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ON THE DUAL PETROV-GALERKIN FORMULATION OF THE KDV EQUATION ON A FINITE INTERVAL

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Abstract. An abstract functional framework is developed for the dual Petrov-Galerkin formulation of the initial-boundary-value problems with a third-order spatial derivative. This framework is then applied to study the wellposedness and decay properties of the KdV equation in a finite interval.

1. Introduction

There is a large body of literature on various aspects of the Korteveg de Vries (KdV) equation. Although most of these studies are concerned with initial-value (or periodic initial-value) problems, the initial-boundary-value problems has also received considerable attention. A natural initial-boundary-value problem for the KdV equation is set for x, the spatial variable, to be in a semi-infinite interval (see, for instance, [18, 10, 1, 2] and the references therein). This can be used, for example, to model waves emanating from a wavemaker. However, it is in general difficult to deal with unbounded domains numerically, so one is often interested in using a finite interval with a suitable set of boundary conditions. Even in the absence of accurate transparent boundary conditions, such two-point boundary-value problems can still be used to model, for example, the laboratory studies

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wherein waves are generated by a wavemaker at the left-hand end and propagate down the channel, at least on the time interval before the waves reach the other end of the channel (cf., for instance, [5, 6, 8, 3]).

From a mathematical point of view, it is also of interest to study the mathematical properties of the KdV equation on a finite spatial interval. It is important to point out that there are many fundamental differences between the initial-value problem for the KdV equation and the associated initial-boundary-value problem. Most notably, the solution of the initial-value (or the periodic initial-value) problem for the KdV equation has many conservation properties (e.g., the L^2-norm), while the initial-boundary-value problems may add or dissipate energy at the boundary. Hence, we may want to use different mathematical frameworks to study these two sets of problems.

In a recent work [16], the second author introduced a new framework for the third- and higher-order (including the KdV) equations on a bounded interval. This framework is based on a suitable dual Petrov-Galerkin formulation. The key observation is that the (usually) skew-symmetric differential operator u_{xxx} is actually coercive on a suitable weighted space. It is shown in [16] that the numerical method based on the new dual Petrov-Galerkin formulation leads to very effective numerical algorithms for third- and higher-order differential equations which are usually more difficult to handle numerically. However, although the stability and convergence of the numerical solutions of the finite-dimensional dual Petrov-Galerkin formulation were studied in [16] and [17] assuming that the underlying solutions belong to suitable weighted Sobolev spaces, the wellposedness of the dual Petrov-Galerkin formulation in the infinite-dimensional weighted Sobolev spaces was left as an assumption.

The purpose of this paper is to provide a suitable functional framework to handle the initial-boundary-value problem for the KdV equation in a finite interval. We note that this problem has been studied before in various contexts (see [3] for an updated review and also [8]) where a different set of boundary conditions at the right end were studied. Our point of view differs from the existing theories on this problem. More precisely, we will introduce a suitable functional framework to study the dual Petrov-Galerkin formulation of this problem in weighted Sobolev spaces. This study will not only provide a further justification for the numerical method introduced in [16], but will also provide new insights on the smoothing and decay properties for solutions of the KdV equation in a finite interval.

We would like to remind the readers that the use of suitable weighted spaces, such as $L^2(e^x dx)$, has been known for years for the linear KdV equation (also referred to as the Airy equation in some literature) and the KdV equation on the whole line. We refer here to [11] where the author describes "quasi-parabolic" properties for the KdV equation using suitable weighted spaces, see also [14] where this fact is used. This quasi-parabolic smoothing property is only valid for $t > 0$, and differs from the "global" smoothing property for the KdV equation on the whole line, which asserts that the solution is more regular than the initial condition almost everywhere in t. These properties, used in [11] and in [12], are also valid for negative times. These global smoothing properties read as follows: for initial data in L^2, one can construct a solution that satisfies u_x belonging to $L^2(-T,T;L^2_{loc,x})$. See [12] for numerous other global smoothing properties for the linear KdV equation on the whole line.

We shall now introduce the governing equation.

