ERROR ESTIMATES FOR A FULLY DISCRETIZED SCHEME TO A CAHN-HILLIARD PHASE-FIELD MODEL FOR TWO-PHASE INCOMPRESSIBLE FLOWS

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ABSTRACT. We carry out in this paper a rigorous error analysis for a finite element discretization of the linear, weakly coupled energy stable scheme introduced in [24, 23] for a Cahn-Hilliard phase-field model of two-phase incompressible flows with matching density.

1. Introduction

Phase-field approaches for multi-phase incompressible flows have attracted considerable interests in recent years (cf. [15, 16, 1, 20, 18, 27, 22] and the references therein). For two-phase incompressible flows, the phase-field models consist of either a Navier-Stokes-Cahn-Hilliard (NSCH) system or a Navier-Stokes-Allen-Cahn (NSAC) system. How to design efficient and accurate numerical methods for these coupled nonlinear systems brings great challenge to the scientific computing community.

In this paper, we focus on the following Cahn-Hilliard phase field model with matching density,

\begin{equation}
\begin{aligned}
&\phi_t + u \cdot \nabla \phi - \gamma \Delta w = 0, \quad \text{in } \Omega \subset \mathbb{R}^d, \\
&w = -\Delta \phi + f(\phi), \quad \text{in } \Omega \subset \mathbb{R}^d, \\
&\rho_0(u_t + (u \cdot \nabla)u) - \mu_0 \Delta u + \nabla p - \lambda w \nabla \phi = 0, \quad \text{in } \Omega \subset \mathbb{R}^d, \\
&\nabla \cdot u = 0, \quad \text{in } \Omega \subset \mathbb{R}^d, \\
&u|_{\partial \Omega} = 0, \quad \frac{\partial \phi}{\partial n}|_{\partial \Omega} = 0, \quad \frac{\partial w}{\partial n}|_{\partial \Omega} = 0,
\end{aligned}
\end{equation}

with given initial data \( u(0) = u_0, \phi(0) = \phi_0 \). In the above, \( d = 2 \) or \( 3 \), \( \phi \) is the phase function where \( \phi \approx \pm 1 \) corresponds to two different fluids, \( w \) is the chemical potential, \( u \) is the velocity field and \( p \) is the pressure. \( \rho_0 \) is the density of both fluids; \( \gamma \) is a relaxation constant; \( \lambda \) is the mixing energy density, \( f(\phi) = F'(\phi) \) where \( F(\phi) = \frac{(1-\phi^2)^2}{4\varepsilon^2} \), and the parameter \( \varepsilon > 0 \) represents the interfacial thickness.

The above system satisfies the following energy law

\begin{equation}
\frac{d}{dt} \int_{\Omega} \left( \frac{\rho_0}{2} |u|^2 + \frac{\lambda}{2} |\nabla \phi|^2 + \lambda F(\phi) \right) dx = - \int_{\Omega} (\mu_0 |\nabla u|^2 + \lambda \gamma |\nabla w|^2) dx.
\end{equation}

2010 Mathematics Subject Classification. 35Q30, 65M12, 65M60.

Key words and phrases. Phase-field model, Cahn-Hilliard equation, Navier-Stokes equation, finite element, error estimates.

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While various convergence results and error estimates have been derived for the Navier-Stokes equations [25, 8, 13], there are only a few error estimates available for phase-field models of multi-phase flows. In [4], Feng proved convergence of discrete finite element solutions for a Cahn-Hilliard phase-field model with matching density, and in [5] the authors established similar convergence results for an Allen-Cahn phase-field model with matching density. Most recently, Grün [9] proved convergence results for a Cahn-Hilliard phase-field model with variable densities. Later, Grün et al. [10] established convergence of a convex splitting scheme for a Cahn-Hilliard phase field model with general mass densities, and Gand Diegel et al. [3] obtained error estimates of a fully discretized convex splitting method for a Cahn-Hilliard-Darcy-Stokes model. However, the schemes considered in these papers are fully coupled (velocity and pressure) and/or nonlinear. From a computational point of view, it is more efficient to use a decoupled or weakly coupled linear scheme. In [24, 23], some weakly coupled linear, energy stable schemes are constructed, where the phase equations are discretized by the stabilized scheme [21, 26] and the Navier-Stokes (NS) equations are discretized by a projection scheme [13]. These schemes lead to, at each time step, a weakly coupled elliptic equations for the phase function and velocity, and a decoupled pressure Poisson equation for the pressure. Hence, they are very efficient and easy to implement.

Though various error estimates are available for projection type methods to the Navier-Stokes equations [13] and for the Cahn-Hilliard/Allen-Cahn equations [6, 7, 17, 21], it is highly non trivial to deal with the systems which couple Navier-Stokes and Cahn-Hilliard/Allen-Cahn, since the splitting error in the projection step affects the whole system. The major difficulties arise from the projection step to deal with the incompressibility constraint and from the coupling between the phase function and the velocity. To the best of our knowledge, error estimates for such schemes in fully-discrete form are not yet available. In a recent work [2], we carried out an error analysis for the schemes presented [24, 23] in semi-discrete-in-time form. However, the analysis in [2] can not be easily extended to the fully discrete case, as the full discretization adds another level of difficulty, particularly in obtaining optimal error estimates for the pressure due to the splitting error in the scheme. The main purpose of this paper is to provide a rigorous error analysis for the energy stable scheme in [24] for the Navier-Stokes-Cahn-Hilliard system in fully-discrete form. To make our analysis applicable to more general settings, we make only standard assumptions on the finite element discretization.

The rest of the paper is organized as follows. In section 2, we recall some preliminary results regarding the Navier-Stokes and the Cahn-Hilliard equations and for the finite-element approximation. In section 3, we introduce the fully discretized scheme for the Cahn-Hilliard phase-field model based on the scheme introduced in [24]. Section 4 is devoted to the error analysis, where we prove the error estimates for phase functions, velocity field and the pressure, under the assumption that the exact solution is sufficiently smooth. In section 5, we show that the numerical solution converges to the weak solution of the continuous problem without assuming extra regularity of the exact solution. We conclude with a few remarks in section 6.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a connected, bounded, open domain with $C^{1,1}$ boundary $\partial \Omega$ (or such that the Stokes problem has $H^2$ regularity, see (2.9) below). We have the following
Sobolev inequalities hold (cf. for instance [19])

\[ \|g\|_{L^1(\Omega)} \leq C(2, \Omega)\|g\|_{L^2(\Omega)}\|g\|_{H^1(\Omega)}, \quad d = 2, \]
\[ \|g\|_{L^2(\Omega)} \leq C(3, \Omega)\|g\|_{L^2(\Omega)}\|g\|_{H^1(\Omega)}, \quad d = 3. \]  

(2.1)

In particular, \( \|g\|_{L^p(\Omega)} \leq C\|g\|_{H^1(\Omega)} \).

Let \( t^n = n\delta t \) (0 ≤ n ≤ N = \([T/\delta t]\)) where \( \delta t \) is the time step. For a sequence of functions \( \varphi^0, \varphi^1, \ldots, \varphi^N \) in some Hilbert space \( E \), we denote the sequence by \( \varphi_{\delta t} \) and define the following discrete norms for \( \varphi_{\delta t} \),

\[ \|\varphi_{\delta t}\|_{l^2(E)} = \left( \delta t \sum_{n=0}^{N} \|\varphi^n\|^2 \right)^{1/2}, \quad \|\varphi_{\delta t}\|_{l^\infty(E)} = \max_{0 \leq n \leq N} (\|\varphi^n\|_E). \]  

(2.2)

Let \( \| \cdot \|_k \) denote the usual \( H^k(\Omega) \) \( (H^k(\Omega)^d) \) norm, and \( \| \cdot \|_{k,p} \) denote the \( W^{k,p} \) norm. In particular, \( \| \cdot \| \) and \( (\cdot, \cdot) \) are the \( L^2(\Omega) \) \( (L^2(\Omega)^d) \) norm and inner product, respectively.

We also denote

\[ X = H_0^1(\Omega)^d, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}, \]  

(2.3)

and the following spaces of incompressible vector fields

\[ H = \left\{ v \in L^2(\Omega)^d; \nabla \cdot v = 0; v \cdot n|_{\partial \Omega} = 0 \right\}, \quad V = \left\{ v \in H^1(\Omega)^d; \nabla \cdot v = 0; v|_{\partial \Omega} = 0 \right\}. \]  

(2.4)

It holds [25]

\[ L^2(\Omega)^d = H \oplus \nabla(H^1(\Omega)), \]  

(2.5)

where \( \nabla(H^1(\Omega)) = \{ \nabla g | g \in H^1(\Omega) \} \). Define \( P_H \) as the \( L^2 \)-orthogonal projector in \( H \), i.e.,

\[ (u - P_H u, v) = 0, \quad \forall u \in L^2(\Omega)^d, \quad v \in H. \]  

(2.6)

We also define the inverse Stokes operator \( S : H^{-1}(\Omega)^d \to V \) as follows. For all \( v \in H^{-1}(\Omega)^d, (S(v), r) \in V \times L_0^2(\Omega) \) is the solution to the following Stokes problem

\[ \begin{cases} 
(\nabla S(v), \nabla w) - (r, \nabla \cdot w) = \langle v, w \rangle, & \forall w \in H_0^1(\Omega)^d, \quad (2.7) \\
(q, \nabla \cdot S(v)) = 0, & \forall q \in L_0^2(\Omega) 
\end{cases} \]  

(2.8)

where \( \langle \cdot, \cdot \rangle \) denote the pairing between \( H^{-1}(\Omega)^d \) and \( H_0^1(\Omega)^d \) and \( r \) is the pressure.

By the assumption on \( \Omega \), we have the following \( H^2 \) regularity results [25]

\[ \|S(v)\|_2 + \|\nabla r\| \leq c\|v\|, \quad \forall v \in L^2(\Omega)^d. \]  

(2.9)

The following properties of \( S \) are shown in [12].

**Lemma 2.1.** For all \( v \in H_0^1(\Omega)^d \) and all \( 0 < \theta < 1 \), we have

\[ (\nabla S(v), \nabla v) \geq (1 - \theta)\|v\|^2 - c(\theta)\|v - v^*\|^2, \quad \forall v^* \in H. \]

In particular

\[ (\nabla S(v), \nabla v) = \|v\|^2, \quad \forall v \in V. \]

**Lemma 2.2.** The bilinear form \( H^{-1}(\Omega)^d \times H^{-1}(\Omega)^d \ni (v, w) \mapsto \langle S(v), w \rangle \in \mathbb{R} \) induces a semi-norm on \( H^{-1}(\Omega)^d \) that we denote by \( | \cdot |_s \), and

\[ |v|_s = \|S(v)\|_1 \leq c_s\|v\|_{-1}, \quad \forall v \in H^{-1}(\Omega)^d. \]
2.1. Variational formulation. Since we are interested in the values of the phase variable \( \phi \) in the range of \([-1, 1] \), it is a common practice to replace \( F(\phi) \) by

\[
F(\phi) = \begin{cases} 
\frac{1}{\varepsilon^2} (\phi - 1)^2, & \phi > 1, \\
\frac{1}{\varepsilon^2} (\phi^2 - 1)^2, & \phi \in [-1, 1], \\
\frac{1}{\varepsilon^2} (\phi + 1)^2, & \phi < -1.
\end{cases}
\] (2.10)

It can be checked that \( f(\phi) = F'(\phi) \) has a bounded first order derivative \( |f'(\phi)| \leq 2/\varepsilon^2 \), \( f'(\phi) \) is Lipschitz and \( |f(\phi)| \leq \frac{2}{\varepsilon^2}(|\phi| + 1) \). Hereafter, we shall assume that \( F(\phi) \) takes the above modified form.

To write the variational formulation, we adopt similar notations as those in [11, 13]. Define the linear, continuous operator \( \tilde{A} : X \to X' = H^{-1}(\Omega)^d \) (resp. bilinear form \( a : X \times X \to \mathbb{R} \)) such that for all \((u, v) \in X \times X\):

\[
\langle Au, v \rangle = a(u, v) := \langle \nabla u, \nabla v \rangle = \sum_{i,j=1}^d \left( \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right). \quad (2.11)
\]

\( \tilde{A} : H^1(\Omega) \to H^{-1}(\Omega) \) (resp. bilinear form \( \tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \)) so that for all \((\phi, \varphi) \in H^1(\Omega) \times H^1(\Omega)\):

\[
\langle \tilde{A}\phi, \varphi \rangle = \tilde{a}(\phi, \varphi) := \langle \nabla \phi, \nabla \varphi \rangle. \quad (2.12)
\]

Introduce the operator \( B : X \to M \) and its transpose \( B^T : M \to X' \) (resp. linear form \( b : X \times M \to \mathbb{R} \)) so that for all \( v \in X \) and \( q \in M \),

\[
(Bv, q) = b(v, q) = -\langle \nabla \cdot v, q \rangle. \quad (2.13)
\]

Define the bilinear operator \( D : X^2 \to X' \) (resp. trilinear form \( d : X \times X \times X \to \mathbb{R} \)) so that for \((u, v, w) \in X \times X \times X\),

\[
\langle D(u, v), w \rangle = d(u, v, w) := \langle (u \cdot \nabla)v, w \rangle + \frac{1}{2} \langle \nabla \cdot u, v \cdot w \rangle, \quad (2.14)
\]

where \( u \cdot v \) denote the Euclidean scalar product. It is easy to check that for any \( u \in X \), \( d(u, \cdot, \cdot) \) is a skew-symmetric bilinear form and thus \( d(u, v, v) = 0 \).

Then, the weak formulation for the continuous problem [1.1]-[1.5] is: Find

\[
\phi \in L^\infty(0, T; H^1(\Omega)), \quad \phi_t \in L^2(0, T; H^{-1}(\Omega)), \quad \text{for all} \quad T > 0,
\]

\[
w \in L^2(0, T; H^1(\Omega)), \quad \text{for all} \quad T > 0,
\]

\[
u \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; H^1(\Omega)^d), \quad u_t \in L^2(0, T; (H^{-1})^d),
\]

\[
p \in L^2(0, T; M), \quad \text{for all} \quad T > 0,
\]

such that

\[
\begin{cases} 
\phi_t + u \cdot \nabla \phi + \gamma \tilde{A}w = 0, \\
w = \tilde{A}\phi + f(\phi), \\
\rho_0(u_t + D(u, u)) + \mu_0 Au + B^T p - \lambda w\nabla \phi = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
Bu = 0,
\end{cases}
\] (2.15-2.18)

with \( \phi(t = 0) = \phi_0 \) and \( u(t = 0) = u_0 \). Initial values of \( p \) and \( w \) can be determined from the equations by \( \phi_0 \) and \( u_0 \).
2.2. FEM discretization. Let $\mathcal{T}_h$ be a regular, quasi-uniform triangulation of $\Omega$ of mesh size $0 < h < 1$ and $\Omega = \bigcup_{K \in \mathcal{T}_h} K$. For a nonnegative integer $r$, denote $P_r(K)$ as the space of polynomials of degree less than or equal to $r$ on $K$.

Let $X_h \subset X$, $M_h \subset M$ be a pair of inf-sup compatible (see (2.25) below) mixed finite element spaces based on the triangulation $\mathcal{T}_h$, and $\Psi_h$ be a finite dimensional subspace of $H^1(\Omega)$. Let $Y_h$ be a finite dimensional subspace of $L^2(\Omega)^d$, and we assume that either $Y_h$ is conformal in

$$H^0_{\text{div}}(\Omega) = \left\{ v \in L^2(\Omega)^d, \quad \nabla \cdot v \in L^2(\Omega), \quad v \cdot n|_{\partial \Omega} = 0 \right\}$$

(2.19)

or $M_h$ is conformal in $H^1(\Omega)$. In particular, we assume $X_h \subseteq Y_h$ and $h(x) \equiv 1 \in \Psi_h$.

