PATTERN FORMATIONS OF 2D RAYLEIGH–BÉNARD CONVECTION WITH NO-SLIP BOUNDARY CONDITIONS FOR THE VELOCITY AT THE CRITICAL LENGTH SCALES

TAYLAN SENGUL, JIE SHEN, SHOUHONG WANG

Abstract. We study the Rayleigh-Bénard convection in a 2-D rectangular domain with no-slip boundary conditions for the velocity. The main mathematical challenge is due to the no-slip boundary conditions, since the separation of variables for the linear eigenvalue problem which works in the free-slip case is no longer possible. It is well known that as the Rayleigh number crosses a critical threshold $R_c$, the system bifurcates to an attractor, which is an $(m-1)$-dimensional sphere, where $m$ is the number of eigenvalues which cross zero as $R$ crosses $R_c$. The main objective of this article is to derive a full classification of the structure of this bifurcated attractor when $m = 2$. More precisely, we rigorously prove that when $m = 2$, the bifurcated attractor is homeomorphic to a one-dimensional circle consisting of exactly four or eight steady states and their connecting heteroclinic orbits. In addition, we show that the mixed modes can be stable steady states for small Prandtl numbers.

1. Introduction

The Rayleigh-Bénard convection problem is one of the fundamental problems in the physics of fluids. The basic phenomena of the Rayleigh-Bénard convection in horizontally extended systems are widely known. The influence of the side walls, although not studied as thoroughly as the horizontally extended case, is of practical importance for engineering applications.

In this paper we study the Rayleigh-Bénard convection in a 2-D rectangular domain with no-slip boundary conditions for the velocity. This problem is also closely related to the problem of infinite channel with rectangular cross-section which has been studied by Davies-Jones [DJ70], Luijkx-Platten [LP81] and Kato-Fujimura [KF00] among others.
The linear aspects of the problem we consider in this paper have been studied by Lee–Schultz–Boyd [LSB89], Mizushima [Miz95] and Gelfgat [Gel99]. In these papers, the critical Rayleigh number and the structure of the critical eigenmodes have been studied for small aspect ratio containers.

From dynamical transition and pattern formation point of view, Ma and Wang [MW04, MW07] proved that under some general boundary conditions, the problem always undergoes a dynamic transition to an attractor $\Sigma_R$ as the Rayleigh number $R$ crosses the first critical Rayleigh number $R_c$. They also proved that the bifurcated attractor, homological to $S^{m-1}$, where $m$ is the number of critical eigenmodes.

In the 2-D setting that we consider, $m$ is either 1 or 2 and the latter case can only happen at the critical length scales where two modes with wave numbers $k$ and $k+1$ become critical simultaneously. When $m = 1$, the structure of $\Sigma_R$ is trivial which is merely a disjoint union of two attracting steady states. Thus our task in this paper is to classify the structure of the attractor when $m = 2$. This has been studied recently in [SW13] for the 3D Rayleigh-Bénard problem where the boundaries were assumed to be free-slip for the velocity and the wave numbers of the critical modes were assumed to be equal.

The main mathematical challenge in this paper is due to the no-slip boundary conditions since the separation of variables for the linear eigenvalue problem which works in the free-slip case is not possible anymore. To overcome this difficulty, the main approach for our study is to combine rigorous analysis and numerical computation using spectral method.

As we know, spectral methods have long been used to address the hydrodynamic instability problems. In fact, in his seminal work [Ors71], Orszag studied the classical Orr-Sommerfeld linear instability problem using a Chebyshev-tau method. In this paper, to treat the linear eigenvalue problem, we employ a Legendre-Galerkin method where compact combinations of Legendre polynomials, called generalized Jacobi polynomials, satisfying all the boundary conditions are used as trial functions. The main advantage of our Legendre-Galerkin method is that the resulting matrices are sparse which allows a very efficient and accurate solution of the linearized problem; see also Hill–Straughan [HS06] and Gheorghiu–Dragomirescu [GD09].

Once the eigenpairs of the linear problem are identified, the transition analysis is carried out by reducing the infinite dimensional system to the center manifold in the first two critical eigendirections. The coefficients of this reduced system are calculated numerically. Our main results are described below.
We first classify the eigenmodes into four classes according to their parities using the symmetry of the problem. Then we numerically show that the first two unstable modes are always parity class one or two. Then we study the transition near the critical length scales where two eigenvalues become positive simultaneously. Next, we rigorously prove that the local attractor at small supercritical Rayleigh numbers is in fact homeomorphic to the circle which has four or eight steady states with half of them as stable points and the rest as saddle points. The critical eigenmodes are always bifurcated steady states on the attractor and when the attractor has eight steady states, the mixed modes which are superpositions of the critical eigenmodes are also bifurcated.

Second, let $\beta_1$ and $\beta_2$ denote the two largest eigenvalues of the linearized problem. We find that a small neighborhood of $\beta_1 = \beta_2 = 0$ in the $\beta_1 - \beta_2$ plane can be separated into several sectors with different asymptotical structures. In particular, we find that there is a critical Prandtl number $Pr_c$ for the first two critical length scales $L = 1.5702$ and $L = 2.6611$, such that for $Pr < Pr_c$, there is a sector in this plane for which mixed modes are stable fixed points of the attractor. For $Pr > Pr_c$, the mixed modes are never stable and instead there is a sector in this plane in which both of the critical eigenmodes coexist as stable steady states. In this case, the initial conditions determine which one of these eigenmodes will be realized. The critical Prandtl number is around 0.14 for the first critical length scale $L = 1.5702$ and around 0.05 for the second critical length scale $L = 2.6611$. For higher critical length scales we found that mixed modes are never stable points of the attractor.

Third, recently Ma–Wang has developed the dynamic transition theory to study transition and bifurcation problems in nonlinear sciences; see [MW]. This paper is a first attempt to combine this theory with the numerical tools of the spectral methods to study the detailed structure of the transition and pattern formation.

The paper is organized as follows. In Section 2, the governing equations and the functional setting of the problem is discussed. In Section 3, linear eigenvalue problem is studied. Section 4 states the main theorem. Section 5 is devoted to the proof of the main theorem. In Section 6, we demonstrate a method to compute the coefficients of the reduced system. And the last section discusses the results obtained by our analysis.
2. Governing Equations and the Functional Setting

Two dimensional thermal convection with no-slip, perfectly conducting boundaries can be modeled by the Boussinesq equations. The governing equations on the rectangular domain $\Omega = (0, L) \times (0, 1) \subseteq \mathbb{R}^2$ read as

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= \Pr(-\nabla p + \Delta u + R \theta \cdot k), \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta &= w + \Delta \theta, \\
\nabla \cdot u &= 0.
\end{align*}
\]

Here $u = (u, w)$ is the velocity field, $\theta$ is the temperature field and $p$ is the pressure field. These fields represent a perturbation around the motionless state with a linear temperature profile. The dimensionless numbers are the Prandtl number $\Pr$ and the Rayleigh number $R$ which is also the control parameter. $k$ represents the unit vector in the z-direction.

With the change of variable

$$\theta = \sqrt{R} \sqrt{\Pr} \theta',$$

and ignoring the primes, the equations (1) become

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\Pr(\nabla p + \Delta u) + \sqrt{R} \sqrt{\Pr} \theta \cdot k, \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta &= \sqrt{R} \sqrt{\Pr} w + \Delta \theta, \\
\nabla \cdot u &= 0.
\end{align*}
\]

The equations (2) are supplemented with no-slip boundary conditions for the velocity and perfectly conducting boundary conditions for the temperature.

\[
\begin{align*}
u = \theta &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

Finally we consider the initial conditions for the velocity and temperature deviation.

\[
\begin{align*}
u(0) &= u_0, \quad \theta(0) = \theta_0.
\end{align*}
\]

For the functional setting, we define the relevant function spaces:

\[
\begin{align*}
H &= \left\{(u, \theta) \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0, u \cdot n |_{\partial \Omega} = 0\right\}, \\
H_1 &= \left\{(u, \theta) \in H^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0, u |_{\partial \Omega} = 0, \theta |_{\partial \Omega} = 0\right\}.
\end{align*}
\]
For $\phi = (u, \theta)$, let $G : H_1 \to H$ and $L_R : H_1 \to H$ be defined by

$$L_R\phi = \begin{bmatrix} \mathcal{P}(\text{Pr}\Delta u + \sqrt{R}\sqrt{\text{Pr}} \theta k) \\ \sqrt{R}\sqrt{\text{Pr}} w + \Delta \theta \end{bmatrix},$$

$$G(\phi) = -\begin{bmatrix} \mathcal{P}(u \cdot \nabla)u \\ (u \cdot \nabla)\theta \end{bmatrix},$$

with $\mathcal{P}$ denoting the Leray projection onto the divergence-free vectors.

The equations (2)–(4) can be put into the following abstract ordinary differential equation:

$$\frac{d\phi}{dt} = L_R\phi + G(\phi), \quad \phi(0) = \phi_0.$$  (7)

The results concerning existence and uniqueness of (7) are classical and we refer the interested readers to Foias, Manley, and Temam [FMT87] for details.