The governing equations. For the sake of simplicity, we consider the space variable x in $\Omega = (-1,1)$, and the KdV equation in the following form

$$u_t - \beta u_x + u_{xxx} + uu_x = 0,\ (t,x) \in (0,+\infty) \times \Omega \tag{1.1}$$

with β being a parameter. Its linear counterpart reads

$$u_t - \beta u_x + u_{xxx} = 0. \tag{1.2}$$

We supplement (1.1) or (1.2) with an initial condition

$$u(0,x) = u_0(x), \tag{1.3}$$

and boundary conditions

$$u(t,1) = u(t,-1) = u_x(t,1) = 0. \tag{1.4}$$

These conditions ensure that some dissipation occurs at the left boundary. Indeed, by taking the inner product of (1.1) or (1.2) with u and integrating over $(0,t)$, we obtain the formal energy equality

$$\int_\Omega |u(t,x)|^2 dx + \int_0^t |u_x(s,-1)|^2 ds = \int_\Omega |u(0,x)|^2 dx. \tag{1.5}$$

Remark 1.1. For the sake of simplicity, we only consider homogeneous boundary conditions in (1.4) in this paper. From a numerical point of view, problems with nonhomogeneous boundary conditions can be easily handled by introducing a lifting function (cf. [16]). However, a rigorous mathematical treatment for problems with nonhomogeneous boundary conditions is much more involved and will not be addressed here.

Remark 1.2. Let $\gamma = \sqrt{|\beta|}$ and perform the change of variables $u(t,x) = \gamma^2 v(\gamma^3 t, \gamma x)$ in (1.1); we find that v satisfies

$$v_t - \text{sign}(\beta) v_x + v_{xxx} + v v_x = 0,\ x \in (-\gamma, \gamma). \tag{1.6}$$

Therefore, $|\beta|$ is related to the length of the interval in which we solve the KdV equation (1.6).

In the following, we denote by L^p, H^m and $W^{m,p}$ the usual Sobolev spaces on Ω. Let $\omega(x) > 0$ be a generalized weight function which is not necessarily in L^1. We denote

$$L^2(\omega) = \Big\{ u \in L^1_{loc} : \int_\Omega u(x)^2 \omega(x) dx < +\infty \Big\}, \tag{1.7}$$

which is the weighted Sobolev space with norm

$$\|u\|_{L^2(\omega)} = \Big(\int_\Omega u(x)^2 \omega(x) dx \Big)^{1/2}.$$

The rest of the paper is organized as follows. In the next section, we introduce the functional framework for our analysis. In the third section we study the initial-boundary-value problems for the linear KdV equation. In Section 4, we study the nonlinear KdV equation using a fixed-point argument.

2. Preliminary Functional Analysis

In [16], a dual Petrov-Galerkin method was introduced to handle the KdV equation on a finite interval. The analysis began with the stationary problem

$$u_{xxx} = f, \tag{2.1}$$

supplemented with boundary conditions

$$u(1) = u(-1) = u_x(1) = 0. \tag{2.2}$$

The idea was to solve the approximated problem

$$\int_\Omega u_x(x) v_{xx}(x) dx = \int_\Omega f(x) v(x) dx, \tag{2.3}$$

for any polynomials u, v in respectively $V_n = \{u = (1-x)^2(1+x)p_{n-3}(x) : p_{n-3}$ is any polynomial of degree $\leq n-3\}$ and $V_n^* = \{u = (1-x)(1+x)^2 p_{n-3}(x) : p_{n-3}$ is any polynomial of degree $\leq n-3\}$. Introducing the weight function $\omega(x) = \frac{1+x}{1-x}$, (2.3) is equivalent to: find $u \in V_n$ such that

$$a(u, \phi) = \int_\Omega f(x) \phi(x) \omega(x) dx,\ \forall \phi \in V_n, \tag{2.4}$$

where the bilinear form a is

$$a(u,\phi) = \int_\Omega u_x(x)(\phi(x)\omega(x))_{xx}dx,\ \forall u,\phi \in V_n. \tag{2.5}$$

Only finite-dimensional spaces V_n were considered in [16]. We introduce in the sequel an abstract infinite-dimensional framework within which this nonstandard approximation fits.