We assume that the finite element spaces satisfy the following approximation properties: There exists $l \geq 1$ and $c > 0$ such that for all $0 \leq r \leq l$

$$\inf_{v_h \in X_h} \{ ||v - v_h|| + h||v - v_h||_1 \} \leq ch^{r+1}||v||_{r+1}, \quad \forall v \in H^{r+1}(\Omega) \cap X,$$

$$\inf_{v_h \in X_h} ||v - v_h||_{1,p} \leq ch^r||v||_{r+1,p}, \quad 2 \leq p \leq \infty, \quad \forall v \in W^{r+1,p}(\Omega)^d \cap X.$$  

(2.20)

and

$$\inf_{\psi_h \in \Psi_h} \{ ||\psi - \psi_h|| + h||\psi - \psi_h||_1 \} \leq ch^{r+1}||\psi||_{r+1}, \quad \forall \psi \in H^{r+1}(\Omega) \cap H^1(\Omega),$$

$$\inf_{\psi_h \in \Psi_h} ||\psi - \psi_h||_{1,p} \leq ch^r||\psi||_{r+1,p}, \quad 2 \leq p \leq \infty, \quad \forall \psi \in W^{r+1,p}(\Omega) \cap H^1(\Omega).$$

(2.21)

There exists some constant $c > 0$ such that for all $0 \leq r \leq l$

$$\inf_{q_h \in M_h} ||q - q_h|| \leq ch^r||q||_r, \quad \forall q \in H^r(\Omega) \cap M.$$  

(2.22)

In addition, for the choice of $M_h \subset H^1(\Omega)$, we assume that there exists some constant $c > 0$, such that for all $0 \leq r \leq l$

$$\inf_{q_h \in M_h} (||q - q_h|| + h||q - q_h||_1) \leq ch^r||q||_r, \quad \forall q \in H^r(\Omega) \cap M,$$  

(2.23)

and the following inverse inequality holds

$$||v_h||_{m,p} \leq ch^{m-n+\frac{d}{p} - \frac{d}{q}} ||v_h||_{m,q}, \forall v_h \in X_h \text{ or } \Psi_h \quad 0 \leq m \leq n \leq 1, \quad 1 \leq q \leq p \leq \infty.$$  

(2.24)

In addition, we also assume the inverse inequality (2.24) holds for $Y_h$ if $Y_h \subset H^0_{\text{div}}$. One possible choice is the following [5]:

$$X_h = Y_h = \{ v_h \in [C^0(\overline{\Omega})]^d \cap [H^1_0(\Omega)]^d; v_h|_K \in [P_2(K)]^d \},$$

$$V_h = \{ v_h \in X_h; (\nabla \cdot v_h, q_h) = 0. \quad \forall q_h \in M_h \},$$

$$M_h = \{ q_h \in L^2_0(\Omega)\cap [P_0(K)]; q_h|_K \in P_0(K) \}, \quad \Psi_h = \{ \psi_h \in C^0(\overline{\Omega}); \psi_h|_K \in P_2(K) \}.$$  

Define the $L^2$-orthogonal projections: $P_{X_h} : L^2(\Omega)^d \rightarrow X_h$, $P_{\Psi_h} : L^2(\Omega) \rightarrow \Psi_h$ and $P_{M_h} : L^2(\Omega) \rightarrow M_h$ such that

$$(v - P_{X_h}v, v_h) = 0, \quad \forall v \in L^2(\Omega)^d, \quad v_h \in X_h,$$

$$(\psi - P_{\Psi_h}\psi, \psi_h) = 0, \quad \forall \psi \in L^2(\Omega), \quad \psi_h \in \Psi_h,$$

$$(q - P_{M_h}q, q_h) = 0, \quad \forall q \in L^2(\Omega), \quad q_h \in M_h.$$  

Define the discrete version of the divergence operator $B_h : X_h \rightarrow M_h$ and its transpose $B_h^T : M_h \rightarrow X_h$ so that for every couple $(v_h, q_h) \in X_h \times M_h$ there holds $B_h v_h, q_h = b(v_h, q_h)$.
and \((v_h, B_h^T q_h) = b(v_h, q_h)\). We also assume \(B_h\) is surjective, i.e. the following \(\inf\sup\) condition holds

\[
\exists \beta > 0, \quad \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{(B_h v_h, q_h)}{\|q_h\|_1 \|v_h\|} \geq \beta.
\] (2.25)

We introduce the following discrete divergence operator on \(Y_h\): Let \(C_h : Y_h \to M_h\) such that for every couple \((v_h, q_h) \in Y_h \times M_h\), either \((C_h v_h, q_h) = -(\nabla \cdot v_h, q_h)\) if \(Y_h \subset H^1_0\) or \((C_h v_h, q_h) = (v_h, \nabla q_h)\) if \(M_h \subset H^1(\Omega)\).

Let \(A_h : X_h \to X_h\) be the linear, continuous operator such that \((A_h u_h, v_h) = a(u_h, v_h)\) for all \((u_h, v_h) \in X_h \times X_h\). Define \(D_h : X_h \times X_h \to X'_h\) such that \((D_h(u_h, v_h), w_h) = d(u_h, v_h, w_h)\) for \((u_h, v_h, w_h) \in X_h \times X_h \times X_h\). Then it holds \(d(u_h, v_h, 0) = 0\).

Define the linear, continuous operator \(\tilde{A}_h : \Psi_h \to \Psi_h \cap L^2_0(\Omega)\) by

\[
\left(\tilde{A}_h \phi_h, \psi_h\right) = (\nabla \phi_h, \nabla \psi_h), \quad \forall \phi_h, \psi_h \in \Psi_h.
\] (2.26)

Let \(i_h : X_h \to Y_h\) be the continuous injection of \(X_h\) into \(Y_h\) and \(i_h^T\) be the transpose of \(i_h\), i.e. the \(L^2\) orthogonal projection onto \(X_h\).

**Proposition 2.1.** (cf. [11]) \(C_h\) is an extension of \(B_h\) and \(B_h = C_h i_h, i_h^T C_h^T = B_h^T\); we have the commutative diagrams as below.

\[
\begin{array}{ccc}
X_h & \xrightarrow{B_h} & M_h \\
\downarrow & & \downarrow \\
Y_h & \xrightarrow{C_h} & M_h \\
\end{array}
\]

\[
\begin{array}{ccc}
X_h & \xleftarrow{B_h^T} & M_h \\
\downarrow & & \downarrow \\
Y_h & \xleftarrow{C_h^T} & M_h \\
\end{array}
\]

Set \(H_h = \ker(B_h)\), we then have the \(L^2\) orthogonal decomposition of \(X_h\) as \(X_h = H_h \oplus B_h^T(\ker(B_h))\) [11]. Similarly, set \(\tilde{H}_h = \ker(C_h), Y_h = \tilde{H}_h \oplus C_h^T(\ker(B_h))\) [11].

3. FULL DISCRETIZATION AND ITS STABILITY

In this section, we will present a full finite element Galerkin approximation based on the stabilized semi-discrete schemes introduced in [24], and show that it is unconditionally stable. Let \(J_{\delta t} = \{t^n\}^N_{n=0}\) be a quasi-uniform partition of \([0, T]\) of mesh size \(\delta t := T/N\). A fully discrete finite element approximation based on the stabilized semi-discrete scheme is defined as: given suitable approximations \((\phi_h^0, w_h^0, u_h^0, p_h^0)\) of initial data \((\phi_0, w_0, u_0, p_0)\) \((w_0\) and \(p_0\) can be determined from \([1.2]\) and \([1.3]\), respectively), find \(\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1} \in \)
\[ \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h \text{ such that} \]

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\phi_h^{n+1} - \phi_h^n}{\delta t} + P_{\Psi_h} ( (u_{h}^{n+1} \cdot \nabla) \phi_h^n ) + \gamma \tilde{A}_hw_h^{n+1} = 0, \\
\frac{w_h^{n+1} - S}{\varepsilon^2} (\phi_h^{n+1} - \phi_h^n) = \tilde{A}_h\phi_h^{n+1} + P_{\Psi_h} f(\phi_h^n), \\
\rho_0 \frac{\delta t}{\delta t} (\tilde{u}_h^{n+1} - i_h u_h^n) + \rho_0 P_{x_h} (D_h (\tilde{u}_h^n , u_h^{n+1}) ) + \mu_0 A_h \tilde{u}_h^{n+1} \\
+ B_h^T p_h^n - \lambda P_{x_h} (w_h^{n+1} \nabla \phi_h^n) = 0, \\
\rho_0 (u_h^{n+1} - i_h \tilde{u}_h^{n+1}) + C_h (p_h^{n+1} - p_h^n) = 0, \\
C_h u_h^{n+1} = 0,
\end{array} \right. \\
(3.1)
\end{align*}
\]

where \( S \geq \frac{1}{2} \) is a stabilizing parameter. Initially, we set \( u_h^0 = \tilde{u}_h^0 \in X_h \subset Y_h \).

Before discussing the numerical procedure, we show below that the above scheme is unconditionally stable. For a sequence \( \{v^n\}_{n=0}^N \), we denote

\[
\delta_t v^n = v^{n+1} - v^n.
\]

**Theorem 3.1.** The fully discrete scheme (3.1)-(3.5) is unconditionally stable and satisfies the following discrete energy law:

\[
\begin{align*}
\left[ \frac{\rho_0}{2} \|u_h^{n+1}\|^2 + \frac{\lambda}{2} \|\nabla \phi_h^{n+1}\|^2 + \lambda (F(\phi_h^{n+1}),1) \right] \quad + \frac{\delta t^2}{2\rho_0} \|C_h^T p_h^{n+1}\|^2 + \mu_0 \delta t \|\nabla \tilde{u}_h^{n+1}\|^2 \\
+ \lambda \gamma \delta t \|\nabla w_h^{n+1}\|^2 \leq \left[ \frac{\rho_0}{2} \|u_h^n\|^2 + \frac{\lambda}{2} \|\nabla \phi_h^n\|^2 + \lambda (F(\phi_h^n),1) \right] \quad + \frac{\delta t^2}{2\rho_0} \|C_h^T p_h^n\|^2, \quad n \geq 0.
\end{align*}
\]

**(3.7)**

**Proof.** Testing (3.1) with \( \lambda \delta_t \tilde{w}_h^{n+1} \in \Psi_h \), (3.2) with \( -\lambda (\phi_h^{n+1} - \phi_h^n) \in \Psi_h \), (3.3) with \( \delta_t \tilde{u}_h^{n+1} \in X_h \), we obtain

\[
\begin{align*}
\lambda (\phi_h^{n+1} - \phi_h^n, w_h^{n+1}) + \lambda \delta_t ( (\tilde{u}_h^{n+1} \cdot \nabla) \phi_h^n , w_h^{n+1} ) + \lambda \gamma \delta t \|\nabla w_h^{n+1}\|^2 = 0, \\
- \lambda (\phi_h^{n+1} - \phi_h^n, w_h^{n+1}) + \frac{\lambda S}{\varepsilon^2} \|\phi_h^{n+1} - \phi_h^n\|^2 = \frac{\lambda}{2} ( \|\nabla \phi_h^{n+1}\|^2 + \|\nabla \delta_t \phi_h^n\|^2 ) \\
- \lambda (f(\phi_h^n), \phi_h^{n+1} - \phi_h^n), \\
\rho_0 \left( \|\tilde{u}_h^{n+1}\|^2 + \|\tilde{u}_h^{n+1} - i_h u_h^n\|^2 - \|i_h u_h^n\|^2 \right) + \mu_0 \delta t \|\nabla \tilde{u}_h^{n+1}\|^2 - \delta_t (B_h^T p_h^n , \tilde{u}_h^{n+1}) \\
- \lambda \delta t (w_h^{n+1} \nabla \phi_h^n , \tilde{u}_h^{n+1}) = 0.
\end{align*}
\]

Summing up the above identities, we find

\[
\begin{align*}
\frac{\lambda}{2} ( \|\nabla \phi_h^{n+1}\|^2 + \|\nabla \delta_t \phi_h^n\|^2 - \|\nabla \phi_h^n\|^2 ) \\
+ \lambda (f(\phi_h^n), \phi_h^{n+1} - \phi_h^n) + \frac{S\lambda}{\varepsilon^2} (\phi_h^{n+1} - \phi_h^n)^2 + \lambda \gamma \delta t \|\nabla w_h^{n+1}\|^2 \\
+ \frac{\rho_0}{2} ( \|\tilde{u}_h^{n+1}\|^2 + \|\tilde{u}_h^{n+1} - i_h u_h^n\|^2 - \|i_h u_h^n\|^2 ) \\
+ \mu_0 \delta t \|\nabla \tilde{u}_h^{n+1}\|^2 - \delta_t (B_h^T p_h^n , \tilde{u}_h^{n+1}) = 0.
\end{align*}
\]

**(3.8)**
Testing \((3.4)\) with \(C^T_h p^n_h \in Y_h,\) in view of \((3.5)\), we obtain

\[
(i_h \tilde{u}^{n+1}_h, C^T_h p^n_h) = (u^{n+1}_h, C^T_h p^n_h) - \frac{\delta t}{\rho_0} \left( C^T_h p^n_h, C^T_h (p^{n+1}_h - p^n_h) \right)
\]

\[
= (C_h u^{n+1}_h, p^n_h) - \frac{\delta t}{2\rho_0} \left( ||C^T_h p^n_h||^2 - ||C^T_h (p^{n+1}_h - p^n_h)||^2 \right)
\]

\[
= - \frac{\delta t}{2\rho_0} \left( ||C^T_h p^{n+1}_h||^2 - ||C^T_h p^n_h||^2 - ||C^T_h (p^{n+1}_h - p^n_h)||^2 \right). \quad (3.9)
\]

Using \(i^T_h C^T_h = B^T_h\) leads to

\[
(i_h \tilde{u}^{n+1}_h, C^T_h p^n_h) = (\tilde{u}^{n+1}_h, i^T_h C^T_h p^n_h) = (\tilde{u}^{n+1}_h, B^T_h p^n_h). \quad (3.10)
\]

Moreover, we rewrite \((3.4)\) as

\[i_h \tilde{u}^{n+1}_h = u^{n+1}_h + \frac{\delta t}{\rho_0} C^T_h (p^{n+1}_h - p^n_h), \quad (3.11)\]

with both sides belonging to \(Y_h \subseteq L^2;\) taking \(L^2\) norm of both sides and using \((3.5)\), we get

\[||i_h \tilde{u}^{n+1}_h||^2 = ||u^{n+1}_h||^2 + \frac{\delta t^2}{\rho_0^2} ||C^T_h (p^{n+1}_h - p^n_h)||^2. \quad (3.12)\]

Combining \((3.8), (3.9), (3.10)\) and \((3.12)\) together and noticing that \(||i_h \tilde{u}^{n+1}_h|| = ||\tilde{u}^{n+1}_h||,\) we find

\[
\frac{\lambda}{2} (||\nabla \phi^{n+1}_h||^2 + ||\nabla \phi^n_h||^2) + \lambda (f(\phi^n_h), \phi^{n+1}_h - \phi^n_h)
\]

\[
+ \frac{S\lambda}{\varepsilon^2} ||\phi^{n+1}_h - \phi^n_h||^2 + \lambda \gamma \delta t \||\nabla w^{n+1}_h||^2 + \frac{\rho_0}{2} \left( ||u^{n+1}_h||^2 + ||\tilde{u}^{n+1}_h - i^T_h u^n_h||^2 - ||i^T_h u^n_h||^2 \right)
\]

\[
+ \frac{\delta t^2}{2\rho_0} \left( ||C^T_h p^{n+1}_h||^2 - ||C^T_h p^n_h||^2 \right) + \mu_0 \delta t ||\nabla \tilde{u}^{n+1}_h||^2 = 0. \quad (3.13)
\]

In addition, Taylor expansion and \((2.10)\) imply that for \(S \geq 1/2,\)

\[
F(\phi^{n+1}_h) - F(\phi^n_h) = f(\phi^n_h)(\phi^{n+1}_h - \phi^n_h) + \frac{f'(\xi)}{2} ||\phi^{n+1}_h - \phi^n_h||^2
\]

\[
\leq f(\phi^n_h)(\phi^{n+1}_h - \phi^n_h) + \frac{S}{\varepsilon^2} ||\phi^{n+1}_h - \phi^n_h||^2.
\]

Hence, we draw the conclusion via substituting the above inequality into \((3.13)\) and noticing the fact that \(i^T_h : Y_h \rightarrow X_h\) is an \(L^2\) orthogonal projection. \qed

**Remark 3.1.** Based on the above stability analysis, if we rewrite \((3.1)-(3.3)\) as a coupled linear system for the unknown \((w^{n+1}_h, \phi^{n+1}_h - \phi^n_h, \tilde{u}^{n+1}_h)^T,\) the matrix of the linear system is then positive definite (but not symmetric). Therefore, the coupled system \((3.1)-(3.3)\) can be solved efficiently by an iterative method such as BICGSTAB (cf. [22] and references therein).

For the projection step \((3.4)-(3.5),\) the following equation is solved in practice by applying \(C_h\) to \((3.4),\)

\[C_h C^T_h (p^{n+1}_h - p^n_h) = \frac{\rho_0}{\delta t} B_h \tilde{u}^{n+1}_h. \quad (3.14)\]

Noticing the inf-sup condition \((2.25)\), the coefficient matrix of \((3.14)\) is positive definite and \((3.14)\) admits a unique solution.

