems that, our method yields better results then the classical trigonometric series expansion, at a cost of computing more expansion coefficients, the trigonometric series can also produce results that are accurate to machine precision. It is clear from Table 5 that for small values of \(q\) both the present Jacobi-Galerkin and the trigonometric series approach produce similar results.
Table 4: Comparison of $b_2(25)$ and $b_{10}(25)$ with some literature results.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_{2n+2}(25)$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-21.314860622249850854314664972573</td>
<td>Present Jacobi-Galerkin formulation</td>
</tr>
<tr>
<td></td>
<td>-21.31486062224985085431466497257381226977</td>
<td>[1]</td>
</tr>
<tr>
<td></td>
<td>-21.3148606222498508543146649725738122697781226977</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td>-21.31486062224985085431466497257381226977</td>
<td>[30]</td>
</tr>
<tr>
<td></td>
<td>-21.31486062224985085431466497257381226977</td>
<td>[24]</td>
</tr>
<tr>
<td></td>
<td>-21.31486062224985085431466497257381226977</td>
<td>[13]</td>
</tr>
<tr>
<td></td>
<td>-21.31486062224985085431466497257381226977</td>
<td>[29]</td>
</tr>
</tbody>
</table>

| 4   | 103.225680042373470004799730544439 | Present Jacobi-Galerkin formulation |
|     | 103.2256800423734700047997305444380455190 | [1] |
|     | 103.2256800423734700047997305444380455190 | [2] |
|     | 103.2256800423734700047997305444380455190 | [30] |
|     | 103.2256800423734700047997305444380455190 | [24] |
|     | 103.2256800423734700047997305444380455190 | [13] |
|     | 103.2256800423734700047997305444380455190 | [29] |

Table 5: Values of the Mathieu function $ce_9(\eta,100)$ at the specified angles $\eta$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$ce_9(\eta,100)$ N = 25</th>
<th>$ce_9(\eta,100)$ N = 26</th>
<th>$ce_9(\eta,100)$ Ref. [8]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0.066266491141916</td>
<td>0.066266491141915</td>
<td>0.066266491106052</td>
</tr>
<tr>
<td>10°</td>
<td>0.152283208659799</td>
<td>0.152283208659799</td>
<td>0.152283208636151</td>
</tr>
<tr>
<td>20°</td>
<td>0.531438482965942</td>
<td>0.531438482965944</td>
<td>0.531438482970954</td>
</tr>
<tr>
<td>30°</td>
<td>1.212916357317376</td>
<td>1.212916357317379</td>
<td>1.212916357348340</td>
</tr>
<tr>
<td>40°</td>
<td>0.842535893157752</td>
<td>0.842535893157752</td>
<td>0.842535893193692</td>
</tr>
<tr>
<td>50°</td>
<td>-0.955912622210367</td>
<td>-0.955912622210367</td>
<td>-0.955912622195211</td>
</tr>
<tr>
<td>60°</td>
<td>0.439824653297739</td>
<td>0.439824653297739</td>
<td>0.439824653279982</td>
</tr>
<tr>
<td>70°</td>
<td>-0.114071528013408</td>
<td>-0.114071528013409</td>
<td>-0.114071528052211</td>
</tr>
<tr>
<td>80°</td>
<td>0.004366328988792</td>
<td>0.004366328988792</td>
<td>0.004366328957511</td>
</tr>
</tbody>
</table>

Table 6 demonstrates the accuracy improvement of $b_{2n+1}(q)$ for the parameter values $q = 1,10^{10}$ and eigenvalue indices $n = 1,1001$. It is clear that the methods described here not only yield accurate results for lower states, but also for higher ones as high as thousand with quite reasonable truncation orders $N$. As a result of separation, the necessary truncation size to print $b_{1001}(1)$ on the screen is $N = 501$. However, $N = 502$ is enough to obtain it with a relative error of order $10^{-15}$. Similarly, $N = 513$ does the same job for $b_{1001}(10^{10})$ revealing the efficiency of our algorithms.

For comparison, the bottom row includes the results obtained from MATSLISE package [29] and Cojocaru [13]. The latter diagonalizes the tri-diagonal matrix resulted from writing the recurrence relation of the expansion coefficients of (1.1) in matrix form. As it is mentioned in Introduction, this type of approaches are suitable for small or moderate
Table 6: Accuracy improvement for $b_{1001}(q)$ for $q = 1$ and $q = 10^{10}$ by using the Jacobi and the Laguerre bases, respectively.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$b_{1001}(1)$</th>
<th>$b_1(10^{10})$</th>
<th>$N$</th>
<th>$b_{1001}(10^{10})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>504</td>
<td>1002001.0000004984</td>
<td>-19999800000.250004</td>
<td>513</td>
<td>-19600301128.153511</td>
</tr>
<tr>
<td>505</td>
<td>1002001.0000004998</td>
<td>-19999800000.250008</td>
<td>514</td>
<td>-19600301128.153587</td>
</tr>
<tr>
<td>506</td>
<td>1002001.0000004988</td>
<td>-19999800000.250000</td>
<td>515</td>
<td>-19600301128.153591</td>
</tr>
<tr>
<td></td>
<td>1002001.0000004990$^a$</td>
<td>-19999800000.250000$^a$</td>
<td></td>
<td>-19600301128.153572$^a$</td>
</tr>
<tr>
<td>505</td>
<td>1002001.0000005000$^b$</td>
<td>1100</td>
<td></td>
<td>19999800000.249931$^b$</td>
</tr>
</tbody>
</table>

$^a$ Reference [29], $^b$ Reference [13]

Figure 2: $ce_{30}(\eta,10^6)$ and $ce_{50}(\eta,10^6)$ corresponding to the characteristic values $a_{30}(10^6) = -1878467.04186782$ and $a_{50}(10^9) = -1799283.43142027$, respectively.