Finally for $\phi_i = (u_i, \theta_i)$, $u_i = (u_i, w_i)$, $i = 1, 2, 3$ we define the following trilinear forms.

$$G(\phi_1, \phi_2, \phi_3) = -\int_\Omega (u_1 \cdot \nabla)u_2 \cdot u_3 dx dz - \int_\Omega (u_1 \cdot \nabla)\theta_2 \cdot \theta_3 dx dz,$$

$$G_s(\phi_1, \phi_2, \phi_3) = G(\phi_1, \phi_2, \phi_3) + G(\phi_2, \phi_1, \phi_3).$$  (8)

3. Linear Analysis

We first study the eigenvalue problem $L_R\phi = \beta\phi$ which reads as

$$\text{Pr}(\Delta u - \frac{\partial p}{\partial x}) = \beta u,$$

$$\text{Pr}(\Delta w - \frac{\partial p}{\partial z}) + \sqrt{R}\sqrt{\text{Pr}} \theta = \beta w,$$

$$\Delta \theta + \sqrt{R}\sqrt{\text{Pr}} w = \beta \theta,$$

$$\text{div} u = 0,$$

$$u = \theta = 0, \quad \text{at } \partial \Omega.$$  (9)

Below we list some of the properties of this eigenvalue problem.

1) The linear operator $L_R$ is symmetric. Hence the eigenvalues $\beta_i$ are real and the eigenfunctions $\phi_i$ are orthogonal with respect to $L^2$–inner product. Moreover there is a sequence

$$0 < R_1 \leq R_2 \leq \cdots$$

such that $\beta_i(R_i) = 0$. $R_i$ is found by setting $\beta = 0$ in (9). In this case the problem becomes an eigenvalue problem with $\sqrt{R}$ as the eigenvalue.
2) The following principal of exchange of stabilities hold true.

\[
\beta_i(R) \begin{cases} 
< 0 & \text{if } R < R_i, \\
0 & \text{if } R = R_i, \\
> 0 & \text{if } R > R_i.
\end{cases}
\]

(10) can be seen by computing the derivative of \(\beta_i\) with respect to \(R\) at \(R = R_i\).

\[
\frac{d\beta_i}{dR} \bigg|_{R=R_i} = \frac{1}{\sqrt{R_i}} \frac{\sqrt{\Pr} \int_{\Omega} \theta_i w_i}{\int_{\Omega} w_i^2 + w_i^2 + \theta_i^2},
\]

where \((u_i, w_i, \theta_i)\) is the \(i\)th eigenfunction. Also at \(R = R_i\), by the third equation in (9), \(w_i = -R_i^{-1/2} \Pr^{-1/2} \Delta \theta_i\) as \(\beta_i(R_i) = 0\).

Plugging these into (11) and integrating by parts, we see that

\[
\frac{d\beta_i}{dR} \bigg|_{R=R_i} = \frac{1}{R_i} \frac{\int_{\Omega} |\nabla \theta_i|^2 dx}{\int_{\Omega} |u_i|^2 + |w_i|^2 + |\theta_i|^2 dx} > 0
\]

3) We denote the critical Rayleigh number \(R_c = R_1\). That is

\[
\beta_i(R) \begin{cases} 
< 0 & \text{if } R < R_c, \\
0 & \text{if } R = R_c, \quad i = 1, \ldots, m \\
> 0 & \text{if } R > R_c. 
\end{cases}
\]

(12) m in (12) does not depend on the Prandtl number \(\Pr\) but only on \(L\). To see this, one makes the change of variable \(\theta = \sqrt{\Pr} \theta'\) so that the solution of (9) for the eigenvalue of \(\beta = 0\) is independent of \(\Pr\). By simplicity of the first eigenvalue (see Theorem 3.7 in Ma–Wang [MW05]), for almost every value of \(L\) except a discrete set of values, \(m\) in (12) is 1.

Introducing the streamfunction \(\psi_z = u, \psi_x = -w\), we can eliminate the pressure \(p\) from the linear eigenvalue problem (9).

\[
\Pr \Delta^2 \psi - \sqrt{R} \sqrt{\Pr} \theta_2 = \beta(R) \Delta \psi,
\]

(13)

\[-\sqrt{R} \sqrt{\Pr} \psi_x + \Delta \theta = \beta(R) \theta,\]

\[
\psi = \frac{\partial \psi}{\partial n} = \theta = 0 \text{ on } \partial \Omega.
\]

The linear equations (9) satisfy several discrete symmetries which may be found from the known non-trivial groups of continuous Lie symmetries of the field equations (2); see (Hydon [Hyd00], Marques–Lopez–Blackburn [MLB04]). However, for the problem we consider, it
can be easily verified that the linear equations have reflection symmetries about the horizontal and vertical mid-planes of the domain. Thus we can classify the solutions of the linear problem into four classes with different parities which are as defined in Table 1 where, for example, \( \psi(o, e) \) means that \( \psi \) is odd in the \( x \)-direction and even in the \( z \)-direction.

The eigenvalue problem can not be solved by the separation of variables and we need to solve it numerically. For this, first we transform the domain with the change of variable

\[
(x, z) \in (0, L) \times (0, 1) \rightarrow (X, Z) = \left( \frac{2x}{L} - 1, 2z - 1 \right) \in (-1, 1)^2.
\]

Then the equations (13) become

\[
\begin{align*}
\Pr \left( \frac{4}{L^4} \partial_{XXX} + \frac{5}{L^2} \partial_{XXXZ} + 2^4 \partial_{ZZZZ} \right) \psi - \frac{2}{L} \sqrt{R \Pr} \theta_X &= \beta(R) \left( \frac{2}{L^2} \partial_{XX} + 2^2 \partial_{ZZ} \right) \psi, \\
\left( \frac{2^2}{L^2} \partial_{XX} + 2^2 \partial_{ZZ} \right) \theta - \frac{2}{L} \sqrt{R \Pr} \psi_X &= \beta \theta, \\
\psi &= \frac{\partial \psi}{\partial n} = \theta = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Slightly abusing the notation, we use \( x \) for \( X \) and \( z \) for \( Z \).

We will employ a Legendre-Galerkin method (cf. Shen [She94], Shen–Tang–Wang [STW11]) to solve the linear eigenvalue problem (14). The approximate solutions \((\psi^N, \theta^N)\) of (13) will be sought in the finite dimensional space

\[
X^N = \text{span}\{(e_j(x)e_k(z), f_l(x)f_m(z)) \mid j, k, l, m = 0, 1, \ldots, N - 1\},
\]

<table>
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<tr>
<th>Parity Class</th>
<th>Parity of Disturbance</th>
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<tr>
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</tr>
<tr>
<td>2</td>
<td>( \psi(o, e), \theta(e, e) )</td>
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<td>( \psi(e, o), \theta(o, o) )</td>
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<td>4</td>
<td>( \psi(o, o), \theta(e, o) )</td>
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Table 1. Possible parity classes of the eigenfunctions of the linear operator.
where \( e_i \) and \( f_i \) are generalized Jacobi polynomials (cf. Guo–Shen–Wang [GSW06], Shen–Tang–Wang [STW11]) which satisfy the boundary conditions
\[
e_i(\pm 1) = De_i(\pm 1) = f_i(\pm 1) = 0.
\]
Here \( D \) denotes the derivative. The polynomials \( e_i \) and \( f_i \) are defined as in Chapter 6 of Shen–Tang–Wang [STW11],
\[
(15) \quad f_i(z) = L_i(z) - L_{i+2}(z),
\]
\[
(16) \quad e_i(z) = \frac{1}{(2i+3)(4i+10)^{1/2}} \left( L_i(z) - \frac{4i+10}{2i+7} L_{i+2}(z) + \frac{2i+3}{2i+7} L_{i+4}(z) \right),
\]
where \( L_i \) is the \( i \)th Legendre polynomial. The coefficient of \( e_i \) guarantees that \( (D^2 e_i, D^2 e_j) = \delta_{ij} \).

We write the approximate solutions of the equation (14) with coefficients to be determined by
\[
(17) \quad \psi^N = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_z-1} \tilde{\psi}_{jk}^N e_j(x)e_k(z), \quad \theta^N = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_z-1} \tilde{\theta}_{jk}^N f_j(x)f_k(z).
\]
Here \( N = 2N_xN_z \) denotes the total degrees of freedom.

Let us define for \( i, j = 0, \ldots, m - 1, \)
\[
(A^m_1)_{ij} = (D^2 e_i, D^2 e_j) = \delta_{ij},
\]
\[
(A^m_2)_{ij} = (D^2 e_i, e_j) = -(De_j, De_i),
\]
\[
(A^m_3)_{ij} = (e_i, e_j),
\]
\[
(A^m_4)_{ij} = (e_i, f_j),
\]
\[
(A^m_5)_{ij} = (D^2 f_i, f_j) = -(Df_i, Df_j),
\]
\[
(A^m_6)_{ij} = (f_i, f_j),
\]
\[
(A^m_7)_{ij} = (Df_i, e_j),
\]
and
\[
\tilde{\psi}^N = \{ \tilde{\psi}_{jk}^N \}, \quad j = 0, \ldots, N_x - 1, k = 0, \ldots, N_z - 1,
\]
\[
\tilde{\theta}^N = \{ \tilde{\theta}_{jk}^N \}, \quad j = 0, \ldots, N_x - 1, k = 0, \ldots, N_z - 1.
\]
The matrices \( A^m_1, \ldots, A^m_7 \) are banded, and except for \( A^m_4 \) and \( A^m_7 \), they are symmetric. Using the properties of the Legendre polynomials, the coefficients of these matrices can be determined explicitly and they are provided in the Appendix.