From Remark 3.1, the practical procedure for implementing scheme \((3.1)-(3.5)\) can be describes as:
ERROR ESTIMATES FOR FULL DISCRETIZATIONS OF PHASE-FIELD MODELS

4. Error estimates

4.1. Preparations. Let \((\phi(t), w(t), u(t), p(t))\) be the exact solution. As usual, we will compare the numerical solution with the interpolates defined below. Let \((u_h(t), p_h(t)) \in X_h \times M_h\) be the solution of the following discrete Stokes problem

\[
\begin{align*}
\mu_0 (\nabla u_h(t), \nabla v_h) + (B_h^T p_h(t), v_h) &= \mu_0 (\nabla u(t), \nabla v_h) - (p(t), \nabla \cdot v_h), \quad \forall v_h \in X_h, \\
(B_h u_h(t), r_h) &= - (\nabla \cdot u(t), r_h), \quad \forall r_h \in M_h,
\end{align*}
\]

and define \((\phi_h(t), w_h(t)) \in \Psi_h \times \Psi_h\) as the solution of the following discrete elliptic problem

\[
\begin{align*}
(\nabla \phi_h(t), \nabla \psi_h) &= (\nabla \phi(t), \nabla \psi_h), \quad \forall \psi_h \in \Psi_h, \\
(\phi_h(t), 1) &= (\psi(t), 1), \\
(\nabla w_h(t), \nabla \varphi_h) &= (\nabla w(t), \nabla \varphi_h), \quad \forall \varphi_h \in \Psi_h, \\
(w_h(t), 1) &= (w(t), 1).
\end{align*}
\]

Using the \(H^2\)-regularity of the Stokes operator in regular domains with classical duality argument, we have the following results concerning interpolates \((u_h(t), p_h(t))\):

**Lemma 4.1.** Under the assumptions \(u(t) \in L^s([0, T]; H^1(\Omega) \cap M)\) for \(1 \leq s \leq \infty\), there exists \(c > 0\) such that

\[
\|u - u_h\|_{L^s([0, T]; L^2(\Omega))} + h \left[\|u - u_h\|_{L^s([0, T]; H^1(\Omega))} + \|p - p_h\|_{L^s([0, T]; L^2(\Omega))}\right] \\
\leq c h^{l+1} \left[\|u\|_{L^s([0, T]; H^l+1(\Omega))} + \|p\|_{L^s([0, T]; H^l(\Omega))}\right].
\]

**Lemma 4.2.** Under the assumptions \(u(t) \in L^s([0, T]; H^2(\Omega) \cap V), p(t) \in L^s([0, T]; H^1(\Omega) \cap M)\) for \(1 \leq s \leq \infty\), there exists \(c > 0\) such that

\[
\|u_h\|_{L^s([0, T]; W^{0,\infty}(\Omega) \cap W^{1,3}(\Omega))} \leq c \left(\|u\|_{L^s([0, T]; H^2(\Omega))} + \|p\|_{L^s([0, T]; H^1(\Omega))}\right).
\]

**Lemma 4.3.** Under the assumptions \(u(t) \in L^s([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap M\) for \(1 \leq s \leq \infty\), there exists \(c > 0\) such that

\[
\|C_h^T p_h\|_{L^s([0, T]; L^2(\Omega))} \leq c \left(\|u\|_{L^s([0, T]; H^2(\Omega))} + \|p\|_{L^s([0, T]; H^1(\Omega))}\right).
\]

For \((\phi_h(t), w_h(t)) \in \Psi_h \times \Psi_h\), we have
Lemma 4.4. Under the assumptions \( \phi(t), w(t) \in L^s([0, T]; H^{l+1}(\Omega)) \), for \( 1 \leq s \leq \infty \), there exists \( c > 0 \) such that

\[
\|\phi - \phi_h\|_{L^s([0, T]; L^2(\Omega))} + h\|\phi - \phi_h\|_{L^s([0, T]; H^1(\Omega))} \leq c h^{l+1} \|\phi\|_{L^s([0, T]; H^{l+1}(\Omega))},
\]

\[
\|w - w_h\|_{L^s([0, T]; L^2(\Omega))} + h\|w - w_h\|_{L^s([0, T]; H^1(\Omega))} \leq c h^{l+1} \|w\|_{L^s([0, T]; H^{l+1}(\Omega))}.
\]

Lemma 4.5. Under the assumptions \( \phi(t), w(t) \in L^s([0, T]; H^2(\Omega)) \), for \( 1 \leq s \leq \infty \), there exists \( c > 0 \) such that

\[
\|\phi_h\|_{L^s([0, T]; W^{0, \infty}(\Omega) \cap W^{1, 3}(\Omega))} \leq c \|\phi\|_{L^s([0, T]; H^2(\Omega))},
\]

\[
\|w_h\|_{L^s([0, T]; W^{0, \infty}(\Omega) \cap W^{1, 3}(\Omega))} \leq c \|w\|_{L^s([0, T]; H^2(\Omega))}.
\]

Discrete inverse Laplace operator with Neumann boundary condition. Define the discrete inverse Laplace operator \( L_h : H^{-1}(\Omega) \to \Psi_h \). For all \( v \in H^{-1}(\Omega) \), \( L_h(v) \in \Psi_h \cap L_0^2(\Omega) \) is the solution to the following problem

\[
(\nabla L_h(v), \nabla \varphi_h) = (v - v_0, \varphi_h), \quad \forall \varphi_h \in X_h,
\]

where \( v_0 = \frac{1}{|\Omega|} (v, 1) \).

\( L_h \) can be viewed as an elliptic projection of the usual inverse Laplace operator \( L : H^{-1}(\Omega) \to H^1(\Omega) \cap L_0^2(\Omega) \), where

\[
(\nabla L(\psi), \nabla \varphi) = (\psi - \psi_0, \varphi), \quad \forall \varphi \in H^1(\Omega),
\]

with \( \psi_0 = \frac{1}{|\Omega|} (\psi, 1) \).

In particular, using the \( H^2 \) regularity, we have

\[
\|L(v) - L_h(v)\|_{L^2(\Omega)} + h\|L(v) - L_h(v)\|_{H^1(\Omega)} \leq c_1 h^2 \|L(v)\|_{H^2(\Omega)} \leq c_2 h^2 \|v\|_{L^2(\Omega)}.
\]

Discrete inverse Stokes operator. Define the discrete inverse Stokes operator \( S_h : H^{-1}(\Omega)^d \to X_h \). For all \( v \in H^{-1}(\Omega)^d \), \( (S_h(v), r_h) \in X_h \times M_h \) is the solution to the following problem

\[
\begin{cases}
(A_h(S_h(v), v_h) + (B_h^T r_h, v_h) = (v, v_h), \quad \forall v_h \in X_h, \\
(B_h S_h(v), w_h) = 0. \quad \forall w_h \in M_h.
\end{cases}
\]

\( (S_h(v), r_h) \) can be viewed as an interpolate of \( (S(v), r) \) similar to (4.1).

In the sequel, the following inequalities, which are valid for \( d = 2, 3 \), will be frequently used:

\[
|((u \cdot \nabla) v, w)| \leq c \|v\|_{L^2} \|u\|_{H^1} \|w\|_{L^2}^{1/2} \|w\|_{H^1}^{1/2}, \quad \forall u, v, w \in H^1(\Omega)^d,
\]

\[
|((u \cdot \nabla) v, w)| \leq \|u\|_{L^2} \|v\|_{L^2} \|w\|_{H^1}, \quad \forall u, v, w \in H^1(\Omega)^d,
\]

\[
|d(u, v, w)| \leq c \|v\|_{L^\infty} + \|v\|_{W^{1, 3}} \|w\|_{L^2} \|w\|_{H^1}, \quad \forall v \in H^2(\Omega)^d, u \in H^1(\Omega)^d, w \in L^2(\Omega)^d.
\]

We note that (4.12) and (4.13) hold for \( u \in H^1(\Omega)^d \) with \( v, w \in H^1(\Omega) \). Thanks to integration by parts, we also have

\[
|d(u, v, w)| \leq c (\|u\|_{L^\infty} + \|u\|_{W^{1, 3}}) \|v\|_{L^2} \|w\|_{H^1}, \quad \forall u \in H^2(\Omega)^d \cap H^1_0(\Omega)^d, v, w \in H^1(\Omega)^d,
\]

and

\[
|((u \cdot \nabla) v, w)| \leq c (\|u\|_{L^\infty} + \|u\|_{W^{1, 3}}) \|v\|_{L^2} \|w\|_{H^1}, \quad \forall u \in H^2(\Omega)^d \cap H^1_0(\Omega)^d, v, w \in H^1(\Omega).
\]

\]

\[
|((u \cdot \nabla) v, w)| \leq c (\|u\|_{L^\infty} + \|u\|_{W^{1, 3}}) \|v\|_{L^2} \|w\|_{H^1}, \quad \forall u \in H^2(\Omega)^d \cap H^1_0(\Omega)^d, v, w \in H^1(\Omega).
\]
Obviously, we have
\[ \|u\|_{L^\infty} + \|u\|_{W^{1,3}} \leq c\|u\|_{H^2}. \] (4.17)

4.2. Error analysis. Let \((\phi_n^0, w_h^n, u_h^n, \tilde{u}_h^n, p_h^n) \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h\) be the numerical solutions and intermediate velocity obtained from scheme (3.1)-(3.5). We define the error functions for \(n = 0, 1, 2, \ldots, N\) as
\[ e_\phi^n = \phi_h^n(t^n) - \phi_h^n, \quad e_w^n = w_h^n(t^n) - w_h^n, \quad e_u^n = u_h^n(t^n) - u_h^n, \]
\[ e_{\tilde{u}}^n = u_h^n(t^n) - \tilde{u}_h^n, \quad e_p^n = p_h^n(t^n) - p_h^n, \] (4.18)
and we denote by \(e_{u,\delta t}, e_{w,\delta t}, e_{u,\delta t}, e_{\phi,\delta t}, e_{p,\delta t}\) the corresponding sequence of error functions.

**Assumption A.** We assume that the solution \((\phi(t), w(t), u(t), p(t))\) of the Cahn-Hilliard phase field model (1.1)-(1.5) is sufficiently smooth such that for some \(l \geq 1:\)
\[ \phi, w, \partial_t \phi \in L^\infty([0,T]; H^{l+1}), \quad u, \partial_t u \in L^\infty([0,T]; (H^{l+1})^d), \quad p, \partial_t p \in L^\infty([0,T]; H^l), \]
\[ \partial_{tt} \phi \in L^\infty([0,T]; H^l), \quad \partial_{tt} u \in L^\infty([0,T]; (H^l)^d), \quad \partial_{tt} p \in L^\infty([0,T]; L^2). \]

In addition, we assume the scheme is initialized such that
\[ \|e_\phi^0\| + \delta t^{1/2}\|\nabla e_\phi^0\| + \|e_w^0\| + \|\nabla e_w^0\| + \delta t\|C_h e_p^0\| \lesssim \delta t + h^l, \] (4.19)
e.g. choose \(u_h^0 = \tilde{u}_h^0 = P_{X_h} u_0, \phi_h^0 = P_{\Psi_h} \phi_0, p_h^0 = 0\). Then, the following error estimates hold:

**Theorem 4.1.** Under Assumption A, for sufficiently small \(\delta t\), the finite element approximate solution to the scheme (3.1)-(3.5) satisfies
\[ \|\phi - \phi_h,\delta t\|_{L^\infty(L^2(\Omega))} + \|u - u_h,\delta t\|_{L^\infty(L^2(\Omega)^d)} + \|u - \tilde{u}_h,\delta t\|_{L^\infty(L^2(\Omega)^d)} \lesssim \delta t + h^l, \]
\[ \|\phi - \phi_h,\delta t\|_{L^\infty(H^l(\Omega))} + \|u - \tilde{u}_h,\delta t\|_{H^l(\Omega)^d} + \|w - w_h,\delta t\|_{H^l(\Omega)^d} \lesssim \delta t + h^l, \]
\[ \|p - p_h,\delta t\|_{L^2(\Omega)} \lesssim (1 + h/\sqrt{\delta t})(\delta t + h^l). \]

If \(X_h = Y_h\), the error on pressure becomes
\[ \|p - p_h,\delta t\|_{L^2(\Omega)} \lesssim \delta t + h^l. \]

The proof of the above results will require a sequence of intermediate results that we establish below.

By definition, we have the following equations for the interpolates \((u_h(t), p_h(t), \phi_h(t), w_h(t)):\)
\[
\begin{aligned}
\phi_h(t^{n+1}) - \phi_h(t^n) &= \gamma \hat{A}_h w_h(t^{n+1}) = P_{\Psi_h} \left[ \tilde{R}_{\phi}^{n+1} - (u(t^n) \cdot \nabla) \phi(t^n) \right], \\
\frac{w_h(t^{n+1}) - \frac{S}{\varepsilon} (\phi_h(t^{n+1}) - \phi_h(t^n))}{\delta t} &= \hat{A}_h \phi_h(t^{n+1}) + P_{\Psi_h} \left[ \tilde{R}_w^{n+1} + f(\phi(t^n)) \right], \\
\frac{\rho_0}{\delta t} (u_h(t^{n+1}) - u_h(t^n)) + \mu_0 \hat{A}_h u_h(t^{n+1}) + B_h^T p_h(t^{n+1}) &= P_{X_h} \left[ \tilde{R}_u^{n+1} - \rho_0 D(u(t^n), u(t^{n+1})) + \lambda w(t^{n+1}) \nabla \phi(t^n) \right], \\
B_h u_h(t^{n+1}) &= 0,
\end{aligned}
\] (4.20)
(4.21)
(4.22)
(4.23)
where

\[
\tilde{R}^{n+1}_{\phi} = \frac{\delta_t \phi_h(t^n)}{\delta t} - \partial_t \phi(t^{n+1}) - (u(t^{n+1}) \cdot \nabla)\phi(t^{n+1}) + (u(t^{n+1}) \cdot \nabla)\phi(t^n),
\]

\[
\tilde{R}^{n+1}_w = - \frac{S}{\varepsilon^2} (\phi_h(t^{n+1}) - \phi_h(t^n)) + f(\phi(t^{n+1})) - f(\phi(t^n)) + w(t^{n+1}) - w_h(t^{n+1}),
\]

\[
\tilde{R}^{n+1}_{u} = \rho_0 \frac{\delta_t u_h(t^n)}{\delta t} - \rho_0 \partial_t u(t^{n+1}) - \rho_0 (u(t^{n+1}) \cdot \nabla) u(t^{n+1}) + \lambda w(t^{n+1}) \nabla \phi(t^{n+1})
\]

\[
+ \rho_0 D(u(t^n), u(t^{n+1})) - \lambda w(t^{n+1}) \nabla \phi(t^n).
\]

Using the properties for the interpolates defined in (4.1) and (4.2), we have the following results.

**Lemma 4.6.** Under Assumption A, we have

\[
\|\tilde{R}^{n+1}_{\phi}\| + \|\tilde{R}^{n+1}_{u}\| + \|\tilde{R}^{n+1}_w\| \lesssim \delta t + h^{l+1}, \quad \|\tilde{R}^{n+1}_{w}\|_1 \lesssim \delta t + h^l, \quad \forall 0 \leq n \leq \frac{T}{\delta t} - 1. \quad (4.25)
\]

**Proof.** We will only prove the bound for \(\|\tilde{R}^{n+1}_{\phi}\|\), and omit the others as the arguments are similar. Noticing that

\[
\tilde{R}^{n+1}_{\phi} = \partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1}) - (u(t^{n+1}) \cdot \nabla) (\phi(t^{n+1}) - \phi(t^n))
\]

\[
- \delta t \int_0^1 \int_0^\theta \partial_{tt} \phi_h(t_{n+1} - s\delta t) \, ds \, d\theta,
\]

applying the properties of interpolates as well as the regularity assumption, we have

\[
\|\tilde{R}^{n+1}_{\phi}\| \lesssim \|\partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1})\| + \delta t \|u(t^{n+1})\|_{L^\infty} \|\partial_t \phi\|_{L^\infty(0,T;H^1)} + \delta t \|\partial_{tt} \phi\|_{L^\infty(0,T;H^1)}
\]

\[
\lesssim h^{l+1} \|\partial_t \phi(t^{n+1})\|_{L^\infty(H^{l+1})} + \delta t \lesssim \delta t + h^{l+1},
\]

where we have differentiated (4.2) in time \(t\) once and used approximation property in Lemma 4.4 to control \(\|\partial_t \phi_h(t^{n+1}) - \partial_t \phi(t^{n+1})\|\). \(\Box\)

Subtracting (3.1)-(3.3) from (4.20)-(4.22) respectively, we obtain the following error equations governing the growth of \(\tilde{e}^{n+1}_u, \tilde{e}^{n+1}_\phi, \) and \(e^{n+1}_w:

\[
\left\{ \begin{array}{l}
\frac{e^{n+1}_\phi - e^n_\phi}{\delta t} + \gamma A_h e^{n}_w = P_{\psi_h} \left[ \tilde{R}^{n+1}_{\phi} + R^{n+1}_{\phi} \right], \\
\frac{e^{n+1}_w - S}{\varepsilon^2} (e^{n+1}_\phi - e^n_\phi) = A_h e^{n+1}_w + P_{\psi_h} \left[ \tilde{R}^{n+1}_w + R^{n+1}_w \right], \\
\frac{\rho_0}{\delta t} \left( \tilde{e}^{n+1}_p - i_s \tilde{e}^{n}_u \right) + \mu_0 A_h e^{n+1}_u + \beta_T e^{n}_p = P_{\chi_h} \left[ \tilde{R}^{n+1}_{u} + R^{n+1}_{u,w} + R^{n+1}_{u,\phi} \right],
\end{array} \right. \quad (4.26, 4.27, 4.28)
\]

where

\[
P^{n+1}_{\phi} = - (u(t^{n+1}) \cdot \nabla)\phi(t^n) + (\tilde{u}^{n+1}_h \cdot \nabla)\phi^h,
\]

\[
P^{n+1}_w = f(\phi(t^n)) - f(\phi^n_h) = (\phi(t^n) - \phi^n_h) \int_0^1 f'(s\phi(t^n)) + (1 - s)\phi^n_h) \, ds,
\]

\[
\tilde{e}^{n}_p = p_h(t^{n+1}) - p^n_h = \delta_t p_h(t^n) + e^n_p,
\]

\[
P^{n+1}_{u,u} = -\rho_0 D(u(t^n), u(t^{n+1})) + \rho_0 D_h(u^h, u^{n+1}_h),
\]

\[
P^{n+1}_{u,\phi} = \lambda w(t^{n+1}) \nabla \phi(t^n) - \lambda w^{n+1}_h \nabla \phi^n_h.
\]

\[
(4.29)
\]
In addition, since \( u_h(t^{n+1}) \in X_h, B_h u_h(t^{n+1}) = 0 \) and \( C_h \) is an extension of \( B_h \), we have the error equation

\[
\begin{cases}
\frac{\partial}{\partial t} (e_u^{n+1} - i_h e_u^{n+1}) + C_h^T (e_p^{n+1} - \tilde{e}_p) = 0, \\
C_h e_u^{n+1} = 0.
\end{cases}
\] (4.30)