2.1. **Density results and Hardy inequalities in weighted Sobolev spaces.** The following notation will be used:

$$\begin{aligned} &H = L^2(\omega),\ V = \Big\{u \in H_0^1(\Omega):\ u_x \in L^2(\dot\omega)\Big\},\\ &W = \{u \in V : u_{xx} \in L^2(\frac{\omega^2}{\dot\omega})\}. \end{aligned} \tag{2.6}$$

For $\omega = \frac{1+x}{1-x}$, we have $\dot\omega = \frac{2}{(1-x)^2}$ and $\frac{\omega^2}{\dot\omega} = \frac{(1+x)^2}{2}$.

Lemma 2.1. *The space V endowed with the norm $||u_x||_{L^2(\dot\omega)}$ is a Hilbert space. The embeddings $C_0^\infty(\Omega) \hookrightarrow V \hookrightarrow H$ are dense and continuous. Moreover, the following Hardy inequality is valid:*

$$\|(1-x)^{-2}u\|_{L^2} \le \frac{2}{3}\|(1-x)^{-1}u_x\|_{L^2},\ \forall u \in V. \tag{2.7}$$

Proof. It is clear that $\|u\|_V = \|u_x\|_{L^2(\dot\omega)}$ is a norm on V. We first prove that $C_0^\infty(\Omega)$ is dense in V. Consider u in V such that

$$\int_{-1}^1 u_x(x)\phi_x(x)\frac{dx}{(1-x)^2} = 0$$

for any test function ϕ. Then $\partial_x(\frac{u_x(x)}{(1-x)^2}) = 0$. Therefore, there exist two constants a and b such that $u(x) = a(1-x)^3 + b$. The boundary conditions $u(1) = u(-1) = 0$ lead to $a = b = 0$. Therefore, $C_0^\infty(\Omega)^\perp = \{0\}$.

Given $\phi \in V$, let ϕ_n be a sequence of functions in $C_0^\infty(\Omega)$ converging to ϕ in V. Then, for $h \le 2$ and any p,

$$\begin{aligned} 0 &\le \int_\Omega \Big(\phi_n'(x) + p\frac{\phi_n(x)}{(1-x)}\Big)^2\frac{dx}{(1-x)^h}\\ &= \int_\Omega \frac{\phi_n'(x)^2}{(1-x)^h}dx + 2p\int_\Omega \frac{\phi_n(x)\phi_n'(x)}{(1-x)^{h+1}}dx + p^2\int_\Omega \frac{\phi_n(x)^2}{(1-x)^{h+2}}dx. \end{aligned} \tag{2.8}$$

Integrating by parts in the second term leads to

$$2p\int_\Omega \frac{\phi_n(x)\phi_n'(x)}{(1-x)^{h+1}}dx = -p(h+1)\int_\Omega \frac{\phi_n(x)^2}{(1-x)^{h+2}}dx.$$

Taking $h = 2$ and $p = \frac{3}{2}$, and letting $n \to +\infty$, we derive the desired result. To prove that $V \hookrightarrow H$, we only have to observe that

$$||u||_{L^2(\omega)} \leq \frac{3\sqrt{3}}{4}\|(1-x)^{-2}u\|_{L^2}. \tag{2.9}$$

□

The Hardy inequality (2.7) complements the classical estimate

$$||(1+x)^{-1}u||_{L^2} \leq 2||u_x||_{L^2}, \tag{2.10}$$

that is valid for any function in $H_0^1(\Omega)$.

We now state a similar result for W.

Lemma 2.2. *W endowed with $||u_{xx}||_{L^2((1+x)^2)}$ is a Hilbert space. The embeddings $C_0^\infty(\Omega) \hookrightarrow W \hookrightarrow V$ are dense and continuous. Moreover, the following Hardy inequality is valid*

$$\|(1-x)^{-1}u_x\|_{L^2} \leq \|(1+x)u_{xx}\|_{L^2},\ \forall u \in W. \tag{2.11}$$