We put (17) into (14), multiply the resulting equations by \( e_m(x)e_n(z) \) and \( f_m(x)f_n(z) \) respectively and integrate over \(-1 \leq x \leq 1, -1 \leq z \leq 1\).
1 to obtain

\[
B^N \bar{x}^N - \sqrt{R}C^N \bar{x}^N = \beta^N(R)D^N \bar{x}^N.
\]

Here:

\[
B^N = \begin{bmatrix}
Pr X_1 & 0 \\
0 & X_3
\end{bmatrix}_{N \times N},
C^N = \begin{bmatrix}
0 & \sqrt{Pr} X_2 \\
-\sqrt{Pr} X_2^T & 0
\end{bmatrix}_{N \times N},
D^N = \begin{bmatrix}
X_4 & 0 \\
0 & X_5
\end{bmatrix}_{N \times N},
\bar{x}^N = \begin{bmatrix}
\text{vec}(\tilde{\psi}^N) \\
\text{vec}(\tilde{\theta}^N)
\end{bmatrix}_{N \times 1},
\]

where

\[
X_1 = \frac{2^4}{L^4} A_3^{N_z} \otimes A_1^{N_z} + \frac{2^5}{L^2} A_2^{N_z} \otimes A_2^{N_z} + 2^4 A_1^{N_z} \otimes A_3^{N_z},
X_2 = \frac{2}{L} A_4^{N_z} \otimes (A_7^{N_z})^T,
X_3 = \frac{2^2}{L^2} A_6^{N_z} \otimes A_5^{N_z} + 2^2 A_6^{N_z} \otimes A_6^{N_z},
X_4 = \frac{2^2}{L^2} A_3^{N_z} \otimes A_2^{N_z} + 2^2 A_2^{N_z} \otimes A_3^{N_z},
X_5 = A_6^{N_z} \otimes A_6^{N_z}.
\]

In (19) and (20) we use the following notations. For a \( m \times k \) matrix \( M \), \( \text{vec}(M) \) is the \( mk \times 1 \) column vector obtained by concatenating the columns \( M_i \) of \( M \), i.e.

\[
\text{vec}\left( \begin{bmatrix}
M_1 & M_2 & \cdots & M_k
\end{bmatrix} \right) = \begin{bmatrix}
M_1 \\
M_2 \\
\vdots \\
M_k
\end{bmatrix}.
\]

0 stands for the zero matrix and \( A \otimes B = \{a_{ij} B\}_{i,j=0,1,\ldots,q} \) is the Kronecker product of \( A \) and \( B \). To obtain (18), we used the following properties of the Kronecker product.

\[
\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X),
(A \otimes B)^T = A^T \otimes B^T.
\]

We note that the matrices \( B^N \), \( C^N \) and \( D^N \) in (18) are sparse, \( B^N \) and \( D^N \) are symmetric while \( C^N \) is skew-symmetric.

From our linear analysis, we find the following results.
Figure 1.
$\psi$ (on the left) and $\theta$ (on the right) of the first critical mode for
different length scales $L = 1, \ldots, 5$.

- Our numerical analysis suggest that $N_x = 6 + 2k \approx 6 + 2L$ and $N_z = 8$ is enough to resolve the critical Rayleigh number and the first critical mode which has $k$ rolls in its stream function. We have checked that increasing $N_x$ and $N_z$ by two only modifies the fourth or fifth significant digit of the result.

- In Figure 1, the first critical mode is shown for the length scales $L = 1, \ldots, 5$. Note that the first critical stream function and the temperature distribution has always even parity in the $z$-direction while their $x$-parity alternates between odd and even as the length scale increases. As observed in Mizushima [Miz95], we also verify the existence of the Moffatt vortices on the corners of the domain which are due to corner singularities as shown in Figure 2.

- For $L < 21$ we observed that $m$ in (12) is either 1 or 2. Moreover, $m = 2$ only at the critical length scales which are given in Table 2. The results found are in agreement with those in Mizushima [Miz95] and Lee-Schultz-Boyd [LSB89].

- The marginal stability curves of the first few critical eigenvalues are given in Figure 3 and Figure 4. In Figure 3 we also show how the wave numbers of the first two critical modes change with changing length scale $L$. Figure 4 demonstrates that the parities of the first two critical modes can only be of parity class one or two as given by Table 1.

The left figure shows the plot of the first critical stream function for $L = 1$. The right figure shows the enlarged plot at the corner.

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<td>26</td>
<td>8</td>
<td>20.9197</td>
<td>20</td>
<td>1713.5226</td>
<td>46</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 2. At $L = L_c$, two modes become unstable. One of the modes has $k$ and the other one has $k + 1$ rolls in $x$-direction in their stream functions. The critical Rayleigh number at this length scale is $R_c$. $N_x$ and $N_z$ are the number of polynomials used in the $x$ and $z$ directions respectively.
Figure 3. The marginal stability curves of the first few eigenvalues. The numbers with arrows indicate the number of rolls of the critical streamfunction. Light/dark color indicates odd/even number of rolls in the streamfunction.

Also it is seen in these figures that there is a repulsion of the eigenvalues. Namely the neutral stability curves of the same parity type do not intersect each other. Such a repulsion does not occur for free-slip boundary conditions. This repulsion arises from a structural instability of the transform of matrices into a Jordan canonical form and a detailed analysis can be found in Mizushima–Nakamura [MN02].

Using the data in Table 2 in the range $15 < L < 21$, we can get the following asymptotic expansions:

$$R_c - R_\infty = 2628.09 \times L^{-2.0137}, \quad \text{as } L \to \infty,$$

$$q_c - q_\infty = 5.9984 \times L^{-0.942}, \quad \text{as } L \to \infty,$$

where $R_\infty = 1707.7618$ is the critical Rayleigh number and $q_\infty = 3.117$ is the critical wave length for the horizontally extended case ($L \to \infty$). Here $q_c = \pi L_c/k$ where $k$ is as given in Table 2.
4. Main Theorem

Let $m$ be the number of modes which become critical as the first Rayleigh number $R_c$ is crossed as given by (12). Ma and Wang [MW04, MW07] proved that under some general boundary conditions, the problem has an attractor $\Sigma_R$ which bifurcates from $(0, R_c)$ as $R$ crosses $R_c$. They also proved that the dimension of the bifurcated attractor is $m - 1 \leq \dim(\Sigma_R) \leq m$. When $m = 1$, the structure of $\Sigma_R$ is trivial which is merely a disjoint union of two attracting steady states.

As stated before, in our problem $m$ is either 1 or 2. And the latter case can only happen at a critical length scale $L_c$ where two eigenmodes with consecutive wave numbers become critical.

Numerically, it turns out that the critical Rayleigh numbers for modes with parity 3 or 4 are much greater than those for modes with parity 1 or 2. This can be seen from the Figure 4.

We will assume the following.

\[
\begin{align*}
1. & \quad (\beta_1, \phi_1), (\beta_2, \phi_2) \text{ are the first two critical eigenpairs.} \\
2. & \quad \phi_1 \text{ has wave number } k, \phi_2 \text{ has wave number } k + 1 \\
& \quad \text{where } k \text{ is a positive integer.} \\
3. & \quad \text{One of the eigenmodes } \{\phi_1, \phi_2\} \text{ is of parity class 1,} \\
& \quad \text{while the other is of parity class 2 as given in Table 1.}
\end{align*}
\]

Let $y_1$ and $y_2$ be the amplitudes of $\phi_1$ and $\phi_2$ respectively. Then in the proof of the main theorem, we show that the dynamics of the
system close to $R = R_c$ and $L = L_c$ is governed by the equations

$$\begin{align*}
\frac{dy_1}{dt} &= \beta_1 y_1 + y_1 (a_{11} y_1^2 + a_{13} y_2^2) + o(3), \\
\frac{dy_2}{dt} &= \beta_2 y_2 + y_2 (a_{22} y_2^2 + a_{24} y_1^2) + o(3).
\end{align*}$$

Here $\beta_i$ is the eigenvalue corresponding to mode $i$, and $o(3)$ denotes $o(3) = o(\max |(y_1, y_2)|^3) + O(\max |(y_1, y_2)|^3 \max_{i=1,2} |\beta(R)|)$.

Let us define

$$\begin{align*}
D_1 &= a_{22} \beta_1 - a_{11} \beta_2, \\
D_2 &= a_{13} \beta_2 - a_{24} \beta_1, \\
D_3 &= a_{11} a_{24} - a_{13} a_{22}.
\end{align*}$$

To state our main theorems, we assume the following non-degeneracy conditions

$$a_{11} \neq 0, a_{24} \neq 0, D_1 \neq 0, D_2 \neq 0, D_3 \neq 0.$$ 

Finally, let us define the following.

$$\begin{align*}
\varphi_i &= (-1)^i \phi_1, \quad i = 1, 2, \quad \text{(modes with wavenumber $k$)} \\
\varphi_i &= (-1)^i \phi_2, \quad i = 3, 4, \quad \text{(modes with wavenumber $k + 1$)} \\
\varphi_i &= c_i \phi_1 + d_i \phi_2, \quad i = 5, 6, 7, 8, \quad \text{(mixed modes)}
\end{align*}$$

where $c_5 = c_6 = -c_7 = -c_8$ and $d_5 = -d_6 = d_7 = -d_8$.