It is convenient to write \( R_{\phi}^{n+1} \) and \( R_{u,\phi}^{n+1} \) as

\[
R_{\phi}^{n+1} = - ((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla) \phi(t^n) - (\tilde{e}_u^{n+1} \cdot \nabla) \phi_h^n - (u_h(t^{n+1}) \cdot \nabla) e_u^n \\
- (u_h(t^{n+1}) \cdot \nabla)(\phi(t^n) - \phi_h(t^n)),
\]

\[
R_{u,\phi}^{n+1} = \lambda(w(t^{n+1}) - w_h(t^{n+1})) \nabla \phi(t^n) + \lambda \epsilon_w^{n+1} \nabla \phi_h^n + \lambda w_h(t^{n+1}) \nabla e_u^n \\
+ \lambda w_h(t^{n+1}) \nabla (\phi(t^n) - \phi_h(t^n)).
\] (4.31, 4.32)

Taking the inner product of (4.26) with \( \lambda e_{\phi}^{n+1} \in \Psi_h, \lambda \epsilon_{w}^{n+1} \in \Psi_h, L_h (\epsilon_{\phi}^{n+1} - \epsilon_{\phi}^{n}) \in \Psi_h \cap L^2_0(\Omega) \) and \( 1 \in \Psi_h \), respectively, we obtain

\[
\frac{\lambda}{2 \delta t} \left( \| e_{\phi}^{n+1} \|^2 + \| e_{\phi}^{n+1} - e_{\phi}^{n} \|^2 - \| e_{\phi}^{n} \|^2 \right) + \lambda \gamma (\nabla \epsilon_w^{n+1}, \nabla \epsilon_{\phi}^{n+1}) = \lambda (\tilde{R}_{\phi}^{n+1} + R_{\phi}^{n+1}, e_{\phi}^{n+1}),
\] (4.33)

\[
\frac{\lambda}{\delta t} \left( e_{\phi}^{n+1} - e_{\phi}^{n}, \epsilon_w^{n+1} \right) + \lambda \gamma (\nabla \epsilon_w^{n+1}, \nabla \epsilon_{\phi}^{n+1}) = \lambda (\tilde{R}_{\phi}^{n+1} + R_{\phi}^{n+1}, e_{\phi}^{n+1}),
\] (4.34)

\[
\frac{\| \nabla L_h (\delta_t e_{\phi}^{n}) \|^2}{\delta t} + \gamma (e_{\phi}^{n+1} - e_{\phi}^{n}, 1) = \gamma (\tilde{R}_{\phi}^{n+1} + R_{\phi}^{n+1}, L_h (\delta_t e_{\phi}^{n})),
\] (4.35)

\[
(\delta_t e_{\phi}^{n}, 1) = \int_\Omega (e_{\phi}^{n+1} - e_{\phi}^{n}) = \delta t \int_\Omega (\tilde{R}_{\phi}^{n+1} + R_{\phi}^{n+1}).
\] (4.36)

Testing (4.27) with \( \epsilon_w^{n+1} \in \Psi_h \) and \( e_{\phi}^{n+1} - e_{\phi}^{n} \in \Psi_h \), respectively, we find

\[
\| \epsilon_w^{n+1} \|^2 - \frac{S}{\varepsilon^2} (e_{\phi}^{n+1} - e_{\phi}^{n}, \epsilon_w^{n+1}) = (\nabla \epsilon_w^{n+1}, \nabla \epsilon_{\phi}^{n+1}) + (R_{w}^{n+1} + \tilde{R}_{w}^{n+1}, \epsilon_{w}^{n+1}),
\] (4.37)

\[
\frac{1}{2} (\| \nabla \epsilon_{\phi}^{n+1} \|^2 + \| \nabla \delta_t e_{\phi}^{n} \|^2 - \| \nabla \epsilon_{\phi}^{n} \|^2) = (e_{\phi}^{n+1}, \delta_t e_{\phi}^{n}) - \frac{S}{\varepsilon^2} \| \delta_t e_{\phi}^{n} \|^2 - (R_{w}^{n+1} + \tilde{R}_{w}^{n+1}, \delta_t e_{\phi}^{n}).
\] (4.38)

Then taking the inner product of (4.28) with \( \delta t \tilde{e}_u^{n+1} \in X_h \), we get

\[
\frac{\rho_0}{2} (\| \tilde{e}_u^{n+1} \|^2 - \| i_h \tilde{e}_u^{n} \|^2 + \| \tilde{e}_u^{n+1} - i_h \tilde{e}_u^{n} \|^2) + \mu_0 \delta t \| \nabla \tilde{e}_u^{n+1} \|^2 + \delta t (B_h^T \tilde{e}_p, \tilde{e}_u^{n+1})
\]

\[= \delta t (\tilde{R}_{u}^{n+1} + R_{u,\phi}^{n+1} + R_{u,u}^{n+1}, e_u^{n+1}).
\] (4.39)
Combining $\delta t \cdot (4.33) + \delta t \cdot (4.34) + (4.35) + \delta t \lambda \gamma \cdot (4.37) + \left( \lambda + \gamma \right) \cdot (4.38) + (4.39)$, we arrive at

$$
\frac{\lambda}{2} (\|e^{n+1}_\varphi\|^2 - \|e^n_\varphi\|^2) + \frac{\rho_0}{2} (\|\tilde{e}^{n+1}_u\|^2 - \|\tilde{e}^n_u\|^2) + \frac{\rho_0}{2} (\|\tilde{e}^{n+1}_e\|^2 - \|\tilde{e}^n_e\|^2)
+ \lambda \left( \|\nabla e^{n+1}_\varphi\|^2 - \|\nabla e^n_\varphi\|^2 \right) + \frac{1}{\delta t} \|\nabla L_h(\delta e^n_\varphi)\|^2 + \frac{S(\lambda + \gamma)}{\varepsilon^2} \|\delta e^n_\varphi\|^2
+ \mu_0 \delta t \|\nabla \tilde{e}^{n+1}_u\|^2 + \lambda \delta t \|e^{n+1}_w\|^2 + \lambda \gamma \delta t \|\nabla e^{n+1}_u\|^2 + \delta t (B_h^T \tilde{e}^{n}_e, \tilde{e}^{n+1}_u)
= \frac{S \delta t \lambda \gamma}{\varepsilon^2} (\delta e^n_\varphi, e^{n+1}_u) + \lambda \delta t (R^{n+1}_\varphi, e^{n+1}_u) + \delta t (R^{n+1}_u, e^{n+1}_u)
+ \delta t \lambda \gamma (\tilde{R}^{n+1}_w, e^{n+1}_w) + \delta t \gamma (R^{n+1}_w, e^{n+1}_w) + \lambda \delta t (\tilde{R}^{n+1}_\varphi, e^{n+1}_\varphi) + \delta t (R^{n+1}_u, e^{n+1}_\varphi)
+ \lambda \delta t (\tilde{R}^{n+1}_\varphi, e^{n+1}_\varphi) + \lambda \delta t (R^{n+1}_w, e^{n+1}_\varphi) + \gamma (\delta e^n_\varphi, 1)(e^{n+1}_w, 1)
+ (\tilde{R}^{n+1}_\varphi + R^{n+1}_\varphi, L_h(\delta e^n_\varphi)) - (\lambda + \gamma)(\tilde{R}^{n+1}_w, \delta e^n_\varphi) - (\lambda + \gamma)(R^{n+1}_w, \delta e^n_\varphi).
$$

(4.40)

In order to control the right hand side of (4.40), we establish below bounds for those non-trivial terms. As the first step, we provide the estimates for terms involving $R^{n+1}_\varphi$ including $(R^{n+1}_\varphi, e^{n+1}_w)$, $(R^{n+1}_w, e^{n+1}_\varphi)$ and $(R^{n+1}_w, L_h(\delta e^n_\varphi))$ as:

Lemma 4.7. Under Assumption $A$, for any $\varphi \in \Psi_h$ and $\kappa, \kappa' > 0$, there holds

$$(R^{n+1}_\varphi, \varphi_h) \leq c_{\kappa, \kappa'} (h^{2(l+1)} + \|e^n_\varphi\|^2 + \|\varphi_h\|^2) + \kappa \|\nabla \varphi_h\|^2 + \kappa' \|\nabla \tilde{e}^{n+1}_u\|^2,$$

(4.41)

and

$$(R^{n+1}_\varphi, \varphi_h) \leq c_{\kappa, \kappa'} (h^{2(l+1)} + \|e^n_\varphi\|^2 + \|\tilde{e}^{n+1}_u\|^2) + \kappa \|\varphi_h\|^2 + \kappa' \|\nabla \tilde{e}^{n+1}_u\|^2.$$

(4.42)

In particular, we have

$$|(R^{n+1}_\varphi, 1)| \leq c (h^{l+1} + \|\tilde{e}^{n+1}_u\| + \|\tilde{e}^n_\varphi\|).$$

(4.43)

Proof. For any $\varphi \in \Psi_h$, in view of (4.31), we have

$$(R^{n+1}_\varphi, \varphi_h) = - ((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla) \phi(t^n), \varphi_h) - ((\tilde{e}^{n+1}_u \cdot \nabla) \tilde{e}^n_\varphi, \varphi_h)
- ((u_h(t^{n+1}) \cdot \nabla) (\phi(t^n) - \phi_h(t^n)), \varphi_h) - ((u_h(t^{n+1}) \cdot \nabla) e^{n}_\varphi, \varphi_h).$$

We estimate each term on the RHS as follows:

$$- ((u(t^{n+1}) - u_h(t^{n+1})) \cdot \nabla) \phi(t^n), \varphi_h) \leq c \|u(t^{n+1}) - u_h(t^{n+1})\| \|\phi(t^n)\|_{W^{1,3}} \|\varphi_h\|_1
\leq c h^{l+1} (\|u\|_{L^\infty(H^{l+1})} + \|p\|_{L^\infty(H^l)}) \|\varphi_h\|_1
\leq \frac{c}{\kappa} h^{2(l+1)} + \frac{\kappa}{3} \|\varphi_h\|^2_1.$$

Using the $a$ priori bound for $\|\nabla \phi^n_h\|$ in Theorem 3.1, we derive

$$- ((\tilde{e}^{n+1}_u \cdot \nabla) \tilde{e}^n_\varphi, \varphi_h) \leq c \|\nabla \phi^n_h\| \|\tilde{e}^{n+1}_u\|_1 \|\varphi_h\|^{1/2} \|\varphi_h\|^{1/2}
\leq c_{\kappa, \kappa'} \|\varphi_h\|^2 + \frac{\kappa}{3} \|\nabla \varphi_h\|^2 + \kappa' \|\nabla \tilde{e}^{n+1}_u\|^2;$$
Under Assumption A, we have

$$- ((u_h(t^{n+1}) \cdot \nabla)(\phi(t^n) - \phi_h(t^n)), \varphi_h)$$

and

$$- ((u_h(t^{n+1}) \cdot \nabla)e_{\phi}^n, \varphi_h) \leq \|u_h(t^{n+1})\|_L^\infty \|\nabla e_{\phi}^n\|_1 \leq c \|\nabla e_{\phi}^n\|^2 + \|\varphi_h\|^2.$$

Combining the above inequalities, we obtain (4.41). The other desired results in Lemma 4.7 can be derived in the same manner.

**Lemma 4.8.** Under Assumption A, we have

$$|(R_{u,u}^{n+1}, e_{u}^{n+1})| \leq c (h^{2(l+1)} + \|e_{u}^{n+1}\|^2) + \frac{\mu_0}{8} \|\nabla e_{u}^{n+1}\|^2 + \frac{\mu_0}{16} \|\nabla e_{u}^{n+1}\|^2. \quad (4.44)$$

**Proof.** Using the skew-symmetric property of $d(u^n, \cdot, \cdot)$, we obtain

$$(R_{u,u}^{n+1}, e_{u}^{n+1}) = - \rho_0 [d(u(t^n), u(t^{n+1}), e_{u}^{n+1}) - d(u_h(t^n), u_h(t^{n+1}), e_{u}^{n+1})]$$

and

$$= - \rho_0 d(u(t^n) - u_h(t^n), u(t^{n+1}) - u_h(t^{n+1}), e_{u}^{n+1}) - \rho_0 d(u(t^n), u(t^{n+1}) - u_h(t^{n+1}), e_{u}^{n+1}).$$

Using Cauchy-Schwarz inequality, Sobolev inequalities, $\|e_{u}^{n+1}\|_1 \leq c \|\nabla e_{u}^{n+1}\|$ and properties of the interpolates, we estimate each term on the RHS as follows:

$$|\rho_0 d(u(t^n) - u_h(t^n), u(t^{n+1}) - u_h(t^{n+1}), e_{u}^{n+1})| \leq c \|u(t^{n+1})\|_2 \|u(t^n) - u_h(t^n)\|_1$$

$$\leq c \|u(t^{n+1})\|_{L^\infty(H^{l+1})} + \|p\| \|u(t^{n+1})\|_{L^\infty(H^l)} \|e_{u}^{n+1}\|_1$$

$$\leq c \|e_{u}^{n+1}\|^2 + \frac{\mu_0}{16} \|\nabla e_{u}^{n+1}\|^2;$$

and

$$|\rho_0 d(e_{u}^{n}, u_h(t^{n+1}), e_{u}^{n+1})| \leq c \|u_h(t^{n+1})\|_{L^\infty} + \|u_h(t^{n+1})\|_{W^{1,3}} \|e_{u}^{n+1}\|_1$$

$$\leq c \|u_h(t^{n+1})\|_{L^\infty(H^l)} + \|p\| \|L^\infty(H^l)\| \|e_{u}^{n+1}\|_1$$

$$\leq c \|e_{u}^{n+1}\|^2 + \frac{\mu_0}{16} \|\nabla e_{u}^{n+1}\|^2;$$

Then the conclusion follows. □

**Lemma 4.9.** Under Assumption A, we have

$$(R_{u,\phi}^{n+1}, e_{u}^{n+1}) \leq c (h^{2(l+1)} + \|e_{u}^{n+1}\|^2 + \|\nabla e_{\phi}^n\|^2) + \frac{\mu_0}{4} \|\nabla e_{u}^{n+1}\|^2 + \frac{\lambda \gamma}{8} (\|e_{w}^{n+1}\|^2 + \|\nabla e_{w}^{n+1}\|^2).$$
Proof. Noticing (4.32), we have
\[
(R^{n+1}_u, e^{n+1}_u) = \lambda \left( (w(t^{n+1}) - w_h(t^{n+1})) \nabla \phi(t^n), e^{n+1}_u \right) + \lambda (e^{n+1}_w \nabla \phi_h, e^{n+1}_u) \\
+ \lambda (w_h(t^{n+1}) \nabla (\phi(t^n) - \phi_h(t^n)), e^{n+1}_u) + \lambda (w_h(t^{n+1}) \nabla e^n_\phi, e^{n+1}_u).
\]
Using \(\|e^{n+1}_u\|_1 \leq c\|\nabla e^{n+1}_u\|\), we bound each term on the RHS as:
\[
\lambda((w(t^{n+1}) - w_h(t^{n+1})) \nabla \phi(t^n), e^{n+1}_u) \leq \|w(t^{n+1}) - w_h(t^{n+1})\| \|\nabla \phi(t^n)\| L^1 \|e^{n+1}_u\|_1 \\
\leq ch^{l+1} \|w\|_{L^\infty(H^{l+1})} \|\phi(t^n)\|_{H^2} \|e^{n+1}_u\|_1 \\
\leq ch^{2(l+1)} + \frac{\mu_0}{16} \|\nabla e^{n+1}_u\|^2.
\]
Recalling the stability of the scheme, which implies the a priori bound for \(\|\nabla \phi_h^n\|\), we have
\[
\lambda \left( e^{n+1}_w \nabla \phi_h^n, e^{n+1}_u \right) \leq c\|e^{n+1}_w\|_1 \|\nabla \phi_h^n\| \|e^{n+1}_u\|^{1/2} \|e^{n+1}_u\|^{1/2} \\
\leq c\|e^{n+1}_u\|^2 + \frac{\mu_0}{16} \|\nabla e^{n+1}_u\|^2 + \frac{\lambda \gamma}{8} \left( \|e^{n+1}_w\|^2 + \|\nabla e^{n+1}_w\|^2 \right); \\
\lambda(w_h(t^{n+1}) \nabla (\phi(t^n) - \phi_h(t^n)), e^{n+1}_u) \\
= -\lambda(w_h(t^{n+1})(\phi(t^n) - \phi_h(t^n)), \nabla \cdot e^{n+1}_u) - \lambda((e^{n+1}_u \cdot \nabla)w_h(t^{n+1}), \phi(t^n) - \phi_h(t^n)) \\
\leq c\|w_h(t^{n+1})\|_{L^\infty} \|\phi(t^n) - \phi_h(t^n)\| \|\nabla e^{n+1}_u\| + c\|w_h(t^{n+1})\|_{W^{1,3}} \|\phi(t^n) - \phi_h(t^n)\| \|e^{n+1}_u\|_1 \\
\leq ch^{l+1} \|\phi\|_{L^\infty(H^{l+1})} \|w\|_{L^\infty(H^2)} \|e^{n+1}_u\|_1 \\
\leq ch^{2(l+1)} + \frac{\mu_0}{16} \|\nabla e^{n+1}_u\|^2;
\]
and
\[
\lambda(w_h(t^{n+1}) \nabla e^n_\phi, e^{n+1}_u) \leq c\|\nabla e^n_\phi\| \|w_h(t^{n+1})\|_1 \|e^{n+1}_u\|_1 \\
\leq c\|\nabla e^n_\phi\| \|e^{n+1}_u\|_1 \|w(t^{n+1})\|_1 \\
\leq c\|\nabla e^n_\phi\|^2 + \frac{\mu_0}{16} \|\nabla e^{n+1}_u\|^2.
\]
This completes the proof. \(\square\)

Lemma 4.10. Under Assumption \([\text{A}]\), we have
\[
|(e^{n+1}_w, 1)| \leq c(\delta t + h^{l+1} + \|\delta t e^n_\phi\| + \|e^n_\phi\|). \tag{4.45}
\]
Proof. We have from (4.27) that
\[
(e^{n+1}_w, 1) = \frac{S}{\varepsilon^2} (\delta t e^n_\phi, 1) + (\tilde{R}^{n+1}_w, 1) + (R^{n+1}_w, 1).
\]
The term \((R^{n+1}_w, 1)\) can be bounded as
\[
|(R^{n+1}_w, 1)| \leq |f'|_{L^\infty} (|\phi(t^n) - \phi_h^n|, 1) \leq c(|\phi(t^n) - \phi_h(t^n)| + \|e^n_\phi\|) \\
\leq ch^{l+1} \|\phi\|_{L^\infty(H^{l+1})} + c\|e^n_\phi\| \leq c(h^{l+1} + \|e^n_\phi\|).
\]
Using Lemma 4.6, we obtain the desired inequality. \(\square\)

We can now establish the following convergence results for \(\tilde{u}_h^n, u_h^n, \phi_h^n, w_h^n\).
Lemma 4.11. Under Assumption A, for sufficiently small $\delta t$, the finite element approximate solution to the scheme (3.1)-(3.5) satisfies
\[
\|\phi - \phi_h,\delta t\|_{L^2(\Omega)} + \|u - uh,\delta t\|_{L^2(\Omega)} + \|u - \bar{u}_h,\delta t\|_{L^2(\Omega)} \lesssim \delta t + h^l,
\]
\[
\|\phi - \phi_h,\delta t\|_{L^2(H^1(\Omega))} + \|w - wh,\delta t\|_{L^2(H^1(\Omega))} + \|u - \bar{u}_h,\delta t\|_{L^2(H^1(\Omega))} \lesssim \delta t + h^l.
\]

Proof. We start from (4.40).