Proof. Notice that $W \subset V$ so we first prove that $C_0^\infty(\Omega)$ is dense in W. Consider u in W such that

$$\int_\Omega u_{xx}(x)\phi_{xx}(x)(1+x)^2 dx = 0, \tag{2.12}$$

for any test function ϕ. Straightforward computations lead to $u_{xx}(x) = \frac{a}{1+x} + \frac{b}{(1+x)^2}$. Integrating once leads to

$$u_x(x) = a\ln(x+1) - \frac{b}{(1+x)} + c.$$

The fact that $u_x \in L^2$ implies $b = 0$. Integrating once again, we get

$$u(x) = a(1+x)(\ln(x+1) - 1) + c(1+x) + d.$$

Then the boundary conditions $u(\pm 1) = 0$ lead to $d = 0$ and $a(\ln 2 - 1) + c = 0$. Hence, $u(x) = a(1+x)\ln(\frac{x+1}{2})$. Finally, we derive from (2.7) that $a = 0$, and hence $C_0^\infty(\Omega)^\perp = \{0\}$.

We now prove the Hardy inequality (2.11). For $u \in W$ (or a smooth $C_0^\infty(\Omega)$ approximation of u),

$$\begin{aligned} 0 &\leq \int_\Omega \Big(u_{xx}(x)(1+x) + \frac{u_x}{1-x}\Big)^2 dx \\ &= \int_\Omega \Big(u_{xx}^2(1+x)^2 + (u_x^2)_x\frac{1+x}{1-x} + \frac{u_x^2}{(1-x)^2}\Big)dx. \end{aligned} \tag{2.13}$$

Integrating by parts the second term leads to the desired result. Moreover, (2.11) implies that $||u_{xx}||_{L^2((1+x)^2)}$ is a norm on W. □

2.2. **A generalized Lax-Milgram theorem.** The aim of this subsection is to solve the equation (2.4) for f in H and to define an unbounded operator A by setting $Au = f$. The first difficulty is that we are not in the classical framework defined by a bilinear operator that satisfies the Lax-Milgram theory. To handle this difficulty, we recall the following generalized Lax-Milgram theorem due to J.L. Lions (cf. [13]; see also Lemma 4.4.4.1 in [9]):

Theorem 2.3. *Consider two Hilbert spaces $W \subset V$ with continuous and dense embedding. Consider a bilinear form $a(v, w)$ that is defined on $V \times W$. Assume that there exist $m, M > 0$ such that*

$$a(v, w) \leq M\|v\|_V \|w\|_W, \ \forall v \in V, \ w \in W, \tag{2.14}$$

$$a(w, w) \geq m\|w\|_V^2, \ \forall w \in W. \tag{2.15}$$

Then, for any $f \in V'$ (the dual space of V), there exists $v \in V$ such that

$$a(v, w) = f(w), \ \forall w \in W. \tag{2.16}$$

Remark 2.1. Observe that there is no uniqueness result in this theorem. In fact, there exists at least one solution in V, and at most one solution in W.

Next, we check that our abstract framework satisfies the assumptions of this theorem.

It is seen from (2.5) that

$$a(u, \phi) = \int_\Omega u_x(x)\phi_{xx}(x)\omega(x)dx + 2\int u_x(x)\phi_x(x)\dot{\omega}(x)dx + \int u_x(x)\phi(x)\ddot{\omega}(x)dx, \ \forall u \in V, \ \phi \in W.$$

The first term can be handled as follows:

$$\int_\Omega u_x(x)\phi_{xx}(x)\omega(x)dx \leq ||u||_V ||\phi||_W. \tag{2.17}$$

The second one can be bounded by $C\|u\|_V \|\phi\|_V$. For the third term,

$$\int u_x(x)\phi(x)\ddot{\omega}(x)dx \leq \|u\|_V \|\phi\|_{L^2((1-x)^{-4})} \leq C\|u\|_V \|\phi\|_V. \tag{2.18}$$

Since $W \subset V$, combining these inequalities leads to (2.14).

We now prove the weak coercivity (2.15). Notice that

$$\begin{aligned} a(u,u) &= -\int_\Omega u_{xx}(x)u_x(x)\omega(x)dx - \int u_{xx}(x)u(x)\dot\omega(x)dx \\ &= \frac{3}{2}\int_\Omega u_x^2(x)\dot\omega(x)dx - \frac{1}{2}\int_\Omega u^2(x)\dddot\omega(x)dx, \quad \forall u \in W. \end{aligned} \tag{2.19}$$

Since $\dot\omega = \frac{2}{(1-x)^2}$ and $\dddot\omega = \frac{12}{(1-x)^4}$, we derive from (2.7) that

$$a(u,u) = 3\int_\Omega \frac{u_x^2(x)}{(1-x)^2}dx - 6\int_\Omega \frac{u^2(x)}{(1-x)^4}dx \geq \frac{1}{3}\|u\|_V^2. \tag{2.20}$$

Hence, (2.15) is valid with $m = \frac{1}{3}$.