**Theorem 4.1.** Under the assumptions (21) and (24), there is an attractor $\Sigma_R$ bifurcating as $R$ crosses $R_c$ which is homeomorphic to the circle $S^1$ when $L$ is sufficiently close to a critical length scale $L_c$. $\Sigma_R$ consists of steady states and their connecting heteroclinic orbits. Moreover depending on the system parameters $L$, $Pr$ and $R$, $\Sigma_R$ can only be one of the structures as shown in the Figure 5.

(i) If $D_1 < 0$, $D_2 < 0$, $D_3 < 0$, $\Sigma_R$ contains eight steady states, four of which are stable and the rest are unstable. Stable steady states are topologically equivalent to the pure modes $\varphi_i$, $i = 1, 2, 3, 4$ and the unstable steady states are topologically equivalent to the mixed modes $\varphi_i$, $i = 5, 6, 7, 8$.

(ii) If $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, $\Sigma_R$ contains eight steady states, four of which are stable and the rest are unstable. Stable steady states are topologically equivalent to the mixed modes $\varphi_i$, $i = 5, 6, 7, 8$ and the unstable steady states are topologically equivalent to the pure modes $\varphi_i$, $i = 1, 2, 3, 4$. 

(iii) If $D_1 < 0$, $D_2 > 0$, $\Sigma_R$ contains four steady states, two of which are stable and the rest are unstable. Stable steady states are topologically equivalent to the pure modes $\phi_i$, $i = 1, 2$ and the unstable steady states are topologically equivalent to the pure modes $\phi_i$, $i = 3, 4$.
(iv) If $D_1 > 0$, $D_2 < 0$, $\Sigma_R$ contains four steady states, two of which are stable and the rest are unstable. Stable steady states are topologically equivalent to the pure modes $\phi_i$, $i = 3, 4$ and the unstable steady states are topologically equivalent to the pure modes $\phi_i$, $i = 1, 2$.

According to Theorem 4.1, the structure of the attractor depends on $D_1$, $D_2$ and $D_3$ which in turn depends on the coefficients of the reduced equations. By (23), $D_3$ has a definite sign whereas $D_1$ and $D_2$ vanish at the criticality $\beta_1 = \beta_2 = 0$. In the proof of Theorem 4.1 we analytically prove that the coefficients $a_{11}$, and $a_{24}$ are negative. Our numerical computations indicate that $a_{13}$ is also always negative. We observed that $a_{22}$ and $D_3$ can be both positive and negative.
That gives three possible cases depending on the signs of $a_{22}$ and $D_3$. In Figure 6, we classify these cases in a small neighborhood of $\beta_1 = \beta_2 = 0$ in the $\beta_1-\beta_2$ plane according to our main theorem and the following observations.

- If $a_{22} > 0$ then $D_3 > 0$, $D_1 > 0$ but $D_2$ changes sign in the first quadrant.
- If $a_{22} < 0$ and $D_3 > 0$ then $D_1$ and $D_2$ changes sign in the first quadrant. Moreover the case where both $D_1 < 0$ and $D_2 < 0$ is not possible.
- If $a_{22} < 0$ and $D_3 < 0$ then again $D_1$ and $D_2$ changes sign in the first quadrant. This time the case where both $D_1 > 0$ and $D_2 > 0$ is not possible.

5. Proof of the Main Theorem

We will give the proof in several steps.

**STEP 1. The Reduced Equations.** When there are two critical modes $\phi_1, \phi_2$, the center manifold is a two dimensional manifold embedded in the infinite dimensional space. We denote the center manifold function by:

$$\Phi = y^2_1 \Phi_1 + y_1 y_2 \Phi_2 + y^2_2 \Phi_3 + o(y^2), \quad \Phi_i = \begin{bmatrix} \Psi_i \\ \Theta_i \end{bmatrix}.$$  

To study the dynamics on the center manifold, we plug in

$$\phi = y_1 \phi_1 + y_2 \phi_2 + y^2_1 \Phi_1 + y_1 y_2 \Phi_2 + y^2_2 \Phi_3 + o(2),$$

into

$$\frac{d\phi}{dt} = L_R \phi + G(\phi),$$

and take the inner product with $\phi_1, \phi_2$ and use the orthogonality of the eigenvectors, thanks to the self-adjointness of the linear operator. The reduced equations read

$$\frac{dy_1}{dt} = \beta_1(R)y_1 + \frac{1}{(\phi_1, \phi_1)}(G(\phi), \phi_1),$$

$$\frac{dy_2}{dt} = \beta_2(R)y_2 + \frac{1}{(\phi_2, \phi_2)}(G(\phi), \phi_2).$$

We normalize the first two eigenfunctions so that

$$(\phi_1, \phi_1) = (\phi_2, \phi_2) = 1.$$
Figure 6. In the above cases we assume that $a_{13} < 0$ which is due to our numerical observations. The arrows on the lines $D_1 = 0, D_2 = 0$ indicate in which directions $D_1$ and $D_2$ increase. First and second eigenmodes correspond to the eigenmodes with wavenumber $k$ and $k + 1$ respectively.

Now if we expand the nonlinear terms in (28), we get

\[
\begin{align*}
\frac{dy_1}{dt} &= \beta_1 y_1 + (a_{11}y_1^3 + a_{12}y_1^2y_2 + a_{13}y_1y_2^2 + a_{14}y_2^3) + o(3), \\
\frac{dy_2}{dt} &= \beta_2 y_2 + (a_{21}y_1^3 + a_{22}y_1^2y_2 + a_{23}y_1y_2^2 + a_{24}y_2^3) + o(3),
\end{align*}
\]
where
\[ a_{k1} = G_s(\phi_1, \Phi_1, \phi_k), \]
\[ a_{k2} = G_s(\phi_1, \Phi_2, \phi_k) + G_s(\phi_2, \Phi_1, \phi_k), \]
\[ a_{k3} = G_s(\phi_1, \Phi_3, \phi_k) + G_s(\phi_2, \Phi_2, \phi_k), \]
\[ a_{k4} = G_s(\phi_2, \Phi_3, \phi_k). \]

**STEP 2. Parities of the center manifold functions.** To compute the center manifold approximation, we will use the following formula which was introduced in Ma–Wang [MW05].

\[ -\mathcal{L}_R \Phi = y_1^2 P_2 G(\phi_1, \phi_1) + y_1 y_2 [P_2 G(\phi_1, \phi_2) + P_2 G(\phi_2, \phi_1)] + y_2^2 P_2 G(\phi_2, \phi_2) + o(2). \]

Here
\[ P_2 : H \to E_2, \]
\[ E_1 = \text{span}\{\phi_1, \phi_2\}, \]
\[ E_2 = E_1^\perp, \]
\[ \mathcal{L}_R = L|_{E_2} : E_2 \to \bar{E}_2. \]

Thus the center manifold functions are the solutions of the following linear non-homoogenous equations.

\[ -\mathcal{L}_R \Phi_1 = P_2 G(\phi_1, \phi_1), \]
\[ -\mathcal{L}_R \Phi_2 = P_2 [G(\phi_1, \phi_2) + G(\phi_2, \phi_1)], \]
\[ -\mathcal{L}_R \Phi_3 = P_2 G(\phi_2, \phi_2). \]

We will need the following basic properties of symmetric functions. Let
\[ X = \{ f \in C(\Omega) | f(-x, z) = \pm f(x, z) \text{ and } f(x, -z) = \pm f(x, z) \}, \]
and let \( s : X \to \{ \pm 1 \}^2 \) denote the parity function:
\[ s(f) = (s_x(f), s_z(f)), \]
where
\[ s_x(f) = \pm 1 \text{ if } f(-x, z) = \pm f(x, z), \]
\[ s_z(f) = \pm 1 \text{ if } f(x, -z) = \pm f(x, z). \]
Lemma 5.1. If \( G = 1 \), \( i,j \)

Under the assumption \( \text{Lemma 5.3.} \)

Note that \( \partial_x f = (-s_x(f), s_z(f)), \) \( \partial_z f = (s_x(f), -s_z(f)) \),

Proof.

(36) \( \int_{\Omega} f(x, y) \, dx \, dy = 0 \) if \( s_x(f) = -1 \) or \( s_z(f) = -1 \),

2. \( s(f + g) = s(f) \) if \( s(f) = s(g) \),

3. \( s(fg) = (s_x(f)s_x(g), s_z(f)s_z(g)) \),

4. \( \partial_x f = (-s_x(f), s_z(f)) \), \( \partial_z f = (s_x(f), -s_z(f)) \),

5. \( s(\Delta f) = s(f) \).

Let us define for \( \phi_i = (u_i, w_i, \theta_i) \), \( i = 1, 2 \) the following.