(a) First, we bound the terms on the RHS of (4.40), except the last two terms, as follows.
\[
\frac{S\delta t\gamma}{\varepsilon^2}(\delta t e^n\phi, e^n w) = c\varepsilon t\|\delta t e^n\phi\|\|e^n w\| \leq c\varepsilon t\|\delta t e^n\phi\|^2 + \frac{\delta t\gamma}{8}\|e^n w\|^2.
\]

From Lemmas 4.7 and 4.9 (choosing $\phi_h = e^n w$ in (4.42)),
\[
\lambda\delta t(R^n\phi, e^n w) + \delta t(R^n u, e^n u) \leq c\delta t(h^{2l+1} + \|e^n\phi\|^2 + \|e^n u\|) + \frac{3\mu_0\delta t}{8}\|\nabla e^n u\|^2
\]
\[
+ \frac{3\lambda\delta t}{8}(\|e^n w\|^2 + \|\nabla e^n w\|^2).
\]

Applying Lemma 4.6 we have
\[
\delta t\lambda\gamma(R^n w, e^n w) \leq \lambda\gamma\delta t\|\bar{R}^n w\|\|e^n w\| \leq \delta t(\delta t^2 + h^{2l+2}) + \frac{\lambda t\delta t}{8}\|e^n w\|^2.
\]

Using the expression of $R^n w$ and the assumption on $f(\cdot)$, we get
\[
\delta t\lambda\gamma(R^n w, e^n w) \leq \delta t\lambda\gamma(R^n w, e^n w) \leq c\delta t(\|\phi(t^n) - \phi_n(t^n)\| + \|e^n\phi\|)\|e^n w\|
\]
\[
\leq c\delta t(h^l+1(\|\phi\|_{L^\infty(H^1)} + \|w\|_{L^\infty(H^1)}) + \|e^n\|)\|e^n w\|
\]
\[
\leq c\delta t(h^{2l+1} + \|e^n\|^2) + \frac{\lambda\gamma\delta t}{8}\|e^n w\|^2.
\]

Lemma 4.6 leads to
\[
\lambda\delta t(R^n\phi, e^n w) \leq \lambda\delta t(R^n\phi, e^n w) \leq c\delta t(\delta t + h^{l+2}) + \frac{\lambda\gamma\delta t}{8}\|e^n w\|^2,
\]
and
\[
\delta t(R^n u, e^n u) \leq \delta t(R^n u, e^n u) \leq c\delta t(\delta t + h^{l+2}) + \delta t\|e^n u\|^2.
\]

Lemma 4.8 implies
\[
|\delta t(R^n u, e^n u)| \leq c\delta t(h^{2l+1} + \|e^n u\|^2) + \frac{\mu_0\delta t}{8}\|\nabla e^n u\|^2 + \frac{\mu_0\delta t}{16}\|\nabla e^n u\|^2.
\]

By Lemma 4.7 choosing $\phi_h = e^n w$ in (4.41), we have
\[
\lambda\delta t(R^n\phi, e^n w) \leq c\delta t(h^2(l+1) + \|e^n\phi\|^2 + \|e^n u\|^2) + \frac{\mu_0\delta t}{16}\|\nabla e^n u\|^2.
\]

Based on Lemma 4.6 the following term can be bounded as
\[
\lambda\delta t(R^n\phi, e^n w) \leq c\delta t(\delta t^2 + \|e^n\phi\|^2 + \|e^n u\|^2).
\]

Using (4.36), we have
\[
(\delta t e^n\phi, 1) = \delta t(R^n\phi, 1) + \delta t(R^n u, 1),
\]
and we derive from Lemmas 4.7, 4.10 and 4.6 that
\[
-\gamma(\delta t e^n\phi, 1)(e^n w) \leq c\delta t(h^2(l+1) + \|e^n w\|^2 + \|\nabla e^n\phi\|(\delta t + h^{l+1} + \|\delta t\| + \|e^n\phi\|))
\]
\[
\leq c\delta t(h^{2l+1} + \|\delta t\|^2 + \|\delta t\| + \|e^n\|^2 + \|\nabla e^n\|^2 + \|e^n w\|^2).
\]
Next, we deal with the first term in the last line of Eq. (4.40). By the definition of $L_h$, we have

$$\|L_h(\delta_t e^n_\phi)\| \leq c\|\nabla(L_h(\delta_t e^n_\phi))\| \leq c\|\delta_t e^n_\phi - (\delta_t e^n_\phi, 1)/\Omega\| \leq c\|\delta_t e^n_\phi\|.$$ 

Applying Lemma 4.7 and Sobolev inequality (choose $\varphi_h = \frac{1}{\delta t} L_h(\delta_t e^n_\phi)$ in (4.42)), we derive

$$\left(R_{\phi}^{n+1} + R_{\phi}^{n+1} + L_h(\delta_t e^n_\phi)\right) \leq c\|R_{\phi}^{n+1}\||\nabla(L_h(\delta_t e^n_\phi))| + \delta t(R_{\phi}^{n+1} + \frac{1}{\delta t} L_h(\delta_t e^n_\phi))$$

$$\leq \left[c \delta t (h^{2(l+1)} + ||e^{n+1}_u||^2 + ||e^n_\phi||^2) + \frac{1}{10} \delta t ||\nabla L_h(\delta_t e^n_\phi)||^2 + \frac{\mu_0 \delta t}{16} ||\nabla e^{n+1}_u||^2\right]$$

$$\leq c\delta t + h^{2(l+1)} + ||e^{n+1}_u||^2 + ||e^n_\phi||^2 + \frac{1}{5 \delta t} ||\nabla L_h(\delta_t e^n_\phi)||^2 + \frac{\mu_0 \delta t}{16} ||\nabla e^{n+1}_u||^2.$$

(b) It remains to estimate the last two terms on the RHS of (4.40). The $H^1$ stability of $P_{\psi_h}$ is implied by the inverse inequality. Recalling that

$$-(\lambda + \gamma)(\tilde{R}_{w}^{n+1}, \delta_t e^n_\phi) = -(\lambda + \gamma)(\nabla(P_{\psi_h} \tilde{R}_{w}^{n+1}), \nabla L_h(\delta_t e^n_\phi)) - (\lambda + \gamma)(\tilde{R}_{w}^{n+1}, 1)(\delta_t e^n_\phi, 1),$$

by using (4.46), we derive

$$-(\lambda + \gamma)(\tilde{R}_{w}^{n+1}, \delta_t e^n_\phi) = -(\lambda + \gamma)(\nabla(P_{\psi_h} \tilde{R}_{w}^{n+1}), \nabla L_h(\delta_t e^n_\phi)) - (\lambda + \gamma)(\tilde{R}_{w}^{n+1}, 1)(\delta_t e^n_\phi, 1)$$

$$\leq c\|\tilde{R}_{w}^{n+1}\| \||\nabla L_h(\delta_t e^n_\phi)|| + \delta t(||\tilde{R}_{w}^{n+1}\| + h^{l+1} + ||e^{n+1}_u|| + ||\nabla e^n_\phi||)||\tilde{R}_{w}^{n+1}\|$$

$$\leq c\|\tilde{R}_{w}^{n+1}\| \||\nabla L_h(\delta_t e^n_\phi)|| + \delta t(h^{l+1} + ||e^{n+1}_u|| + ||\nabla e^n_\phi||)||\tilde{R}_{w}^{n+1}\|$$

$$\leq \delta t \left(h^{2l} + \delta t^2 + ||e^{n+1}_u||^2 + ||\nabla e^n_\phi||^2\right) + \frac{\|\nabla L_h(\delta_t e^n_\phi)\|^2}{5 \delta t}.$$

To deal with the last term in (4.40), we split $R_{w}^{n+1}$ as

$$R_{w}^{n+1} = R_{w,a}^{n+1} + R_{w,b}^{n+1},$$

where

$$R_{w,a}^{n+1} = f(\phi(t^n)) - f(\phi(t^n)), \quad R_{w,b}^{n+1} = f(\phi(t^n)) - f(\phi(t^n)).$$

It is obvious that, given the properties of $f$, interpolates $\phi_h(t^n)$ and exact solution $\phi$, we have $|R_{w,a}^{n+1}| \leq c|\phi(t^n) - \phi_h(t^n)|$ and $|R_{w,b}^{n+1}| \leq c|e^n_\phi|, |\nabla R_{w,b}^{n+1}| \leq c(|\nabla e^n_\phi| + |\nabla \phi(t^n)||e^n_\phi|)$. In addition, as $f'$ is bounded and Lipschitz, we have

$$|\nabla R_{w,a}^{n+1}| \leq c(|\nabla(\phi(t^n) - \phi_h(t^n))| + |\phi(t^n) - \phi_h(t^n)||\nabla \phi(t^n)|).$$
We can then estimate \((R_{w,a}^{n+1}, \delta_t e^p_{\phi})\) by using the discrete inverse Laplacian \(L_h\) as follows,

\[
-(\lambda + \gamma)(R_{w,a}^{n+1}, \delta_t e^p_{\phi})
\]

\[
= -(\lambda + \gamma)(\nabla(P_{\phi} R_{w,a}^{n+1}), \nabla L_h(\delta_t e^p_{\phi})) - (\lambda + \gamma)(R_{w,b}^{n+1}, 1)(\delta_t e^p_{\phi}, 1)
\]

\[
\leq c\|\nabla L_h(\delta_t e^p_{\phi})\| - (\lambda + \gamma)(R_{w,a}^{n+1}, 1)(\delta_t e^p_{\phi}, 1)
\]

\[
\leq c(\|\phi(t^n) - \phi_h(t^n)\| + \|\phi(t^n) - \phi_h(t^n)\|\|\nabla \phi(t^n)\|L^3\|\nabla L_h(\delta_t e^p_{\phi})\|
\]

\[
+ c\delta_t h^{l+1} + \|\delta_t e^p_{\phi}\| + \|\nabla \phi(t^n)\|\|\phi(t^n) - \phi_h(t^n)\|
\]

\[
\leq c\delta_t h^{l+1} + \|\delta_t e^p_{\phi}\|^2 + \|\nabla \phi(t^n)\|^2 + \frac{1}{\delta_t h^{l+1}}\|\nabla L_h(\delta_t e^p_{\phi})\|^2.
\]

On the other hand, we have

\[
-(\lambda + \gamma)(R_{w,b}^{n+1}, \delta_t e^p_{\phi})
\]

\[
= -(\lambda + \gamma)(\nabla(P_{\phi} R_{w,b}^{n+1}), \nabla L_h(\delta_t e^p_{\phi})) - (\lambda + \gamma)(R_{w,b}^{n+1}, 1)(\delta_t e^p_{\phi}, 1)
\]

\[
\leq c\|\nabla L_h(\delta_t e^p_{\phi})\| + c\delta_t h^{l+1} + \|\delta_t e^p_{\phi}\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2\|\nabla L_h(\delta_t e^p_{\phi})\|
\]

\[
\leq c\|\delta_t e^p_{\phi}\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2\|\nabla L_h(\delta_t e^p_{\phi})\|^2.
\]

Thus, combining all previous estimates, \([4.40]\) implies that

\[
\frac{\lambda}{2}(\|e^p_{\phi}\|^2 - \|e^p_{\phi}\|^2) + \frac{\lambda}{8}(\|\delta_t e^p_{\phi}\|^2) + \frac{\rho}{2}(\|e^p_{\phi}\|^2 - \|i^T_h e^p_{\phi}\|^2) + \frac{1}{\delta t}h^{l+1} + \|\delta_t e^p_{\phi}\|^2 + \frac{S(\lambda + \gamma)}{\epsilon^2}\|\nabla L_h(\delta_t e^p_{\phi})\|^2 + \frac{5\mu h^{l+1}}{16}\|\nabla e^p_{\phi}\|^2 + \frac{5\lambda h^{l+1}}{8}\|\nabla e^p_{\phi}\|^2 + \frac{5\lambda h^{l+1}}{8}\|\nabla e^p_{\phi}\|^2 + \delta t(B^T_h \hat{e}_p, e^p_{\phi})
\]

\[
\leq c\delta_t h^{l+1} + \|\delta_t e^p_{\phi}\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2
\]

\[
\leq c\delta_t h^{l+1} + \|\nabla e^p_{\phi}\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2 + \|\nabla \phi(t^n)\|^2.
\]

(c) Now, we want to bound \(\delta t(B^T_h \hat{e}_p, e^p_{\phi})\) on the LHS of \([4.47]\) to complete the estimates. Testing \([4.30]\) with \(\delta tC^T_h \hat{e}_p \in Y_h\), we have

\[
-\delta t\rho_0(B^T_h \hat{e}_p, e^p_{\phi}) + \frac{\delta t^2}{2}\|e^p_{\phi}\|^2 = \frac{\delta t^2}{2}\|C^T_h (\hat{e}_p - e^p_{\phi})\|^2 + \frac{\delta t^2}{2}\|C^T_h \hat{e}_p\|^2
\]

\[
= \frac{\rho_0}{2}(\|e^p_{\phi}\|^2 - \|i_h e^p_{\phi}\|^2) + \frac{\delta t^2}{2}\|C^T_h \hat{e}_p\|^2
\]

Applying Lemma \([4.3]\) we obtain

\[
-\delta t\rho_0(B^T_h \hat{e}_p, e^p_{\phi}) + \frac{\delta t^2}{2}\|C^T_h \hat{e}_p\|^2 + \frac{\delta t^2}{2}\|C^T_h \hat{e}_p\|^2
\]

\[
\leq c(\|C^T_h \hat{e}_p\|^2 + \|C^T_h \delta t p_h(t^n)\|^2)\|C^T_h \delta t p_h(t^n)\|^2
\]

\[
\leq c(\|C^T_h \hat{e}_p\|^2 + \|C^T_h \delta t p_h(t^n)\|^2)\|C^T_h \delta t p_h(t^n)\|^2
\]

\[
\leq \delta t^4\|C^T_h \hat{e}_p\|^2 + \delta t^4.
\]
Testing (4.30) with $\delta t e_{u}^{n+1} \in Y_{h}$, we get
\[\frac{\rho_{0}}{2} \| e_{u}^{n+1} \|^2 + \frac{\rho_{0}}{2} \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \|^2 - \frac{\rho_{0}}{2} \| \tilde{e}_{u}^{n+1} \|^2 = 0. \] (4.49)