Proposition 2.4. *For any f in H there exists a unique u in W such that* (2.4) *is valid. Hence, we can set $Au = f$, where A, defined through* (2.4), *is an unbounded operator whose domain is $D(A) = \{u \in W : Au \in H\}$. Moreover, A is maximal dissipative in H in the sense that*

$$\text{Range}(I_d + A) = H, \tag{2.21}$$

(*where I_d is the identity operator*) *and*

$$(Au, u)_H \geq 0, \quad \forall u \in D(A). \tag{2.22}$$

Proof. Applying Theorem 2.3 with $f \in H \subset V'$ leads to the existence of u in V that solves (2.4). To prove the uniqueness, we need to show that u is in W.

Consider f in $C_0^\infty(\Omega)$ and u in V that solves (2.4). Assuming for the moment that the following inequality holds

$$\|u_{xx}\|^2_{L^2(\frac{\omega^2}{\dot\omega})} \leq 2\left(\|f\|^2_{L^2(\omega)} + \|u_x\|^2_{L^2(\dot\omega)}\right), \tag{2.23}$$

then a limiting argument implies that if f is in H, then u is in W, and the uniqueness result follows.

Let us now prove (2.23). Assume that f is in $C_0^\infty(\Omega)$. Then $u_{xxx} = f$. The fact that u is in V implies that $u(x) = a(1-x)^2$ in a neighborhood of $x = 1$, and $u(x) = (cx+d)(1+x)$ in a neighborhood of $x = -1$. Integrating by parts and noticing that $u_x u_{xx}(1+x)^2$ and $u_x^2(1+x)$ vanish at the boundary, we obtain

$$\begin{aligned} 2\|u_{xx}\|^2_{L^2(\frac{\omega^2}{\dot\omega})} &= \int_\Omega u_{xx}^2(x)(1+x)^2dx \\ &= -\int_\Omega u_x(x)u_{xxx}(x)(1+x)^2dx - 2\int_\Omega u_x(x)u_{xx}(x)(1+x)dx \end{aligned}$$

$$= -\int_\Omega u_x(x)u_{xxx}(x)(1+x)^2dx + \int_\Omega u_x^2(x)dx.$$

By the Cauchy-Schwarz inequality,

$$\Big|\int_\Omega u_x(x)u_{xxx}(x)(1+x)^2dx\Big| \le \frac{1}{\sqrt{2}}\|u_{xxx}\|_{L^2(\omega)}\|u_x\|_{L^2(\dot\omega)}$$
$$= \frac{1}{\sqrt{2}}\|f\|_{L^2(\omega)}\|u_x\|_{L^2(\dot\omega)}.$$

The proof of (2.23) is then straightforward.

Applying once again Theorem 2.3 to $Id + A$, we infer (2.21). The dissipativity (2.22) derives from the weak coerciveness (2.15). This completes the proof of the proposition. □

3. Cauchy problem for the linear KdV equation

Due to the results in the previous section, we can now apply the Hille-Yosida theorem to solve

$$u_t + Au = f(t),\ u(0) = u_0 \tag{3.1}$$

with suitable data $f(t)$ and u_0 respectively in $L^2(0,T,V')$ and H.

Proposition 3.1. *The operator A is the infinitesimal generator of a semi-group of contraction e^{-tA} in H, and*

$$\|e^{-tA}u_0\|_{L^2(\omega)} \le e^{-\frac{16t}{9}}\|u_0\|_{L^2(\omega)}. \tag{3.2}$$

For any $f \in L^2(0,T;V')$, there exists a unique strong solution $u \in C([0,T];H) \cap L^2(0,T;V)$ of (3.1). Moreover, the energy inequality

$$\|u(t)\|^2_{L^2(\omega)} + \frac{1}{3}\int_0^t \|u_x(s)\|^2_{L^2(\dot\omega)}ds \le \|u_0\|^2_{L^2(\omega)} + 3\|f\|^2_{L^2(0,T;V')} \tag{3.3}$$

is valid.