(34) \( G(\phi_i, \phi_j) = \begin{bmatrix} g_1(\phi_i, \phi_j) \\ g_2(\phi_i, \phi_j) \\ g_3(\phi_i, \phi_j) \end{bmatrix} = \begin{bmatrix} -u_i \frac{\partial u_i}{\partial x} - w_i \frac{\partial w_i}{\partial z} \\ -u_i \frac{\partial w_i}{\partial x} - w_i \frac{\partial w_i}{\partial z} \\ -u_i \frac{\partial \theta_i}{\partial x} - w_i \frac{\partial \theta_i}{\partial z} \end{bmatrix} \).

The following lemma can be proved using the properties listed in (33).

Lemma 5.1. If \( \phi_i = (u_i, v_i, \theta_i) \in X^3 \cap H_1 \), \( i = 1, 2 \) then for \( i, j, k = 1, 2 \),

1) \( -s_1 g_1(\phi_i, \phi_j) = s_1 g_2(\phi_i, \phi_j) = s_1 g_3(\phi_i, \phi_j) = (s_z(w_iw_j), -s_z(w_iw_j)) \).

2) \( s_1 g_k(\phi_i, \phi_j) = s_1 g_k(\phi_j, \phi_i) \).

Hereafter without loss of generality we will assume

(35) \( \phi_1 \) is of parity class 1 and \( \phi_2 \) is of parity class 2,

which are as given in Table 1.

Using the Lemma 5.1 and the properties (33), we can prove

Lemma 5.2. Under the assumption (35),

\( s_1 g_2(\phi_1, \phi_1) = s_1 g_2(\phi_2, \phi_2) = (1, -1), \ s_1 g_2(\phi_1, \phi_2) = (-1, -1). \)

Lemma 5.3. Under the assumption (35), \( P_2 G(\phi_i, \phi_j) = G(\phi_i, \phi_j) \) for \( i, j = 1, 2 \).

Proof. Note that \( P_2 G(\phi_i, \phi_j) = G(\phi_i, \phi_j) \) if \( (G(\phi_i, \phi_j), \phi_k) = 0 \) for \( i, j, k = 1, 2 \).

Now (36)

\( (G(\phi_i, \phi_j), \phi_k) = \int_{\Omega} (g_1(\phi_i, \phi_j)u_k + g_2(\phi_i, \phi_j)w_k + g_3(\phi_i, \phi_j)\theta_k) \, dx \, dy \)

By Lemma 5.1 and Lemma 5.2, \( g_1(\phi_i, \phi_j) \) is even in the \( z \)-direction while \( g_2(\phi_i, \phi_j) \) and \( g_3(\phi_i, \phi_j) \) are odd in the \( z \)-direction. Since \( u_k \) is odd and \( w_k \) and \( \theta_k \) are even in the \( z \)-direction, the integral in (36) must vanish over \( \Omega \). \( \square \)
Thus by the Lemma 5.3 and the equation (32), \( \Phi_i \) \((i = 1, 2, 3)\) are solutions of

\[
- \mathcal{L}_R \Phi_1 = P_2 G(\phi_1, \phi_1) = G(\phi_1, \phi_1),
\]

(37) \[
- \mathcal{L}_R \Phi_2 = P_2 [G(\phi_1, \phi_2) + G(\phi_2, \phi_1)] = G(\phi_1, \phi_2) + G(\phi_2, \phi_1),
\]

\[
- \mathcal{L}_R \Phi_3 = P_2 G(\phi_2, \phi_2) = G(\phi_2, \phi_2).
\]

In other words, the center manifold functions \( \Phi_i = (U_i, V_i, \Theta_i) \) are the solutions of the non-homogenous linear equations with right hand sides given by (37).

\[
\Pr(\Delta U - \frac{\partial P}{\partial x}) = -g_1,
\]

\[
\Pr(\Delta W - \frac{\partial P}{\partial z} + \sqrt{R} \sqrt{Pr} \Theta) = -g_2,
\]

(38) \[
\sqrt{R} \sqrt{Pr} W + \Delta \Theta = -g_3,
\]

\[
\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0,
\]

\[
U = W = \Theta = 0, \text{ on } \partial \Omega.
\]

Using the streamfunction \( \Psi_z = U, \Psi_x = -W \), we eliminate the pressure from (38).

\[
\Pr \Delta^2 \Psi - \sqrt{R} \sqrt{Pr} \frac{\partial^2 \Theta}{\partial x} = -\frac{\partial g_1}{\partial z} + \frac{\partial g_2}{\partial x},
\]

(39) \[
-\sqrt{R} \sqrt{Pr} \frac{\partial \Psi}{\partial x} + \Delta \theta = -g_3,
\]

\[
\Psi = \frac{\partial \Psi}{\partial n} = \Theta = 0 \text{ on } \partial \Omega.
\]

**Lemma 5.4.** Under the assumption (35), the center manifold functions have the parity as given in Table 3.

**Proof.** We can eliminate \( \Theta \) from the first equation of (39) to obtain

\[
\Pr \Delta^3 \Psi - RPr \frac{\partial^2 \Psi}{\partial x^2} = \Delta(-\frac{\partial g_1}{\partial z} + \frac{\partial g_2}{\partial x}) - \sqrt{R} \sqrt{Pr} \frac{\partial g_3}{\partial x},
\]

(40) \[
\Delta \Theta = -g_3 + \sqrt{R} \sqrt{Pr} \frac{\partial \Psi}{\partial x}.
\]

Now using the properties (33), the Lemma 5.1 and Lemma 5.2, we see that \( s(\Psi) = (-s_x(g_2), s_z(g_2)) \) and \( s(\Theta) = s(g_2) \).

A direct computation using the Table 3 shows that the integrands in \( a_{12}, a_{14}, a_{21}, a_{23} \) are all odd functions of \( z \) and hence we have the following.
Table 3. Parities of the first two critical modes and the center manifold functions

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>$\theta_1$</td>
<td>$\psi_2$</td>
<td>$\theta_2$</td>
<td>$\psi_1$</td>
</tr>
<tr>
<td>(e,e)</td>
<td>(o,e)</td>
<td>(o,e)</td>
<td>(e,e)</td>
<td>(o,o)</td>
</tr>
</tbody>
</table>

Lemma 5.5. Under the assumption (35), we have

$$a_{12} = a_{14} = a_{21} = a_{23} = 0,$$

in (30).

By Lemma 5.5, we obtain the reduced equations (22).

**STEP 3. Attractor Bifurcation.** Now, we will prove that the bifurcated attractor is homeomorphic to $S^1$. For this we will need the following result.

Theorem 5.6 (Ma–Wang [MW05]). Let $v$ be a two dimensional $C^r$ ($r \geq 1$) vector field given by

$$v_\lambda(x) = \beta(\lambda) x - h(x, \lambda),$$

for $x \in \mathbb{R}^2$. Here $\beta(\lambda)$ is a continuous function of $\lambda$ satisfying

$$\beta(R) = \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0. \end{cases}$$

and

$$h(x, \lambda) = h_k(x, \lambda) + o(|x|^k),$$

$h_k(\cdot, \lambda)$ is a $k$-multilinear field, $C_1|x|^{k+1} \leq (h_k(x, \lambda), x)$,

for some odd integer $k \geq 3$, and some constant $C_1 > 0$. Then the system

$$\frac{dx}{dt} = v_\lambda(x), \quad x \in \mathbb{R},$$

bifurcates from $(x, \lambda) = (0, \lambda_0)$ to an attractor $\Sigma_\lambda$, which is homeomorphic to $S^1$, for $\lambda_0 < \lambda < \lambda_0 + \epsilon$, for some $\epsilon > 0$. Moreover, one and only of the following is true.

1. $\Sigma_\lambda$ is a periodic orbit.
2. $\Sigma_\lambda$ consists of an infinite number of singular points, or,
3. $\Sigma_\lambda$ contains at most $2(2k+1)$ singular points, consisting of $2N$ saddle points, $2N$ stable node points and $n(\leq 2(k+1) - 4N)$ singular points with index zero.
Now let
\[ h(y_1, y_2) = \begin{bmatrix} y_1(a_{11}y_1^2 + a_{13}y_2^2) \\ y_2(a_{22}y_1^2 + a_{24}y_2^2) \end{bmatrix}. \]

**Lemma 5.7.** Assume that \( \Phi_i \neq 0 \) for \( i = 1, 2, 3 \). Then for any \( y = (y_1, y_2) \),
\[ (h(y), y) = a_{11}y_1^4 + (a_{13} + a_{22})y_1^2y_2^2 + a_{24}y_2^4 \leq C|y|^4, \]
where \( C < 0 \).

**Proof.**
\[ a_{11} = G_s(\phi_1, \Phi_1, \phi_1) = G(\phi_1, \Phi_1, \phi_1) + G(\Phi_1, \phi_1, \phi_1) \]
\[ = G(\phi_1, \Phi_1, \phi_1) = -G(\phi_1, \phi_1, \Phi_1) = -G(\phi_1, \Phi_1) = (\mathcal{L}_R \Phi_1, \Phi_1) \]
Here we used (37) and the following properties of Navier-Stokes non-linearity
\[ G(\phi, \tilde{\phi}, \phi^*) = G(\phi, \phi^*, \tilde{\phi}), \]
\[ G(\phi, \tilde{\phi}, \tilde{\phi}) = 0, \]
and \( -\mathcal{L}_R \Phi_1 = G(\phi_1, \phi_1) \) which is due (37).