Summing up (4.47) + $\frac{1}{\rho_{0}}$ (4.48) + (4.49), we obtain
\begin{align*}
\lambda \left(\| e_{\phi}^{n+1} \|^2 - \| e_{\phi}^{0} \|^2 \right) + \frac{\lambda}{8} \| \delta_{e_{\phi}} \|^2 &+ \frac{\rho_{0}}{2} \left(\| e_{u}^{n+1} \|^2 - \| i_{h}^{T} e_{u}^{n+1} \|^2 + \| \tilde{e}_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \|^2 \right) \\
+ \frac{\lambda + \gamma}{2} \left(\| \nabla e_{\phi}^{n+1} \|^2 - \| \nabla e_{\phi}^{0} \|^2 \right) &+ \frac{\delta t^{2}}{2 \rho_{0}} \left(\| C_{h}^{T} e_{p}^{n+1} \|^2 - \| C_{h}^{T} e_{p}^{0} \|^2 \right) + \frac{1}{4 \delta t} \| \nabla L_{h} (\delta e_{\phi}) \|^2 + \frac{S(\lambda + \gamma)}{\varepsilon^{2}} \| \delta_{e_{\phi}} \|^2 \\
+ \frac{5 \mu_{0} \delta t}{16} \| \nabla e_{u}^{n+1} \|^2 + \frac{\lambda \gamma \delta t}{8} \| e_{w}^{n+1} \|^2 &+ \frac{5 \lambda \gamma \delta t}{8} \| \nabla e_{w}^{n+1} \|^{2} \\
\leq c \delta t \left(\| e_{\phi}^{0} \|^2 + \| e_{\phi}^{0} \|^2 + \| e_{u}^{n+1} \|^{2} + \| e_{u}^{n+1} \|^{2} + \| e_{w}^{n+1} \|^{2} + \| e_{w}^{n+1} \|^{2} + \| e_{w}^{n+1} \|^{2} + \| e_{w}^{n+1} \|^{2} \right) + \frac{\mu_{0} \delta t}{16} \| \nabla e_{u}^{n} \|^2. \tag{4.50}
\end{align*}

Summing up (4.50) for $n = 0, \ldots, m$, and noticing $\| \tilde{e}_{u}^{n+1} \| \leq \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n} \| + \| i_{h}^{T} e_{u}^{n} \|$ and $\| e_{u}^{n+1} \|^2 = \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \|^2 + \| i_{h}^{T} e_{u}^{n+1} \|^2$, we obtain
\begin{align*}
\frac{\lambda}{2} \| e_{\phi}^{n+1} \|^2 + \frac{\rho_{0}}{2} \| i_{h}^{T} e_{u}^{n+1} \|^2 &+ \frac{\lambda + \gamma}{2} \| \nabla e_{\phi}^{n+1} \|^2 + \frac{\delta t^{2}}{2 \rho_{0}} \| C_{h}^{T} e_{p}^{n+1} \|^2 + \sum_{n=0}^{m} \left[\frac{\lambda}{8} \| \delta_{e_{\phi}} \|^2 \\
+ \frac{\rho_{0}}{2} \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \| + \frac{\rho_{0}}{2} \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \| + \frac{1}{4 \delta t} \| \nabla L_{h} (\delta e_{\phi}) \|^2 &+ \frac{S(\lambda + \gamma)}{\varepsilon^{2}} \| \delta_{e_{\phi}} \|^2 + \frac{5 \mu_{0} \delta t}{4} \| e_{w}^{n+1} \|^2 + \frac{\lambda \gamma \delta t}{8} \| e_{w}^{n+1} \|^2 + \frac{5 \lambda \gamma \delta t}{8} \| \nabla e_{w}^{n+1} \|^{2} \right] \\
\leq c \delta t \sum_{n=0}^{m} \left(\| e_{\phi}^{n+1} - i_{h}^{T} e_{u}^{n} \|^{2} + \| \delta_{e_{\phi}} \|^2 \right) &+ c \left(\| e_{\phi}^{0} \|^2 + \| e_{w}^{0} \|^2 + \| e_{w}^{0} \|^2 + \| e_{w}^{0} \|^2 + \| e_{w}^{0} \|^2 + \| e_{w}^{0} \|^2 \right) + \mu_{0} \delta t (\delta t + h^{1})^{2}. \tag{4.51}
\end{align*}

Denote
\begin{align*}
I^{n} &= \frac{\lambda}{2} \| e_{\phi}^{n+1} \|^2 + \frac{\rho_{0}}{2} \| i_{h}^{T} e_{u}^{n+1} \|^2 + \frac{\lambda + \gamma}{2} \| \nabla e_{\phi}^{n+1} \|^2 + \frac{\delta t^{2}}{2 \rho_{0}} \| C_{h}^{T} e_{p}^{n+1} \|^2, \\
S^{n} &= \frac{\lambda}{8} \| \delta_{e_{\phi}} \|^2 + \frac{S(\lambda + \gamma)}{\varepsilon^{2}} \| \delta_{e_{\phi}} \|^2 + \frac{\rho_{0}}{2} \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \|^{2}, \\
T^{n} &= \frac{1}{4 \delta t} \| \nabla L_{h} (\delta e_{\phi}) \|^2 + \frac{\rho_{0}}{2} \| e_{u}^{n+1} - i_{h}^{T} e_{u}^{n+1} \|^{2} + \frac{\mu_{0} \delta t}{4} \| \nabla e_{u}^{n+1} \|^2 + \frac{\lambda \gamma \delta t}{8} \| e_{w}^{n+1} \|^2 + \frac{5 \lambda \gamma \delta t}{8} \| \nabla e_{w}^{n+1} \|^{2} \\
&+ \frac{5 \lambda \gamma \delta t}{8} \| \nabla e_{w}^{n+1} \|^{2} + (\lambda + \gamma) \| \nabla \delta e_{\phi} \|^2.
\end{align*}

Then, we have
\begin{align*}
I^{m} + \sum_{n=0}^{m} (S^{n} + T^{n}) &\leq c_{1} \delta t \sum_{n=0}^{m} (I^{n} + S^{n}) + c_{2} (\delta t + h^{1})^{2} + c_{3} (\| e_{\phi}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2}) + \delta t^{2} \| C_{h}^{T} e_{p}^{0} \|^{2}.
\end{align*}

Since the initial errors at $t = 0$ satisfy
\begin{align*}
\| e_{\phi}^{0} \|^{2} + \| \nabla e_{\phi}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} + \| e_{w}^{0} \|^{2} &\leq h^{2t} + \delta t^{2},
\end{align*}
by applying the discrete Gronwall inequality, we obtain the following estimates for sufficiently small $\delta t$:

$$I^n + \sum_{k=0}^{n} (S^k + T^k) \lesssim (\delta t + h^l)^2. \quad (4.52)$$

The desired results then follow from the properties of the interpolates. □

**Remark 4.1.** In the proof of Lemma 4.11, we notice that the projection step implies that

$$\rho_0 \frac{e_u^{n+1} - e_u^n}{\delta t} + C_h T^n + \bar{c}_p^{n+1} - \bar{c}_p^n = \rho_0 \frac{i_h e_u^{n+1} - e_u^n}{\delta t}. \quad (4.53)$$

Taking $L^2$ norm of both sides, using $C_h e_u^{n+1} = C_h e_u^n = 0$ and the properties of $i_h$, we find

$$\|e_u^{n+1} - e_u^n\|^2 \leq \|i_h e_u^{n+1} - e_u^n\|^2 = \|e_u^{n+1} - i_h e_u^n\|^2 + \|e_u^n - i_h \bar{c}_u e_u^n\|^2, \quad (4.54)$$

from which we derive that for $n \leq [T/\delta t] - 1$,

$$\|\delta t e_u^n\| \lesssim \delta t + h^l, \quad \left( \delta t \sum_{k=0}^{[T/\delta t]-1} \|\delta t e_u^k\|^2 \right)^{1/2} \lesssim \delta t^{1/2}(\delta t + h^l).$$

### 4.3. Error on pressure

As in the case of pressure-correction scheme for Navier-Stokes equations [14], the key idea to obtain the optimal error estimate on the pressure is to use the inf-sup condition/inverse Stokes operator and improved estimates of the time increments for the errors.

In order to obtain error estimates on $\|e_p^{n+1}\|$, we introduce the following semi-norm for $v \in H^{-1}(\Omega)^d$

$$|v|_{*,h} = \sup_{v_h \in \mathbb{X}_h} \frac{\langle v, v_h \rangle}{\|v_h\|_{1}} \quad (4.55)$$

Denote $H_{J=0}^1(\Omega) = H^1(\Omega) \cap M$, and we have the $L^2$ orthogonal decomposition

$$L^2(\Omega)^d = H \oplus \nabla (H_{J=0}^1(\Omega)). \quad (4.56)$$

**Lemma 4.12.** For $v \in \mathbb{Y}_h$ with $C_h v = 0$, there exists constant $c > 0$ independent of $h$ such that

$$|v|_{*,h} = |\bar{i}_h v|_{*,h} \leq c(\|\nabla S_h(v)\| + h\|v\|). \quad (4.57)$$

In particular, if $X_h = \mathbb{Y}_h$, for $v \in \mathbb{Y}_h$ with $C_h v = 0$, there holds

$$|v|_{*,h} \leq c\|\nabla S_h(v)\|. \quad (4.58)$$

**Proof.** By the definition of $S_h$ and $\mathcal{S}$, we have for some $p_h \in M_h, p \in M$, $B_h S_h(v) = 0$ and the following holds

$$\langle \nabla S_h(v), \nabla v_h \rangle + (B_h^{T} p_h, v_h) = \langle v, v_h \rangle, \forall v_h \in \mathbb{X}_h,$nabla \mathcal{S}(v), \nabla \mathcal{S}(v) + (p, \nabla \cdot \mathcal{S}) = \langle v, v_h \rangle, \forall \mathcal{S}_h \in H_{0}^{1}(\Omega)^d. \quad (4.59)$$

For $v_h \in \mathbb{X}_h$, we have $(B_h^{T} p_h, v_h) = (p_h, \nabla \cdot v_h)$. Using the $L^2$ decomposition, we set $v_h = P_H v_h + \nabla R_h$ with $R_h \in H_{J=0}^1(\Omega)^d$.

(i) If $\mathbb{Y}_h \subset H_0^{\text{div}}$. We have

$$\langle v, v_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla R_h \rangle = \langle v, P_H v_h \rangle + \langle \nabla \cdot v, R_h - r_h \rangle, \forall r_h \in M_h.$$
Making use of the inverse inequality, $H^1$ stability of $P_H$ and approximation properties of $M_h$ and properties of $S_h$ and $S$, we deduce that $\|R_h\|_2 \leq c\|v_h\|_1$, and

$$\langle v, v_h \rangle = \langle v, P_H v_h \rangle + \langle \nabla \cdot v, R_h - r_h \rangle = \langle \nabla S(v), \nabla P_H v_h \rangle + \langle \nabla \cdot v, R_h - r_h \rangle \leq c(\|\nabla S_h(v)\| + \|\nabla S_h(v) - \nabla S(v)\|)\|\nabla v_h\| + ch^{-1}\|v\|h\|R_h\|_2 \leq c(\|\nabla S_h(v)\| + h\|v\|)\|\nabla v_h\| + ch^{-1}\|v\|h\|R_h\|_2 \leq c(\|\nabla S_h(v)\| + h\|v\|)\|\nabla v_h\| + ch^{-1}\|v\|h\|R_h\|_2 \leq c(\|\nabla S_h(v)\| + h\|v\|)\|\nabla v_h\|_1.$$ (ii) if $M_h \subset H^1$. We have

$$\langle v, v_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla R_h \rangle = \langle v, P_H v_h \rangle + \langle v, \nabla (R_h - r_h) \rangle, \ \forall r_h \in M_h.$$ Using the same arguments as above, we derive

$$\langle v, v_h \rangle \leq c(\|\nabla S_h(v)\|\|\nabla v_h\| + \|v\| \inf_{r_h \in M_h} \|R_h - r_h\|_1 \leq c(\|\nabla S_h(v)\| + h\|v\|)\|\nabla v_h\| + c\|v\|h\|R_h\|_2 \leq c(\|\nabla S_h(v)\| + h\|v\|)\|v_h\|_1.$$ Hence, we obtain (4.57).

If $X_h = Y_h$, testing (4.59) with $v \in X_h$, making use of $C_h v = 0$ and inverse inequality, we have

$$\|v\|^2 = (\nabla S_h(v), \nabla v) \leq \|\nabla S_h(v)\| \|\nabla v\| \leq c/h\|\nabla S_h(v)\| \|v\|,$$

and so $h\|v\| \leq c\|\nabla S_h(v)\|$, which would imply the conclusion (4.58) in light of (4.57). □

**Lemma 4.13.** Under Assumption A, we have the following estimates

$$\|\nabla S_h(\delta_t e^n_u)\| \leq \delta t^{1/2}(\delta t + h^l), \quad \left(\delta t \sum_{n=0}^N \|\nabla S_h(\delta_t e^n_u)\|^2\right)^{1/2} \leq \delta t(\delta t + h^l). \quad (4.60)$$

**Proof.** Summing $i_h^T (4.30) + (4.28)$, we have

$$\frac{\rho_0}{\delta t} (i_h^T e^{n+1}_u - i_h^T e^n_u) + \mu_0 A_h e^{n+1}_u \ast B_h e^{n+1}_p = P_X_h \left[ \tilde{R}^{n+1}_u + R^{n+1}_{u,u} + R^{n+1}_{u,\phi} \right], \quad (4.61)$$

$$C_h e^{n+1}_u = 0. \quad (4.62)$$

Testing (4.61) with $S_h(i_h^T e^{n+1}_u - i_h^T e^n_u) = S_h(e^{n+1}_u - e^n_u) \in X_h$, noticing that $B_h(S_h(\delta_t e^n_u)) = 0$, we get

$$\rho_0 \frac{\|\nabla S_h(\delta_t e^n_u)\|^2}{\delta t} + \mu_0 (\nabla e^{n+1}_u, \nabla S_h(\delta_t e^n_u)) = \left( \tilde{R}^{n+1}_u + R^{n+1}_{u,u} + R^{n+1}_{u,\phi}, S_h(\delta_t e^n_u) \right).$$

Using the same arguments as those in the proof of Lemma 4.11, the RHS can be easily bounded as

$$\left( \tilde{R}^{n+1}_u + R^{n+1}_{u,u} + R^{n+1}_{u,\phi}, S_h(\delta_t e^n_u) \right) \leq \frac{\rho_0}{4\delta t} \|\nabla S_h(\delta_t e^n_u)\|^2 + c\delta t \left[ (\delta t + h^{l+1})^2 + \|e^{n+1}_u\|^2 \right] + \|e^n_\phi\|^2_1 + \|e^{n+1}_w\|^2_1 + \|e^n_w\|^2_1.$$
We also have
\[-\mu_0(\nabla e^{n+1}_u, \nabla S_h(\delta_t e^n_u)) \leq \frac{\rho_0}{4\delta t} \|\nabla S_h(\delta_t e^n_u)\|^2 + c\delta t\|e^{n+1}_u\|^2_1.\]

Thus, we get
\[\frac{\rho_0}{2\delta t}\|\nabla S_h(\delta_t e^n_u)\|^2 \leq c\delta t(\|e^{n+1}_u\|^2 + \|e^n_u\|^2 + \|e^{n+1}_u\|^2_1 + \|e^n_u\|^2_1) + c\delta t(\delta t + h^4)^2.\]

Applying Lemma 4.11, we draw the conclusion.

Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** We only need to bound the error on pressure. By inf-sup condition and (4.61), there holds
\[\|e^{n+1}_p\| \leq c \sup_{v_h \in X_h} \frac{(B^T_{\delta t} e^{n+1}_p, v_h)}{\|v_h\|_1} \leq c(\frac{1}{\delta t} \|B^T_{\delta t} \delta_t e^n_u\|_{\star, h} + |A_h \delta_t e^{n+1}_u|_{\star, h} + |\tilde{R}^{n+1}_u + R^{n+1}_{u, u} + R^{n+1}_{u, \phi}|_{\star, h}).\]

From Remark 4.1, Lemmas 4.12 and 4.13 we obtain
\[\frac{1}{\delta t} |\delta_t e^n_u|_{\star, h} = \frac{1}{\delta t} |\delta_t e^n_u|_{\star, h} \leq c \left(\frac{1}{\delta t} \|\nabla S_h(\delta_t e^n_u)\| + \frac{\delta t}{\delta t} \|\delta_t e^n_u\|\right) \lesssim \frac{\delta t^{1/2} + h}{\delta t} (\delta t + h^4),\]
\[\left(\delta t \sum_{n=0}^{N-1} \frac{1}{\delta t^2} |\delta_t e^n_u|_{\star, h}^2 \right)^{1/2} \lesssim (1 + h/\sqrt{\delta t})(\delta t + h^4), \quad N = \lceil T/\delta t \rceil.\]

From the proof of Lemma 4.11 it can be easily seen that
\[|A_h \delta_t e^{n+1}_u|_{\star, h} + |\tilde{R}^{n+1}_u + R^{n+1}_{u, u} + R^{n+1}_{u, \phi}|_{\star, h} \lesssim \frac{1}{\delta t^{1/2}} (\delta t + h^4),\]
\[\left(\delta t \sum_{n=0}^{N-1} \left(|A_h \delta_t e^{n+1}_u|_{\star, h} + |\tilde{R}^{n+1}_u + R^{n+1}_{u, u} + R^{n+1}_{u, \phi}|_{\star, h}\right)\right)^{1/2} \lesssim \delta t + h^4.\]

Then it follows that
\[\|e_{p, \delta t}\|_{L^2(\Omega)} \lesssim (\delta t + h^4) (1 + h/\sqrt{\delta t}).\]

Combining with the estimates for the interpolate and under the condition $h^2 \lesssim \delta t$, we obtain
\[\|p - p_{h, \delta t}\|_{L^2(\Omega)} \lesssim \delta t + h^4.\]

The proof is complete.

**Remark 4.2.** In this approach, since there is no particular assumption on the relation between finite element spaces $X_h$ and $Y_h$ except $X_h \subset Y_h$, the use of inverse Stokes operator will result in the type of estimates in Lemma 4.12. It is noticed that if $X_h = Y_h$, then $h^2 \lesssim \delta t$ is unnecessary. It is possible to study the time increment $\delta_t e^n_u$ and establish higher order convergence for $\delta_t e^n_u$ as in 11, instead of using the discrete inverse Stokes operator.