Remark 3.1. The inequality (3.3) is very important. It describes a *smoothing effect* for the initial boundary-value problem (3.1) in a suitable weighted space. More precisely, for $u_0 \in H$, $u(t)$ is in V almost everywhere in $(0,+\infty)$. This property has already been observed in the article [8], where a different set of boundary conditions, namely $u(1) = u_x(1) = u_{xx}(-1) = 0$, was considered.

Proof. The proof of all elements of this proposition is standard (cf., for instance, [4, 7, 15])) except the energy inequalities (3.2) and (3.3) which are established below.

Let u_0 be in $D(A)$ and f be a smooth function in t that takes values in H. Taking the inner product in H of u with $u_t + Au = f(t)$ leads to

$$\frac{d}{dt}||u||^2_{L^2(\omega)} + \frac{2}{3}||u_x||^2_{L^2(\dot{\omega})} \leq 2(f,u)_{L^2(\omega)} \leq 3||f||^2_{V'} + \frac{1}{3}||u_x||^2_{L^2(\dot{\omega})}. \quad (3.4)$$

Thus, (3.3), for smooth data, follows by integrating in time the above inequality. For general data, we can conclude by a limiting argument since $H \subset V'$ with dense and continuous embedding.

The proof of (3.2) follows from a similar procedure, but using (2.9) and (2.7). □

Remark 3.2. In the case $\beta \neq 0$, setting

$$a_\beta(u,\phi) = a(u,\phi) - \beta \int_\Omega u_x(x)\phi(x)\omega(x)dx,$$

it can be shown that for any $\mu \in [0,1]$,

$$\begin{aligned} a_\beta(u,u) &= 3\int_\Omega \frac{u_x^2(x)}{(1-x)^2}dx - 6\int_\Omega \frac{u^2(x)}{(1-x)^4}dx + \frac{\beta}{2}\int_\Omega \frac{u^2(x)}{(1-x)^2}dx \\ &\geq \frac{3\mu}{4}\int_\Omega \frac{u^2(x)}{(1-x)^4}dx + \frac{1-\mu}{3}\int_\Omega \frac{u_x^2(x)}{(1-x)^2}dx + \frac{\beta}{2}\int_\Omega \frac{u^2(x)}{(1-x)^2}dx. \end{aligned}$$

Thus, a_β is coercive on V if $\beta > -\frac{3}{16}$. The continuity is obvious. For the case $\beta \leq -\frac{3}{16}$ it is an easy exercise to prove that there exists λ_β such that $a_\beta(u,u) + \lambda_\beta\|u\|^2_H$ is coercive on V. In fact, by Young's inequality

$$\int_\Omega \frac{u^2(x)}{(1-x)^2}dx \leq \frac{\varepsilon}{2}\int_\Omega \frac{u^2(x)}{(1-x)^3(1+x)}dx + +\frac{1}{2\varepsilon}\|u\|^2_{L^2(\omega)}. \quad (3.5)$$

The first term in the right-hand side can be bounded by $c\varepsilon\|u\|^2_V$ thanks to (2.7) and (2.10). Then, by the Hille-Yosida theorem, one can define a semi-group that satisfies $\|e^{-tA_\beta}\| \leq e^{\lambda_\beta t}$.

We complete this section with a numerical result illustrating the decay property of the linear KdV equation (1.2) with $\beta = 0$ on the interval $(-40, 40)$.

The initial condition is taken as the following soliton at $t = 0$:

$$u(x,t) = 12\kappa^2 \text{sech}^2(\kappa(x - 4\kappa^2 t)), \ \kappa = 0.3. \quad (3.6)$$

Note that $u(x,0)$ satisfies the boundary conditions to the machine accuracy.