If we write
\[ \Phi_j = \sum_{k=3}^{\infty} c_{j,k} \phi_k, \quad j = 1, 2, 3, \]
then for \( j = 1, 2, 3, \)
\[ (\mathcal{L}_R \Phi_j, \Phi_j) = \sum_{k=3}^{\infty} c_{j,k}^2 \beta_k \|\phi_k\|^2 < 0 \]
since \( \beta_k < 0 \) for \( k \geq 3 \) and by assumption there exists \( k \geq 3 \) such that \( c_{1,k} \neq 0 \). In particular, \( a_{11} < 0 \). As in (43), we can show that
\[ a_{24} = G_s(\phi_2, \Phi_3, \phi_2) = (\mathcal{L}_R \Phi_3, \Phi_3) < 0. \]

Now if \( a_{13} + a_{22} < 0 \) then it is easy to prove (42). Assume otherwise. Using (44) and (37), we can write
\[ a_{13} = (G_s(\phi_1, \Phi_3, \phi_1) + G_s(\phi_2, \Phi_2, \phi_1)) \]
\[ = (G(\phi_1, \Phi_3, \phi_1) + G(\Phi_3, \phi_1, \phi_1) + G_s(\phi_2, \Phi_2, \phi_1)) \]
\[ = (-G(\phi_1, \phi_1, \Phi_3) + G_s(\phi_2, \Phi_2, \phi_1)) \]
\[ = (-G(\phi_1, \Phi_3) + G_s(\phi_2, \Phi_2, \phi_1)) \]
\[ = (\mathcal{L}_R \Phi_1, \Phi_3) + G_s(\phi_2, \Phi_2, \phi_1) \]
A similar computation shows
\begin{equation}
    a_{22} = (L_R \Phi_3, \Phi_1) + G_s(\phi_1, \Phi_2, \phi_2).
\end{equation}

Let us define
\begin{equation}
    \alpha = G_s(\phi_1, \Phi_2, \phi_2) + G_s(\phi_2, \Phi_2, \phi_1)
\end{equation}

By (44) and (37),
\begin{equation}
    \alpha = -(G(\phi_1, \phi_2) + G(\phi_2, \phi_1), \Phi_2) = (L_R \Phi_2, \Phi_2)
\end{equation}

Note that \( \alpha < 0 \) by (45). Using Cauchy-Schwarz inequality and the orthogonality of the eigenfunctions,
\begin{equation}
    (L_R \Phi_3, \Phi_1) = \sum_{k=3}^{\infty} \beta_k c_{1,k} c_{3,k} ||\phi_k||^2
    \leq \left( \sum_{k=3}^{\infty} -\beta_k c_{1,k} ||\phi_k||^2 \right)^{1/2} \left( \sum_{k=3}^{\infty} -\beta_k c_{3,k} ||\phi_k||^2 \right)^{1/2}
    = \sqrt{a_{11} a_{24}}.
\end{equation}

Since, \( (L_R \Phi_3, \Phi_1) = (L_R \Phi_1, \Phi_3) \), we have by (46)–(50),
\begin{equation}
    a_{13} + a_{22} < 2\sqrt{a_{11} a_{24} + \alpha},
\end{equation}

where \( \alpha < 0 \) is given by (48). Thus, there exists \( 0 < \epsilon_1 < -a_{11}, 0 < \epsilon_2 < -a_{24} \) such that
\begin{equation}
    a_{13} + a_{22} < 2\sqrt{a_{11} a_{24} + \alpha} < 2\sqrt{(a_{11} + \epsilon_1)(a_{24} + \epsilon_2)}.
\end{equation}

Since \( 2ab < a^2 + b^2 \), we have,
\begin{equation}
    2\sqrt{(a_{11} + \epsilon_1)(a_{24} + \epsilon_2)} y_1^2 y_2^2 \leq -(a_{11} + \epsilon_1) y_1^4 - (a_{24} + \epsilon_2) y_2^4.
\end{equation}

Now, let \( C = \max\{\epsilon_1, -\epsilon_2\} \). Then \( C < 0 \) and we have
\begin{equation}
    (h(y), y) \leq a_{11} y_1^4 + (a_{13} + a_{22}) y_1^2 y_2^2 + a_{24} y_2^4 \leq -\epsilon_1 y_1^4 - \epsilon_2 y_2^4
    \leq C(x^2 + y^2)^2.
\end{equation}

That finishes the proof. \( \square \)

Thus by Theorem 5.6 and Lemma 5.7, \( \Sigma_R \) is homeomorphic to \( S^1 \). Now we will describe the details of its structure by determining the bifurcated steady states and their stabilities.
STEP 4. The steady states and their stabilities. The possible equilibrium solutions of the truncated equations of (22) are as follows.

\[ R_1 = \left( \sqrt{\frac{\beta_1}{-a_{11}}}, 0 \right), \]

\[ R_2 = \left( 0, \sqrt{\frac{\beta_2}{-a_{24}}} \right), \]

\[ M = \left( \sqrt{\frac{D_2}{D_3}}, \sqrt{\frac{D_1}{D_3}} \right), \]

where \( D_1, D_2 \) and \( D_3 \) are given by (23).

Due to the invariance of the equations (22) with respect to \((x, y) \rightarrow (-x, y)\) and \((x, y) \rightarrow (x, -y)\), we only consider the positive solutions when writing (51).

The eigenvalues of the truncated vector field at these steady states are

\[ \lambda_{R_1}^1 = -2\beta_1, \quad \lambda_{R_2}^1 = -D_1/a_{11}, \]

\[ \lambda_{R_1}^2 = -2\beta_2, \quad \lambda_{R_2}^2 = -D_2/a_{24}. \]

Note that \( R_i \) is always bifurcated for \( \beta_i > 0, i = 1, 2 \). Moreover \( R_i \) is a stable steady state for \( \beta_i > 0 \) if \( D_i < 0 \) for \( i = 1, 2 \). The trace and the determinant of the Jacobian matrix of the truncated vector field at the mixed states \( M \) are

\[ Tr = \frac{2}{D_3} (a_{24}D_1 + a_{11}D_2), \]

\[ Det = \frac{4}{D_3} D_1 D_2. \]

Notice that the steady states \( M \) are bifurcated only when \( D_1, D_2, D_3 \) have the same sign and according to trace-determinant plane analysis, they are saddles if \( D_1 < 0, D_2 < 0, D_3 < 0 \) and are stable if \( D_1 > 0, D_2 > 0, D_3 > 0 \) since \( a_{11} \) and \( a_{24} \) are both negative as shown in Lemma 5.7.

Finally, only the four cases stated in our main theorem can occur. To see this, note that according to the Theorem 5.6 and (51)–(53), the case \( D_1 < 0, D_2 < 0, D_3 > 0 \) is not possible since that would lead to only 4 steady states on the attractor which are all stable. Similarly the case \( D_1 > 0, D_2 > 0, D_3 < 0 \) is not possible either which would lead to 4 steady states which are all unstable.
6. Numerical approximation of the coefficients of the reduced equations

To compute the coefficients of the reduced equations (22), we fix \( L, \text{Pr} \) and \( R \) to compute all the eigenvalues \( \beta_i^N \) and the corresponding eigenvectors of (18) which are given by

\[
\tilde{x}_i^N = \begin{bmatrix}
\text{vec}(\tilde{\psi}_i^N) \\
\text{vec}(\tilde{\theta}_i^N)
\end{bmatrix}, \quad i = 1, \ldots, N = 2N_xN_z.
\]

Here \( \tilde{\psi}_i^N \) and \( \tilde{\theta}_i^N \) are the matrices of the coefficients of the generalized Jacobi polynomials expansion (17) of the \( i \)-th eigenvector of (18).

Numerical computation of the center manifold functions.

Now we will numerically approximate \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) which are the solutions of the equations (32). We will illustrate the method to approximate \( \Phi_1 \) since \( \Phi_2, \Phi_3 \) can be approximated similarly. To determine \( \Phi_1 \), we have to find its stream function \( \Psi \) and its temperature function \( \Theta \).

By (39), \( \Psi \) and \( \Theta \) are the solutions of the equation

\[
\text{Pr}\Delta^2 \Psi - \sqrt{R}/\sqrt{\text{Pr}} \frac{\partial \Theta}{\partial x} = h_1,
\]

\[
-\sqrt{R}/\sqrt{\text{Pr}} \frac{\partial \Psi}{\partial x} + \Delta \theta = h_2,
\]

\[
\Psi = \frac{\partial \Psi}{\partial n} = \Theta = 0 \text{ on } \partial \Omega.
\]

where

\[
h_1 = -\frac{\partial g_1(\phi_1, \phi_1)}{\partial z} + \frac{\partial g_2(\phi_1, \phi_1)}{\partial x},
\]

\[
h_2 = -g_3(\phi_1, \phi_1),
\]

and \( g_i(\phi_1, \phi_1), i = 1, 2, 3 \) are given by (34).