The same proof works for the Allen-Cahn phase field equations, but we will have $L^2$ norm of $e^{n+1}_w$ instead of the $H^1$ norm of $e^{n+1}_w$. 
5. Convergence Analysis

We derived error estimates of the fully discrete scheme \[(3.1)-(3.5)\] under appropriate regularity assumptions of the exact solution. Now we show below that the numerical solution of \[(3.1)-(3.5)\] converges to the exact solution of NSCH system \[(1.1)-(1.5)\] without such assumptions, when \(h, \delta t \to 0\). There are several challenges towards such an analysis. In particular, the choices of the finite element spaces are rather general, where \(X_h \subset Y_h\) may not be identical, and this brings significant difficulties when analyzing the velocity splitting scheme considered here. In the following arguments, when \(X_h \neq Y_h\), we will assume the technical condition \(h^2 \lesssim \delta t\).

From Remark 3.1 and the stability result Theorem 3.1, we could obtain the following a priori bounds on the numerical solution \(\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, \tilde{u}_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1}\).

**Lemma 5.1.** Given the initialization \((\phi_h^0, w_h^0, u_h^0, p_h^0)^T\) and \(\tilde{u}_h^0 = u_h^0\), there exists a unique solution \(\{(\phi_h^{n+1}, w_h^{n+1}, u_h^{n+1}, \tilde{u}_h^{n+1}, p_h^{n+1})\}_{n=0}^{N-1} \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h\) to the fully discrete scheme \[(3.1)-(3.5)\]. Assume \((\phi_h^0, w_h^0, u_h^0, p_h^0)^T\) is initialized such that \(\frac{\alpha_0}{2} \|\phi_h^0\|^2 + \frac{1}{2} \|\phi_h^0\|^2 + \lambda \langle F(\phi_h^0), 1 \rangle + \frac{\delta t^2}{\rho_0} \|C_h P_h\|^2 \leq c_0\), the numerical approximation satisfies the following estimates

\[
\max_{0 \leq n \leq N} \left\{ \|u_h^n\|^2 + \|\nabla \phi_h^n\|^2 + (F(\phi_h^n), 1) + \delta t^2 \|C_h P_h^n\|^2 \right\} \leq c, \tag{5.1}
\]

\[
\delta t \sum_{n=0}^{N-1} \left( \|\nabla \phi_h^n\|^2 + \|u_h^n\|^2 + \|\nabla \phi_h^n\|^2 \right) \leq c, \tag{5.2}
\]

\[
\delta t \sum_{n=0}^{N} \|\phi_h^n\|^2 + \|\tilde{u}_h^n\|^2 \leq c, \tag{5.3}
\]

\[
\delta t \sum_{n=0}^{N-1} \left\| \frac{1}{\delta t} \phi_h^n \right\|_{-1}^{\frac{12}{\nu + \alpha}} \leq c \left( 1 + \left( \frac{h^2}{\delta t} \right)^{\frac{\gamma}{2}} \right), \tag{5.4}
\]

\[
\delta t \sum_{n=0}^{N} \left\| \frac{1}{\delta t} \phi_h^n \right\|_{-1}^{\frac{12}{\nu + \alpha}} \leq c \left( 1 + \left( \frac{h^2}{\delta t} \right)^{\frac{\gamma}{2}} \right), \tag{5.5}
\]

where the constant \(c\) depends on \(c_0, \rho_0, \lambda\) and \(\gamma\). In particular, if \(Y_h\) is chosen as \(Y_h = X_h\), then the estimates \[(5.5)-(5.7)\] become

\[
\frac{1}{\delta t} \sum_{n=0}^{N-1} \left\| \frac{1}{\delta t} \phi_h^n \right\|_{-1}^{\frac{12}{\nu + \alpha}} + \frac{1}{\delta t} \sum_{n=0}^{N-1} \left\| \frac{1}{\delta t} \phi_h^n \right\|_{-1}^{\frac{12}{\nu + \alpha}} + \delta t \sum_{n=0}^{N} \left\| \frac{1}{\delta t} \phi_h^n \right\|_{-1}^{\frac{12}{\nu + \alpha}} \leq c. \tag{5.8}
\]
Proof. (5.1)-(5.3) are direct consequence of the proof in Theorem 3.1 except those estimates on \( \delta_i u_h^n \) and \( \delta_i \tilde{u}_h^n \). Indeed, using the arguments in Remark 4.1 we find from (3.4)

\[
\frac{\rho_0}{\delta t} (u_h^{n+1} - u_h^n) + C_h^T (p_h^{n+1} - p_h^n) = \frac{\rho_0}{\delta t} (i_h \tilde{u}_h^{n+1} - u_h^n).
\]

Since \( C_h (u_h^{n+1} - u_h^n) = 0 \), taking \( L^2 \) norm on both sides, using triangle inequality and the fact that \( i_h^T \) is an \( L^2 \) orthogonal projection, we get

\[
\| u_h^{n+1} - u_h^n \| \leq \| i_h \tilde{u}_h^{n+1} - u_h^n \| \leq \| u_h^n - i_h i_h^T u_h^n \| + \| \tilde{u}_h^{n+1} - i_h^T u_h^n \|, \quad 0 \leq n \leq N - 1,
\]

and the estimates of \( \| \delta_i u_h^n \| \) are implied. For \( \delta_i \tilde{u}_h^n \), \( \delta_i \tilde{u}_h^n = \tilde{u}_h^n - i_h^T u_h^n \), for \( n \geq 1 \), we get

\[
\| \delta_i \tilde{u}_h^n \| \leq \|i_h^T \delta_i u_h^{n-1}\| + \| u_h^{n-1} - i_h^T u_h^{n-1}\| + \| \tilde{u}_h^{n-1} - i_h^T u_h^{n-1}\|.
\]

Hence, we derive the estimates on \( \| \delta_i \tilde{u}_h^n \| \) in (5.2).

For (5.4), the potential term \( F(\cdot) \) (2.10) implies that \( s^2 \leq (1 + 2\varepsilon^2)(F(s) + 1) \) \( (s \in \mathbb{R}) \) and

\[
\| \phi^n_h \|^2 \leq c \left( (F(\phi^n_h), 1) + |\Omega| \right),
\]

where \( |\Omega| \) denotes the total volume of \( \Omega \). The estimates on \( \| \phi^n_h \| \) hold. Testing (3.1) with \( \delta t \phi^n_h \in \Psi_h \) and using previous bounds (5.1)-(5.3), we arrive at

\[
\frac{1}{2} \left( \| \phi^{n+1}_h \|^2 + \| \phi^{n+1}_h - \phi^n_h \|^2 \right) - \frac{1}{2} \| \phi^n_h \|^2 = \delta t \left( (\tilde{u}_h^{n+1} \cdot \nabla) \phi^n_h, \phi^{n+1}_h \right) - \gamma (\nabla w^{n+1}_h, \nabla \phi^{n+1}_h)
\]

\[
\leq c \delta t \left( \| \tilde{u}_h^{n+1} \|_{L^6} \| \nabla \phi^n_h \| \| \phi^{n+1}_h \| + \| \nabla w^{n+1}_h \| \| \nabla \phi^{n+1}_h \| \right)
\]

\[
\leq c \delta t \left( \| \nabla \phi^{n+1}_h \| + \| \phi^{n+1}_h \|^2 \right),
\]

which leads to the estimates for \( \| \delta_i \phi^n_h \| \).

Testing (3.2) with \( u_h \in \Psi_h \) and applying Cauchy inequality, we obtain

\[
\| u_h^{n+1} \|^2 = \frac{S}{\varepsilon^2} \left( \delta t \phi^n_h, w_h^{n+1} \right) + (\nabla \phi^{n+1}_h, \nabla w^{n+1}_h) + (f(\phi^n_h), w_h^{n+1})
\]

\[
\leq \frac{1}{2} \| w_h^{n+1} \|^2 + c \left( \| \delta_i \phi^n_h \|^2 + \| f(\phi^n_h) \|^2 \right) + \| \nabla \phi^{n+1}_h \|^2 + \| \nabla w^{n+1}_h \|^2
\]

\[
\leq \frac{1}{2} \| w_h^{n+1} \|^2 + c \left( \| \delta_i \phi^n_h \|^2 + \| \phi^n_h \|^2 + 1 \right) + \| \nabla \phi^{n+1}_h \|^2 + \| \nabla w^{n+1}_h \|^2,
\]

and the estimates of \( w_h^n \) in (5.4) is true by using (5.1)-(5.3).

Next, we prove (5.5). For \( i_h^T \delta_i u_h^n \in X_h \), using Lemma 4.12 and \( H^1 \) stability of \( P_{X_h} \), we have

\[
\| i_h^T \delta_i u_h^n \|^2 = \sup_{0 \neq v \in H_h^0(\Omega)^d} \frac{\langle i_h^T \delta_i u_h^n, v \rangle}{\| v \|_1} \| v \|_1 = \sup_{0 \neq v \in H_h^0(\Omega)^d} \frac{\langle i_h^T \delta_i u_h^n, P_{X_h} v \rangle}{\| P_{X_h} v \|_1} \| P_{X_h} v \|_1
\]

\[
\leq c \| i_h^T \delta_i u_h^n \|_{\ast, h} \leq c \| \nabla S_h (i_h^T \delta_i u_h^n) \| + h \| i_h^T \delta_i u_h^n \|,
\]

where the estimates of \( \| i_h^T \delta_i u_h^n \| \) can be found in (5.1)-(5.2) as

\[
\sum_{n=0}^{N-1} \| i_h^T \delta_i u_h^n \| \leq \sum_{n=0}^{N-1} \| \delta_i u_h^n \|^2 \leq c.
\]

(5.11)

We only need to estimate \( \| \nabla S_h (i_h^T \delta_i u_h^n) \| \). Applying \( i_h^T \) to (3.4) and adding it to (3.3), we obtain

\[
\frac{\rho_0}{\delta t} i_h^T \delta_i u_h^n + \rho_0 P_{X_h} \left( D_h(\bar{u}_h^n, \bar{u}_h^{n+1}) \right) + \mu_0 A_h \bar{u}_h^{n+1} + B_h P_{X_h} (w_h^{n+1} \nabla \phi^n_h) = 0.
\]

(5.12)
Testing (5.12) with \( v_h = S_h(\delta_t u^n_h) = S_h(i^T_h \delta_t u^n_h) \in X_h \), noticing (5.1)–(5.4), we derive
\[
\frac{1}{\delta t} \| \nabla v_h \|^2 = - \frac{\mu_0}{\rho_0} \langle \nabla \tilde{u}_h^{n+1}, \nabla v_h \rangle - d(\tilde{u}_h^{n+1}, v_h) + (w^n_{h+1} \nabla \phi^n_h, v_h)
\leq \frac{\mu_0}{\rho_0} \| \nabla \tilde{u}_h^{n+1} \| \| \nabla v_h \| + \left( \| \tilde{u}_h^n \|_{L^1} \| \nabla \tilde{u}_h^{n+1} \| + \frac{1}{2} \| \nabla \tilde{u}_h^n \| \| \tilde{u}_h^{n+1} \|_{L^2} \right) \| v_h \|_{L^6}
\quad + \| w^n_{h+1} \|_{L^\infty} \| \nabla \phi^n_h \| \| v_h \|_{L^6}
\leq c \left( \| \nabla \tilde{u}_h^{n+1} \| + \| u^n_{h+1} \|_1 + c \| \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| \right) \| v_h \|_{L^6},
\]
and
\[
\frac{1}{\delta t} \| \nabla v_h \| \leq c \left( \| \nabla \tilde{u}_h^{n+1} \| + \| u^n_{h+1} \|_1 + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| \right).
\]
Combining the above estimates with Young’s inequality, (5.1)–(5.3), (5.11) and (5.10), we have
\[
\delta t \sum_{n=0}^{N-1} \left| \frac{1}{\delta t} i^T_h \delta_t u^n_h \right|_{-1}^{\frac{12}{6+\delta}} \leq c + \delta t \sum_{n=0}^{N-1} \left| \frac{h}{\delta t} i^T_h \delta_t u^n_h \right|^{\frac{12}{6+\delta}} \leq c \left( 1 + \left( \frac{\delta t}{\delta} \right) \frac{3}{12} \right).
\]

Next, we prove the estimates on the pressure \( p^n_h \). Using inf-sup condition (2.25) and similar arguments above as well as those in subsection 4.3 for the pressure error estimates, from (5.12), we have for \( n = 0, \ldots, N-1 \)
\[
\| p^n_{h+1} \| \leq \left| \frac{1}{\delta t} i^T_h \delta_t u^n_h \right|_{-1} + \left( \| \nabla \tilde{u}_h^{n+1} \| + \| u^n_{h+1} \|_1 + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| \right),
\]
and (5.6) follows.

In a similar way, we can prove (5.7) for \( \delta_t \tilde{u}_h^n \). Noticing that \( \tilde{u}_h^0 = u^0_h \) by initialization and for \( n \geq 1 \), applying \( i^T_h \) to (3.4) with time step \( n-1 \) and adding it to (3.3) to eliminate \( i^T_h u^n_h \), we have
\[
\frac{\rho_0}{\delta t} \delta_t \tilde{u}_h^n + \rho_0 P_{X_h} \left( D_h(\tilde{u}_h^n, \tilde{u}_h^{n+1}) \right) + \mu_0 A_h \tilde{u}_h^{n+1} + B^T_h (2p^n_h - p^{n-1}_h) - \lambda P_{X_h} (w^n_{h+1} \nabla \phi^n_h) = 0. \tag{5.13}
\]
We then control
\[
\left| \frac{1}{\delta t} \delta_t \tilde{u}_h^n \right| \lesssim \left( \| \nabla \tilde{u}_h^{n+1} \| + \| u^n_{h+1} \|_1 + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| + \| \nabla \tilde{u}_h^n \|_6 \| \nabla \tilde{u}_h^{n+1} \| \right),
\]
and the estimates on \( \| \delta_t \tilde{u}_h^n \|_1 \) can be derived.

Lastly, for any \( \varphi \in H^1_0 \), testing (3.1) with \( \varphi_h = P_{\Psi_h} \varphi \in \Psi_h \), we get
\[
\frac{1}{\delta t} (\delta_t \varphi^n_h, \varphi) = - \left( (\tilde{u}_h^{n+1} \cdot \nabla) \phi^n_h, \varphi \right) + \gamma \left( \nabla u^n_{h+1}, \nabla \varphi \right)
\leq c \left( \tilde{u}_h^{n+1} \|_{L^2} \| \nabla \phi^n_h \| \| \varphi \|_{H^1} + \| \nabla u^n_{h+1} \| \| \nabla \varphi \|_{H^1} \right)
\leq c \left( \| \nabla \tilde{u}_h^{n+1} \| + \| \nabla u^n_{h+1} \| \right) \| \nabla \varphi \|_{H^1},
\]
and we can derive (5.8) since \( P_{\Psi_h} \) is \( H^1 \) stable. \( \square \)

For fully discrete numerical solution \( \{ (\phi^n_h, w^n_h, u^n_h, \tilde{u}_h^n, p^n_h) \}_{n=0}^N \in \Psi_h \times \Psi_h \times Y_h \times X_h \times M_h \), define the linear interpolation \( (U_{h, \delta t}(x, t), \Phi_{h, \delta t}(x, t)) \) of \( \{ (\tilde{u}_h^n, \phi^n_h) \}_{n=0}^N \) in time as
\[
U_{h, \delta t}(\cdot, t) = \tilde{u}_h^n + \frac{t - t_n}{\delta t} (\tilde{u}_h^{n+1} - \tilde{u}_h^n), \quad \Phi_{h, \delta t}(\cdot, t) = \phi^n_h + \frac{t - t_n}{\delta t} (\phi^{n+1}_h - \phi^n_h), \tag{5.14}
\]
for \( t \in [t_n, t_{n+1}], \ n = 0, 1, \ldots, N - 1 \). We also denote \( \mathbf{P}_{h,\delta t}(x,t), \mathcal{U}_{h,\delta t}(x,t), \Phi_{h,\delta t}(x,t), \overline{W}_{h,\delta t}(x,t) \) and \( \widetilde{U}_{h,\delta t}(x,t) \) as the piecewise constant extensions of \( \{p_h^n\}_{n=0}^N, \{u_h^n\}_{n=0}^N, \{\phi_h^n\}_{n=0}^N, \{w_h^n\}_{n=0}^N \) and \( \{\overline{u}_h^n\}_{n=0}^N \), respectively, i.e.

\begin{align}
\Phi_{h,\delta t}(\cdot, t) &:= p_h^n, \quad t \in [t_n, t_{n+1}], \\
\overline{w}_{h,\delta t}(\cdot, t) &:= u_h^n, \quad t \in [t_n, t_{n+1}], \\
\overline{U}_{h,\delta t}(\cdot, t) &:= \phi_h^n, \quad t \in [t_n, t_{n+1}], \\
\widetilde{U}_{h,\delta t}(\cdot, t) &:= \overline{u}_h^n, \quad t \in [t_n, t_{n+1}], \quad n = 0, 1, \ldots, N - 1. 
\end{align}

**Theorem 5.1.** Assume initial data \((\phi_0, u_0) \in H^1(\Omega) \times V\) and \((\phi_h^0, w_h^0, u_h^0, P_h^0)^T\) is initialized such that \( \frac{\rho}{2} \|u_0^h\|^2 + \frac{1}{2} \|\phi_0^h\|^2 + \lambda(F(\phi_0^h), 1) + \frac{\delta t^2}{\rho_0} \|C T_h P_h^0\|^2 \leq c_0 \). Furthermore, we require \( h^2 \lesssim \delta t \) if \( Y_h \neq X_h \). Let \((u, \phi, w, p)\) denote the unique solution of NSCH system (1.1)–(1.5), and \( (\overline{\Phi}_{h,\delta t}, \overline{W}_{h,\delta t}, \overline{U}_{h,\delta t}, \overline{P}_{h,\delta t}) \) be defined as above. Then there hold

\begin{align}
\lim_{\delta t \to 0} \left( \|\overline{U}_{h,\delta t} - u\|_{L^2(\Omega)} + \|\overline{\Phi}_{h,\delta t} - \phi\|_{L^2(\Omega)} + \|\overline{P}_{h,\delta t} - p\|_{L^2(\Omega)} \right) &= 0, \\
\overline{W}_{h,\delta t} &\to w, \quad \text{weakly in } L^2([0,T]; H^1(\Omega)), \\
\overline{P}_{h,\delta t} &\to p, \quad \text{weakly in } L^{\frac{12}{5+d}}([0,T]; L^2(\Omega)).
\end{align}

**Proof.** We only sketch the proof below as the procedure is similar to that in [3]. Moreover, we work with the case \( X_h \neq Y_h \) as the arguments are almost the same for the \( X_h = Y_h \) case but without the condition \( h^2 \lesssim \delta t \).