$q$ values and not able to produce satisfactory results for very large values of the parameter $q$. Specifically, $N = 2000$ was not enough to approximate the eigenvalue $b_{1001}(10^{10})$. Even for the ground state eigenvalue $b_1(10^{10})$ one needs to diagonalize a matrix of size $N = 1100$. Actually, MATSLISE works for very large values of $q$ as high as $10^{13}$ beyond which it also has difficulties in approximating the Mathieu eigenpairs.

For large values of $q$ it is still possible to approximate the lower eigenpairs by the Jacobi bases at a cost of very high truncation orders. In Fig. 2 we present plots of the Mathieu functions $ce_{30}(\eta,10^6)$ and $ce_{50}(\eta,10^6)$ that are obtained by using both the Jacobi and the Laguerre bases. Note the confinement of the eigenfunctions to a small interval around the point $\pi/2$ for such large values of $q$.

In order to check the efficiency of the Laguerre basis approach, in Table 7 we include several eigenvalues for very large value of $q = 10^{20}$. In this case, the integer part of the eigenvalues occupy more than twenty digits. Therefore, we have implemented our algorithm in Fortran programming language by using quadruple precision arithmetic and tabulate the values $a(q) + 2q$ or $b(q) + 2q$ instead of $a(q)$ and $b(q)$, respectively. To ap-
Table 7: Several eigenvalues of the angular Mathieu equation for $q = 10^{20}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>$a_{2n}(q) + 2q \approx b_{2n+1}(q) + 2q$</th>
<th>$a_{2n+1}(q) + 2q \approx b_{2n+2}(q) + 2q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>19999999999.749999999968</td>
<td>5999999999.749999999972</td>
</tr>
<tr>
<td>55</td>
<td>100</td>
<td>4019999994949.74999936553094</td>
<td>4059999994848.749993464034</td>
</tr>
<tr>
<td>255</td>
<td>500</td>
<td>20019999874749.7492164015471</td>
<td>20059999874248.749211695272</td>
</tr>
<tr>
<td>505</td>
<td>1000</td>
<td>40019999949949.7437406154262</td>
<td>40059999949849.743721827900</td>
</tr>
</tbody>
</table>

Approximate the eigenvalue $a_{1000}(10^{20})$ with the present Laguerre pseudospectral method to the accuracy quoted, we only need to diagonalize a matrix of size $505 \times 505$ which is impressive. It is known that, for finite $n$ and large $q$ values $a_{2n}(q) \approx b_{2n+1}(q)$ and $a_{2n+1}(q) \approx b_{2n+2}(q)$ [33]. This can also be observed from the equations (3.14)-(3.16) and (3.7). In fact, in (3.14) one has $\lambda_n(v,c) = \lambda_n(-v,c)$ since there is a quadratic dependence on $v$. Then, it is easy to see that $a_n(q) \approx b_{n+1}(q)$ by employing this fact in the connection formulas (3.7). Numerical results in Table 7 are in accordance with the asymptotic formula

$$a_n(q) \approx b_{n+1}(q) = -2q + 2q^{1/2}k - 2^{-3}(k^2 + 1) - 2^{-7}q^{-1/2}k(k^2 + 3)$$

$$-2^{-12}q^{-1}(5k^4 + 34k^2 + 405) - 2^{-17}q^{-3/2}k(33k^4 + 410k^2 + 405) + O(q^{-2})$$

(4.1)

up to their last digits where $k = 2n + 1$ [28,33].

Then, in Fig. 3 we present the plots of two eigenfunctions corresponding to the same $q = 10^{20}$. Notice in this case that their nonzero parts are confined to a tiny interval of length $10^{-4}$ around the points $(2k+1)\pi/2$ where $k$ is an integer.

(a) $ce_{10}(\eta,10^{20})$  
(b) $se_{12}(\eta,10^{20})$

Figure 3: Mathieu functions $ce_{10}(\eta,q)$ and $se_{12}(\eta,q)$ corresponding to the characteristic values $a_{10}(q) + 2q = 41999999944.7499999927156$ and $b_{12}(q) + 2q = 4599999993.749999904406$, respectively.
5 Conclusion

In this article, we have constructed accurate and efficient spectral and pseudospectral methods based on the Jacobi and the Laguerre polynomials to approximate the integer order periodic Mathieu functions and the corresponding characteristic values. To this end, the angular Mathieu equation is transformed into several equations resembling the Jacobi and the Laguerre differential equations. These particular transformations motivated us to use the most suitable Jacobi or Laguerre polynomial with specific parameter(s) as basis sets for the approximation of the eigenpairs. It is observed from numerical results that the Jacobi-Galerkin methods are well suited for small $q$ values whereas the Laguerre pseudospectral methods scaled by $(4q)^{1/4}$ are more appropriate for very large values of $q$. Note that, in the Laguerre pseudospectral method a suitable scaling parameter optimizing the accuracy is not chosen by experimentally since it is initially set as $\alpha_{opt} = (4q)^{1/4}$.

The algorithms developed in this paper are implemented in a set of Matlab routines, Mathieu.m, Mathieu_Small_q.m, Mathieu_Large_q.m and eigfunplot.m, which can be downloaded via the link http://www.math.purdue.edu/~shen/pub/mathieu.zip. The routine Mathieu.m serves as the main file which calls the other three to compute the Mathieu eigenpairs. Mathieu_Small_q.m and Mathieu_Large_q.m approximate the eigenpairs for $|q| < 10^6$ and $|q| \geq 10^6$, respectively whereas Eigfunplot.m is responsible for plotting the results obtained from these routines.

The methods developed here will be useful in a variety of physical and engineering applications in which accurate solutions of the angular Mathieu equation are needed.

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References