The Legendre-Galerkin approximation of the problem (55).

As in the linear eigenvalue problem, we discretize the equations (55) using the generalized Jacobi polynomials (15)–(16).

\[
\Psi^N = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} \tilde{\Psi}_{mn}^N e_m(x)e_n(z),
\]

\[
\Theta^N = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} \tilde{\Theta}_{mn}^N f_m(x)f_n(z).
\]
Since we do not have \( h_1 \) and \( h_2 \) exactly, we approximate them by \( h_1^N, h_2^N \) as below

\[
\begin{align*}
    h_1^N &= -\frac{\partial g_1^N}{\partial z} + \frac{\partial g_2^N}{\partial x}, \\
    h_2^N &= -g_3^N,
\end{align*}
\]

where

\[
\begin{align*}
    g_1^N &= -\psi_{1,z}^N \psi_{1,xx}^N + \psi_{1,x}^N \psi_{1,xz}^N, \\
    g_2^N &= -\psi_{1,x}^N \psi_{1,xx}^N + \psi_{1,x}^N \psi_{1,xz}^N, \\
    g_3^N &= -\psi_{1,z}^N \theta_{1,x}^N + \psi_{1,x}^N \theta_{1,z}^N.
\end{align*}
\]

Here \((\psi_1^N, \theta_1^N)\) is the first critical eigenfunction of the discrete problem \([18]\).

\[
\begin{align*}
    \psi_1^N &= \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} \tilde{\psi}_{1,mn}^N e_m(x) e_n(z), \\
    \theta_1^N &= \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} \tilde{\theta}_{1,mn}^N f_m(x) f_n(z).
\end{align*}
\]

We plug in \(\Psi^N, \Theta^N, h_1^N, h_2^N\) for \(\Psi, \Theta, h_1, h_2\) in \([55]\) and multiply the resulting equations by Jacobi polynomials \(e_j(x) e_k(z), f_j(x) f_k(z)\) and integrate over \(-1 \leq x \leq 1, -1 \leq z \leq 1\) to reduce \([55]\) to the following finite dimensional linear equation

\[
(B^N - \sqrt{R} C^N) \bar{x} = \bar{b}.
\]

Here \(B^N\) and \(C^N\) are given by \([19]\) and

\[
\bar{x} = \begin{bmatrix} \text{vec}(\tilde{\Psi}^N) \\ \text{vec}(\tilde{\Theta}^N) \end{bmatrix},
\]

\[
\bar{b} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} \text{vec}(B_1) \\ \text{vec}(B_2) \end{bmatrix}.
\]

For \(0 \leq j \leq N_x - 1, 0 \leq k \leq N_z - 1,\)

\[
\begin{align*}
    (B_1)_{jk} &= \int_{-1}^{1} \int_{-1}^{1} h_1^N(x, z) e_j(x) e_k(z) dx dz, \\
    (B_2)_{jk} &= \int_{-1}^{1} \int_{-1}^{1} h_2^N(x, z) f_j(x) f_k(z) dx dz,
\end{align*}
\]
Now by (16), $e_j$ is a polynomial of degree $j + 4$ and by (57) and (59), $h_i^N$ is a polynomial of degree at most $(2N_x + 6, 2N_z + 6)$. Thus the above integrands are of degree at most $(3N_x + 9, 3N_z + 9)$. Since the Legendre-Gauss-Lobatto quadrature with $N + 1$ quadrature points is exact for polynomials of degree less or equal than $2N - 1$, the integrals in (62) can be replaced by the following discrete inner products.

\[(B_1)_{jk} = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} h_1^N(x_m, z_n) e_j(x_m) e_k(z_n) \omega_m^x \omega_n^z,\]
\[(B_2)_{jk} = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} h_2^N(x_m, z_n) f_j(x_m) f_k(z_n) \omega_m^x \omega_n^z,\]

Here \(\{x_j, w_j^x\}_{j=0}^{3N_x+5}\) and \(\{z_j, w_j^z\}_{j=0}^{3N_z+5}\) are the Legendre-Gauss-Lobatto points and weights in the x-direction and the z-direction.

To compute $h_i^N(x_m, z_n)$, $i = 1, 2$ in (63), we first compute the physical values $D^{(p)}e_m(x_j)$ and $D^{(p)}f_m(x_j)$ of the generalized Jacobi polynomials and their derivatives. These can be obtained by first differentiating the Jacobi polynomials exactly using the formulas in the Appendix A and then evaluating the results at the collocation points $(x_j, z_k)$. Then using the formulas for $D^{(p)}e_m(x_j)$ and $D^{(p)}f_m(x_j)$, we compute the physical values of the first eigenvector and its derivatives.

\[D_x^{(p)} D_z^{(q)} \psi_1^N(x_j, z_k) = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_z-1} \tilde{\psi}_{1,mn}^N D^{(p)}e_m(x_j) D^{(q)}e_n(z_k),\]
\[D_x^{(p)} D_z^{(q)} \theta_1^N(x_j, z_k) = \sum_{m=1}^{N_x} \sum_{n=1}^{N_z} \tilde{\theta}_{1,mn}^N D^{(p)}f_m(x_j) D^{(q)}f_n(z_k),\]

Finally using formulas (64), we compute $h_1^N$ and $h_2^N$ at the collocation points $(x_j, z_k)$.

**Solution of (60).** Since $\Phi_1 \in E_2 = \text{span}\{\phi_1, \phi_2\}^\perp$, we look for a solution of (60) in the form

\[\bar{x} = \sum_{i=3}^{N} x_i \bar{x}_i,\]

where $\bar{x}_i$ are the eigenvectors of

\[B^N \bar{x}_i - \sqrt{RC}^N \bar{x}_i = \beta_i(R) D^N \bar{x}_i\]
and are given by (54). If we multiply (60) by $(D^N)^{-1}$ and use (66), the
left hand side of (60) becomes
\[(67) \quad \sum_{i=3}^{N} x_i \beta_i(R) \bar{x}_i = (D^N)^{-1}(B^N - \sqrt{RC^N}) \bar{x} = (D^N)^{-1} \bar{b} := \bar{f}\]

We determine $\bar{f}$ from
\[D^N \bar{f} = \bar{b},\]
using Gaussian elimination. Once again using Gaussian elimination, we can find the coefficients $f_i$ in the expansion
\[(68) \quad \bar{f} = \sum_{i=1}^{N} f_i \bar{x}_i.\]

In (68), we see that $f_1 = f_2 = 0$ is necessary for the existence of a solution of (60). The solution $\bar{x}$ in (65) can now be obtained by using (67) and (68).

\[(69) \quad x_i = \frac{f_i}{\beta_i(R)}, \quad i = 3, 4, \ldots, N.\]

Thus the Jacobi expansion coefficients in (56) of the center manifold are given by
\[(70) \quad \text{vec}(\tilde{\Psi}^N) = \sum_{i=3}^{N} \frac{f_i}{\beta_i(R)} \text{vec}(\tilde{\psi}^N_i),\]
\[(71) \quad \text{vec}(\tilde{\Theta}^N) = \sum_{i=3}^{N} \frac{f_i}{\beta_i(R)} \text{vec}(\tilde{\theta}^N_i).\]

**Numerical computation of $a_{ij}$ in (30).** We now show how to compute the coefficient $a_{ij}$ in (30). For illustration, we only compute $a_{11}$ and the other coefficients $a_{13}, a_{22}$ and $a_{24}$ can be computed similarly. We approximate $a_{11}$ by
\[(71) \quad a_{11}^N = G_s(\phi_1^N, \Phi_1^N, \phi_1^N).\]

The integrands in $G_s(\phi_1^N, \Phi_1^N, \phi_1^N)$ are polynomials of degree at most $(3N_x + 9, 3N_z + 9)$. Thus to replace the integrals in (71), one needs again $(\frac{3}{2}N_x + 5, \frac{3}{2}N_z + 5)$ quadrature points and nodes in the numerical inner product.

**Remark 6.1.** We observed that increasing $N_x$ and $N_z$ above $N_x = 10 + 2k$ and $N_z = 8$ only changes $a_{ij}^N$ in the seventh digit when the first critical mode which has $k$ rolls and the second critical mode has $k + 1$ rolls in their stream functions.
7. Numerical Results and Discussion

We list in Appendix B the computed coefficients of the reduced equations for various Pr values ranging from 0.1 to $10^4$ at the first three critical length scales and at the critical Rayleigh numbers which are given in Table 2.

As proved in Theorem 4.1, the coefficients $a_{11}$ and $a_{24}$ are always negative. In our numerical calculations, we encountered that $a_{13}$ is also always negative. But the sign of $a_{22}$ and the sign of $D_3$ depends on $L$ and Pr.

For the first critical length scale $L_c = 1.5702$, we found that $a_{22} > 0$ for $Pr < 0.04$ and $a_{22} < 0$ for $Pr > 0.05$. Thus the transition is as described by Figure 6(a) for $Pr < 0.04$. Also $D_3 > 0$ for $Pr < 0.14$ and becomes negative for $Pr > 0.15$. Hence the transitions are as described in Figure 6(b) for $0.05 < Pr < 0.14$ and as in Figure 6(c) for $Pr > 0.15$. Thus the mixed modes can be stable when $Pr < 0.14$ but only the pure modes are stable steady states when $Pr > 0.15$.