**Step 1:** Firstly, we collect some *a priori* estimates and extract convergent subsequences. From Lemma [5.1], we know

\begin{align}
\|\Phi_{h,\delta t}\|_{L^\infty(H^1)} + \|\partial_t \Phi_{h,\delta t}\|_{L^2(H^{-1})} + \|\overline{W}_{h,\delta t}\|_{L^2(H^1)} + \|F(\overline{\Phi}_{h,\delta t})\|_{L^\infty(L^1)} &\leq C, \\
\|\partial_t U_{h,\delta t}\|_{L^{2\frac{12}{5+d}}(H^{-1})} + \|U_{h,\delta t}\|_{L^\infty(L^2)} + \|U_{h,\delta t}\|_{L^2(L^2)} &\leq C \left( 1 + \left(\frac{h^2}{\delta t}\right)^{\frac{d+4}{2d}} \right), \\
\|\overline{P}_{h,\delta t}\|_{L^{\frac{12}{5+d}}(L^2)} &\leq C \left( 1 + \left(\frac{h^2}{\delta t}\right)^{\frac{d+4}{2d}} \right).
\end{align}

We can extract convergent subsequences \( \{\Phi_{h,\delta t}, W_{h,\delta t}, \overline{U}_{h,\delta t}, \overline{P}_{h,\delta t}\} \) (denoted as the original one for simplicity) and find a quadruple \((\phi_*, w_*, u_*, p_*)\) such that

\begin{align}
\phi_* &\in L^\infty([0,T]; H^1(\Omega)) \cap H^1([0,T]; H^{-1}(\Omega)), \quad w_* \in L^2([0,T]; H^1(\Omega)), \\
u_* &\in L^\infty([0,T]; L^2(\Omega)^d) \cap L^2([0,T]; H^1(\Omega)^d) \cap W^{1,12/6+d}([0,T]; H^{-1}(\Omega)^d), \\
p_* &\in L^{12/6+d}([0,T]; L^2(\Omega)),
\end{align}

and

\begin{align}
\Phi_{h,\delta t} &\stackrel{h,\delta t \to 0}{\to} \phi_* \quad \text{weakly in } L^\infty([0,T]; H^1(\Omega)), \\
w_* &\to \phi_* \quad \text{weakly in } H^1([0,T]; H^{-1}(\Omega)), \\
\overline{w}_{h,\delta t} &\to w_* \quad \text{weakly in } L^2([0,T]; H^1(\Omega)).
\end{align}
Recalling (3.14), (3.1)-(3.5) can be written equivalently for converging to $U$ in $L^2([0, T]; \Omega)$, and $\bar{U}$, $\bar{U}$, and $\bar{U}$ converge to $u_*$ strongly in $L^2([0, T]; \Omega)$, and $\bar{U}$, $\bar{U}$, and $\bar{U}$ converge to $u_*$ weakly in $L^2([0, T]; \Omega)$.

Further more, recalling Lemma [5.1] we have

$$\|\Phi_{h, \delta t} - \bar{\Phi}_{h, \delta t}\|_{L^2(H^1)} = \sum_{n=1}^{N} \|\phi^\ast_n - \phi^{n-1}_h\|_{H^1} = \frac{\delta t}{3} \sum_{n=1}^{N} \|\phi^\ast_n - \phi^{n-1}_h\|_{H^1} \to 0,$$

$$\|U_{h, \delta t} - \bar{U}_{h, \delta t}\|_{L^2(L^2)} = \frac{\delta t}{3} \sum_{n=1}^{N} \|\bar{u}^n_{h, \delta t} - \bar{u}^{n-1}_{h, \delta t}\|_{L^2} \to 0,$$

$$\|\bar{u}_{h, \delta t} - \bar{U}_{h, \delta t}\|_{L^2(L^2)} = \delta t \sum_{n=1}^{N} \|\bar{u}^n_{h, \delta t} - \bar{u}^{n-1}_{h, \delta t}\|_{L^2} \to 0.$$

We then conclude from the above computations and the estimates in Lemma [5.1] that $\bar{\Phi}_{h, \delta t}$ converges to $\phi_*$ strongly in $L^2([0, T]; \Omega)$, and $\bar{\Phi}_{h, \delta t}$ converges to $\phi_*$ weakly in $L^\infty([0, T]; H^1(\Omega))$; $\bar{U}_{h, \delta t}$ and $\bar{U}_{h, \delta t}$ converge to $u_*$ strongly in $L^2([0, T]; \Omega)$, and $\bar{U}_{h, \delta t}$ converges to $u_*$ weakly in $L^2([0, T]; H^1(\Omega^d))$.

**Step 2.** Now, we want to pass to limits as $h, \delta t \to 0$ in (3.1)-(3.5) and show ($\phi_*, u_*, v_*, p_*$) is a weak solution of (2.15)-(2.16). Applying $\int_T^T$ to (3.4) with time step $n - 1$, adding it to (3.3) and denoting $p_{h-1} = p_h$, we have

$$\frac{\rho_0}{\delta t} \bar{u}_h^n + \rho_0 P_{X_h} (D_h(u^n_{h, \delta t} + u^{n+1}_{h, \delta t})) + \mu_0 A_h u^{n+1}_{h, \delta t} + B_h (2p_h^n - p_h^{n-1}) - \lambda P_{X_h} (u^{n+1}_h \nabla \phi^n_h) = 0.$$

Recalling (3.14), (3.1)-(3.5) can be written equivalently for $t \in [t_n, t_{n+1}]$ as

$$(\partial_t \Phi_{h, \delta t}(t), \psi_h) + \left( \bar{U}_{h, \delta t}(t + \delta t) \cdot \nabla \Phi_{h, \delta t}(t), \psi_h \right) + \gamma \left( \nabla \bar{W}_{h, \delta t}(t + \delta t), \nabla \psi_h \right) = 0, \quad (5.25)$$

$$(\bar{W}_{h, \delta t}(t + \delta t), \varphi_h) - \frac{S}{\varepsilon^2} (\Phi_{h, \delta t}(t + \delta t) - \Phi_{h, \delta t}(t), \varphi_h) - (\nabla \Phi_{h, \delta t}(t), \nabla \varphi_h) = (f(\Phi_{h, \delta t}(t)), \varphi_h), \quad (5.26)$$

$$\rho_0 (\partial_t U_{h, \delta t}(t), v_h) + \rho_0 (\bar{U}_{h, \delta t}(t), \bar{U}_{h, \delta t}(t + \delta t), v_h) + \mu_0 (\nabla \bar{U}_{h, \delta t}(t + \delta t), \nabla v_h) + (\lambda (\bar{W}_{h, \delta t}(t + \delta t) \nabla \Phi_{h, \delta t}(t), v_h) = 0, \quad (5.27)$$
with

\[
(\nabla \cdot U_{h,\delta t}, q_h) = \frac{1}{\rho_0} \left( \mathcal{P}^C_{h,\delta t}(t + \delta t) - \mathcal{P}^C_{h,\delta t}(t), \nabla q_h \right), \quad \text{if } M_h \subset H^1(\Omega), \tag{5.28}
\]

\[
(\nabla \cdot U_{h,\delta t}, q_h) = \frac{1}{\rho_0} \left( \nabla \cdot (\mathcal{P}^C_{h,\delta t}(t + \delta t) - \mathcal{P}^C_{h,\delta t}(t)), q_h \right), \quad \text{if } Y_h \subset H^1_0(\Omega), \tag{5.29}
\]

for all \((\psi_h, \varphi_h, v_h, q_h) \in \Psi_h \times \Psi_h \times X_h \times M_h\), where \(\mathcal{P}^C_{h,\delta t}\) denotes the piecewise extension of \(\delta t C_h^T p^n_h\). From Lemma 5.1, it is easy to see there exists \(p^n_* \in L^2([0, T]; L^2(\Omega)^d)\) such that a subsequence \(\mathcal{P}^C_{h,\delta t} \to p_*^C\) weakly in \(L^2([0, T]; L^2(\Omega)^d)\) (denoted as the original sequence for simplicity) as \(h, \delta t \to 0\).

For any \(\eta(t) \in C[0, T]\), multiplying (5.25)-(5.29) by \(\eta(t)\) and integrate from 0 to \(T\), we have

\[
\int_0^T \left[ (\partial_t \Phi_{h,\delta t}(t), \psi_h) + (\ddot{U}_{h,\delta t}(t + \delta t) \cdot \nabla \Phi_{h,\delta t}(t), \psi_h) + \gamma \nabla W_{h,\delta t}(t + \delta t), \nabla \psi_h \right] \eta(t) dt = 0,
\]

\[
\int_0^T \left[ \left( \frac{\partial}{\partial t} \Phi_{h,\delta t}(t), \varphi_h \right) - \frac{S_0}{\varepsilon^2} \left( \Phi_{h,\delta t}(t + \delta t) - \Phi_{h,\delta t}(t), \varphi_h \right) - \left( \nabla \Phi_{h,\delta t}(t), \nabla \varphi_h \right) \right] \eta(t) dt = \int_0^T \left( f(\Phi_{h,\delta t}(t)), \varphi_h \right) \eta(t) dt,
\]

\[
\int_0^T \left[ \left( \rho_0 (\partial_t U_{h,\delta t}(t), v_h) + \rho_0 d(\ddot{U}_{h,\delta t}(t), \dot{U}_{h,\delta t}(t + \delta t), v_h) + \mu_0 \left( \nabla \ddot{U}_{h,\delta t}(t + \delta t), \nabla v_h \right) \right) \eta(t) dt + \int_0^T \left[ \left( \left( 2\mathcal{P}^C_{h,\delta t}(t) - \mathcal{P}^C_{h,\delta t}(t - \delta t) \right), \nabla v_h \right) - \lambda \left( \nabla W_{h,\delta t}(t + \delta t) \nabla \Phi_{h,\delta t}(t), v_h \right) \right] \eta(t) dt = 0,
\]

and if \(M_h \subset H^1(\Omega),\)

\[
\int_0^T (\nabla \cdot U_{h,\delta t}, q_h) \eta(t) dt = \frac{1}{\rho_0} \int_0^T \left( \mathcal{P}^C_{h,\delta t}(t + \delta t) - \mathcal{P}^C_{h,\delta t}(t), \nabla q_h \right) \eta(t) dt, \tag{5.30}
\]

or if \(Y_h \subset H^1_0(\Omega),\)

\[
\int_0^T (\nabla \cdot U_{h,\delta t}, q_h) \eta(t) dt = \frac{1}{\rho_0} \int_0^T \left( \nabla \cdot (\mathcal{P}^C_{h,\delta t}(t + \delta t) - \mathcal{P}^C_{h,\delta t}(t)), q_h \right) \eta(t) dt. \tag{5.31}
\]

For any \((\psi, \varphi, v) \in H^1(\Omega) \times H^1(\Omega) \times X\), choose \((\psi_h, \varphi_h, v_h) \in \Psi_h \times \Psi_h \times X_h\) such that when \(h \to 0,\)

\[
\psi_h \to \psi \text{ strongly in } H^1(\Omega); \quad \varphi_h \to \varphi \text{ strongly in } H^1(\Omega); \quad v_h \to v \text{ strongly in } X = H^1_0(\Omega)^d. \tag{5.32}
\]

Since \(U_{h,\delta t} \to u_*\) strongly in \(L^2([0, T]; L^p(\Omega)^d)\) \((1 < p < \frac{2d}{d-2})\) and \(\Phi_{h,\delta t} \to \phi_*\) strongly in \(C([0, T]; L^p(\Omega))\) \((1 < p < \frac{2d}{d-2})\), combing with the results in step 1 and following [4], we can
conclude that \( u_*(t = 0) = u_0, \phi_*(t = 0) = \phi_0 \).

\[
\int_0^T \left[ (\partial_t \phi_*, \psi) + (u_* \cdot \nabla \phi_*, \psi) - \gamma (\nabla w_*, \nabla \psi) \right] \eta(t) \, dt = 0,
\]

\[
\int_0^T [w_*, \varphi] - (\nabla \phi_*, \nabla \varphi) \right] \eta(t) \, dt = \int_0^T f(\phi_*, \varphi) \eta(t) \, dt,
\]

\[
\int_0^T [\rho_0 (\partial_t u_*, v) + \rho_0 d(u_*, u_*, v) + \mu_0 (\nabla u_*, \nabla v) + (p_*, \nabla v) - \lambda (w_*, \nabla \phi_*, v)] \eta(t) \, dt = 0.
\]

Thus \((\phi_*, w_*, u_*, p_*)\) satisfies (2.15)-(2.17) except the divergence free equation (2.18) in view of the fact that \(C([0, T])\) is dense in \(L^2[0, T]\). Next, we prove \(u_*\) actually satisfies (2.18) and separate the proof into two cases.

(1) When \(M_h \subset H^1(\Omega)\), for any \(q \in H^1(\Omega) \cap L^2_0(\Omega)\), we can choose \(q_h \in M_h\) such that \(q_h \to q\) strongly in \(H^1\). Then passing to the limit in (5.30) as \(h, \delta t \to 0\), we have \(\overline{P}_{h, \delta t}(t + \delta t) - \overline{P}_{h, \delta t}(t) \to 0\) weakly in \(L^2([0, T] ; L^2(\Omega)^d)\) and

\[
\int_0^T [(\nabla \cdot u_*, q)] \eta(t) \, dt = 0.
\]

Since \(H^1(\Omega) \cap L^2_0(\Omega)\) is dense in \(L^2(\Omega)\) and \(C([0, T])\) is dense in \(L^2[0, T]\), we find \(u_*\) fulfills (2.18).

(2) When \(Y_h \subset H^1_0(\Omega)\), for any \(q \in L^2(\Omega)\), we can choose \(q_h \in M_h\) such that \(q_h \to q\) strongly in \(L^2\). On the other hand, using (3.14) and the estimates in Lemma 5.1, we have \(\nabla \cdot (\overline{P}_{h, \delta t}(t + \delta t) - \overline{P}_{h, \delta t}(t))\) is uniformly bounded in \(L^2([0, T] ; L^2)\), and we can extract a subsequence (denoted as the original one for simplicity) such that for some \(p_*^\gamma \in L^2([0, T] ; L^2)\), \(\nabla \cdot (\overline{P}_{h, \delta t}(t + \delta t) - \overline{P}_{h, \delta t}(t)) \to p_*^\gamma\) weakly in \(L^2([0, T] ; L^2)\) as \(h, \delta t \to 0\). Since we know \(\overline{P}_{h, \delta t}(t + \delta t) - \overline{P}_{h, \delta t}(t) \to 0\) weakly in \(L^2([0, T] ; L^2(\Omega)^d)\) as \(h, \delta t \to 0\), we identify \(p_*^\gamma = 0\). Thus, passing to the limit in (5.31) as \(h, \delta t \to 0\), we get

\[
\int_0^T [(\nabla \cdot u_*, q)] \eta(t) \, dt = 0,
\]

and then \(u_*\) fulfills (2.18).

Now, we have proved that \((\phi_*, w_*, u_*, p_*)\) is a weak solution of (2.15)-(2.18).

Step 3. Under the assumption that the system (1.1)-(1.5) admits a unique solution, the convergence results in Theorem 5.1 can be obtained by the same arguments in [4] and the detail is omitted here. \(\square\)

6. Concluding remarks

We derived rigorously in this paper error estimates for a fully discretized energy stable scheme of a Cahn-Hilliard phase-field model for two-phase incompressible flow. The full discretization is based on a finite element discretization to the weakly coupled, linear, energy stable scheme introduced in [24]. The main difficulties for the error analysis were introduced by the splitting error in the projection step and the nonlinear coupling between the phase function and velocity. We derived optimal convergence rates for both phase functions and velocity in the \(H^1\) norm and pressure in the \(L^2\) norm, and established qualitative convergence of the numerical solution towards the weak solution of the continuous problem under minimal regularity assumption. To the best of our knowledge, this is the first rigorous error analysis
for a fully discrete scheme involving a projection step for a phase-field model of two phase flows.

REFERENCES