For the second critical length scale $L_c = 2.6611$, we always observed that $a_{22} < 0$. For $Pr < 0.05$, $D_3 > 0$ and for $Pr > 0.06$, $D_3 < 0$. Thus the transition is as described in Figure 6(b) for $Pr < 0.05$ and as described in Figure 6(c) for $Pr > 0.06$. In particular, the mixed modes can be stable when $Pr < 0.05$ but only the pure modes are stable steady states when $Pr > 0.06$.

For higher critical length scales (third and beyond), we found that $a_{22} < 0$ and $D_3 < 0$ for the Prandtl numbers we considered. Thus the transition is as described in Figure 6(c). Either the critical Prandtl number that was observed for the first two critical length scales is now very close to zero or it does not exist at all.

The above analysis depends on the coefficients $a_{ij}$ of the reduced equations and predicts the transitions when both eigenvalues $\beta_1$, $\beta_2$ are close to zero. Now we present an analysis depending on the direct computation of the numbers $D_1$, $D_2$ (both of which vanish when $\beta_1 = \beta_2 = 0$) and $D_3$. We computed $D_1$, $D_2$ and $D_3$ for $L$ and $R$ values around (but not necessarily very close to) the criticality $(L, R) = (L_c, R_c)$ for the first four critical length scales and for Prandtl numbers $Pr = 0.1, 0.71, 7, 130$. The results are shown in Figure 7–10. Although we might have omitted the smallness assumptions of $|L - L_c|$ and $|R - R_c|$ where our main theorem is valid, these figures help us predict the transitions in the $L - R$ plane. The results we obtain are as follows.

For $Pr = 0.71$, $Pr = 7$, $Pr = 130$, transitions are qualitatively same in the $L-R$ plane. For $L > L_c$, the basic motionless state loses its
stability to the eigenmode with wavenumber $k + 1$ as the Rayleigh number crosses the first critical Rayleigh number and further increase of the Rayleigh number does not alter the stability of this steady state. This is in contrast to the situation $L < L_c$ where there is a transition of stabilities as the Rayleigh number is increased. Namely, as the Rayleigh number crosses the first critical Rayleigh number, the eigenmode with wavenumber $k$ becomes stable. As the Rayleigh number is further increased, both eigenmodes coexist as stable steady states and the initial conditions determine which one of these steady states will be realized. Finally as the Rayleigh number is further increased, the eigenmode with wavenumber $k + 1$ becomes stable.

The transition at $Pr = 0.1$ is essentially different than for those at $Pr = 0.71, 7, 130$. In particular, for the first critical length scale $L_c = 1.5702$, for $L < L_c$, subsequently $k$ mode, mixed modes and finally $k + 1$ mode will be realized as the Rayleigh number is increased while for $L > L_c$, $k + 1$ mode is the only stable steady state. The transitions at higher critical length scales are also shown in Figure 7.
Figure 10.

Appendix A. Non-zero Entries of the Matrices $A_j^m$, $j = 1, 2, \cdots, 7$

Using
\[(2i + 3)L_{i+1} = D(L_{i+2} - L_i),\]
it is easy to see that:
\[(73) \quad Dc_i(z) = \frac{L_{i+3} - L_{i+1}}{\sqrt{4i + 10}},\]
and
\[(74) \quad D^2c_i(z) = \sqrt{\frac{2i + 5}{2}}L_{i+2}(z).\]

Hence
\[(D^2c_i, D^2c_j) = \delta_{ij}.\]

Again by (72), one has
\[Df_i = -(2i + 3)L_{i+1},\]
and hence
\[(Df_i, Df_j) = (2i + 3)(2j + 3)(L_{i+1}, L_{j+1}) = 2(2i + 3)\delta_{ij}.\]
For $i, j = 1, \ldots, m$,

$$(A^m_2)_{ij} = \begin{cases} \begin{align*} \frac{1}{\left(7 + 2i\sqrt{(5 + 2i)(9 + 2i)}\right)} & \text{if } j = i + 2, \\ -2\left((3 + 2i)(7 + 2i)\right)^{-1} & \text{if } j = i, \\ (A^m_2)_{ji} & \text{if } j = i - 2, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

$$(A^m_3)_{ij} = \begin{cases} \begin{align*} \frac{1}{\left(\sqrt{(5 + 2i)(13 + 2i)(7 + 2i)(9 + 2i)(11 + 2i)}\right)} & \text{if } j = i + 4, \\ -4\left(\sqrt{(5 + 2i)(9 + 2i)(3 + 2i)(7 + 2i)(11 + 2i)}\right)^{-1} & \text{if } j = i + 2, \\ 6\left((1 + 2i)(3 + 2i)(7 + 2i)(9 + 2i)\right)^{-1} & \text{if } j = i, \\ (A^m_3)_{ji} & \text{if } j = i - 2, \\ (A^m_3)_{ji} & \text{if } j = i - 4, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

$$(A^m_4)_{ij} = \begin{cases} \begin{align*} \sqrt{2}\left(\sqrt{(5 + 2i)(7 + 2i)(9 + 2i)}\right)^{-1} & \text{if } j = i + 4, \\ -3\sqrt{2}\left((3 + 2i)\sqrt{5 + 2i(9 + 2i)}\right)^{-1} & \text{if } j = i + 2, \\ 3\sqrt{2}\left((1 + 2i)\sqrt{5 + 2i(7 + 2i)}\right)^{-1} & \text{if } j = i, \\ -\sqrt{2}\left((1 + 2i)(3 + 2i)\sqrt{5 + 2i}\right)^{-1} & \text{if } j = i - 2, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

$$(A^m_5)_{ij} = \begin{cases} \begin{align*} -4i + 6 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

$$(A^m_6)_{ij} = \begin{cases} \begin{align*} -2(5 + 2i)^{-1} & \text{if } j = i + 2, \\ (12 + 8i)\left((1 + 2i)(5 + 2i)\right)^{-1} & \text{if } j = i, \\ (A^m_6)_{ji} & \text{if } j = i + 2, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

$$(A^m_7)_{ij} = \begin{cases} \begin{align*} -\sqrt{2(7 + 2i)(5 + 2i)^{-1}} & \text{if } j = i + 1, \\ 2\sqrt{(6 + 4i)(5 + 2i)(1 + 2i)^{-1}} & \text{if } j = i - 1, \\ -\sqrt{2(1 + 2i)(-1 + 2i)^{-1/2}} & \text{if } j = i - 1, \\ 0 & \text{otherwise,} \end{align*} \end{cases}$$

**Appendix B. Computed Coefficients of the Reduced Equations**

In this section, we provide the coefficients of the reduced equations at the first three critical length scales for various Prandtl numbers.
<table>
<thead>
<tr>
<th>$\text{Pr}$</th>
<th>$a_{11}$</th>
<th>$a_{13}$</th>
<th>$a_{22}$</th>
<th>$a_{24}$</th>
<th>$D_3$</th>
</tr>
</thead>
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<td>-0.038773</td>
<td>-0.23894</td>
<td>0.076492</td>
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Table 4. The coefficients of the reduced equations for various $\text{Pr}$ values at the first critical length scale $L_c = 1.5702$.

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<th>$\text{Pr}$</th>
<th>$a_{11}$</th>
<th>$a_{13}$</th>
<th>$a_{22}$</th>
<th>$a_{24}$</th>
<th>$D_3$</th>
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<tr>
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<td>-6.0355e-06</td>
<td>-6.0479e-06</td>
<td>-6.1936e-06</td>
<td>-1.0431e-12</td>
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</tbody>
</table>

Table 5. The coefficients of the reduced equations for various $\text{Pr}$ values at the second critical length scale $L_c = 2.6611$. 
Table 6. The coefficients of the reduced equations for various Pr values at the third critical length scale $L_c = 3.7048$.

<table>
<thead>
<tr>
<th>Pr</th>
<th>$a_{11}$</th>
<th>$a_{13}$</th>
<th>$a_{22}$</th>
<th>$a_{24}$</th>
<th>$D_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1e-05</td>
<td>-227.5475</td>
<td>-225.2649</td>
<td>-211.0906</td>
<td>-208.2449</td>
<td>-165.7075</td>
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<tr>
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<td>-22.5178</td>
<td>-21.1016</td>
<td>-20.8168</td>
<td>-1.6554</td>
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<tr>
<td>0.01</td>
<td>-0.22051</td>
<td>-0.218</td>
<td>-0.20518</td>
<td>-0.20214</td>
<td>-0.0001567</td>
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<tr>
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<td>-0.02809</td>
<td>-0.028735</td>
<td>-0.028848</td>
<td>-0.029002</td>
<td>-1.4316e-05</td>
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<td>-0.00059467</td>
<td>-0.0005946</td>
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<td>-6.3084e-09</td>
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<tr>
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<td>-6.0023e-05</td>
<td>-6.0015e-05</td>
<td>-6.1117e-05</td>
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<td>-6.008e-06</td>
<td>-6.0071e-06</td>
<td>-6.1175e-06</td>
<td>-6.4394e-13</td>
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</table>

References