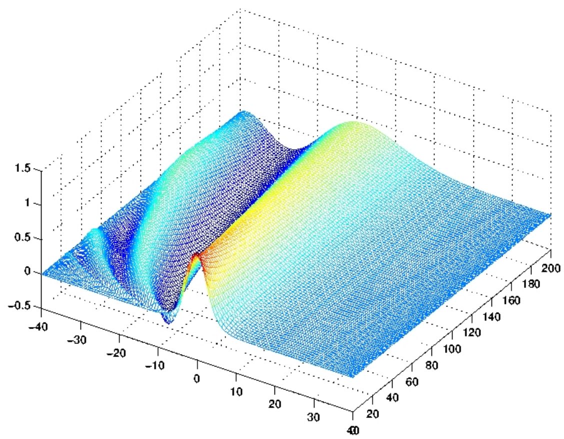

FIGURE 1. Evolution of the linear KdV solution for $t \in [0, 200]$

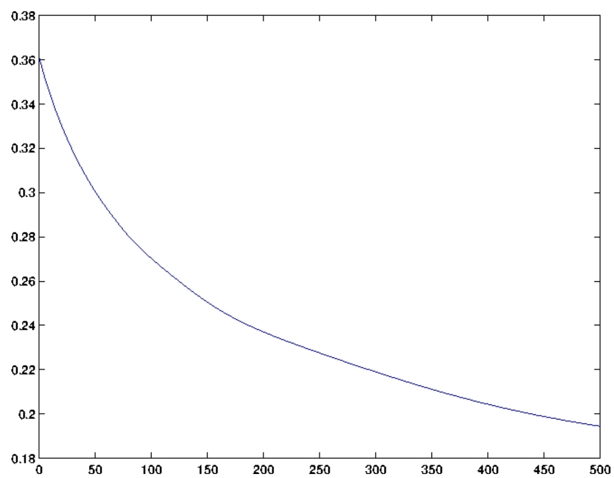

FIGURE 2. Left: decay in the weighted L^2-norm; Right: decay in the standard L^2-norm

4. THE KdV EQUATION

In this section, we consider the KdV equation (1.1) with (1.3)-(1.4). For the sake of simplicity, we assume here that $\beta = 0$. However, it can be shown that all the results which are local in time are valid for $\beta \in \mathbb{R}$, while those results which are global in time can be extended to the cases where the bilinear form is coercive on V, i.e $\beta > -\frac{3}{16}$ (cf. Remark 3.2).

Let $W(t) = e^{-tA}$ where A is defined in the previous section. We plan to apply a fixed-point argument to the Duhamel's form of (1.1) with $\beta = 0$,

$$u(t) = W(t)u_0 - \frac{1}{2}\int_0^t W(t-s)\partial_x(u^2)ds. \tag{4.1}$$

We begin by deriving some estimates on the bilinear form $B(u,v) = (uv)_x$.

4.1. **Bilinear estimate.**

Proposition 4.1. *There exists a constant $C > 0$ such that*

$$\|B(u,v)\|_{V'} \le C\|u\|_H \|v\|_H^{1/2} \|v\|_V^{1/2},\ \forall u, v \in V. \tag{4.2}$$

Proof. Consider a smooth and compactly supported function ϕ in V. We seek an upper bound for

$$\int_\Omega (uv)_x \phi\omega dx = -\int_\Omega u(x)v(x)\phi_x(x)\omega(x)dx - \int_\Omega u(x)v(x)\phi(x)\dot{\omega}(x)dx. \tag{4.3}$$

To bound the first term on the right-hand side of this equality, we use the Cauchy-Schwarz inequality to get

$$\Big|\int_\Omega u(x)v(x)\phi_x(x)\omega(x)dx\Big| \le C\|\phi\|_V\|u\|_H \sup_x \Big(|v(x)|\sqrt{\frac{\omega(x)}{\dot{\omega}(x)}}\Big).$$

At this stage, observe that

$$v(x)^2\frac{\omega(x)}{\dot{\omega}(x)} = 2\int_{-1}^x v(y)v_x(y)(1-y^2)dy - 2\int_{-1}^x v^2(y)ydy.$$

On the one hand, $|2\int_{-1}^x v(y)v_x(y)(1-y^2)dy|$ is bounded by $C\|v\|_H\|v\|_V$ thanks to the Cauchy-Schwarz inequality; on the other hand,

$$\Big|2\int_{-1}^x v^2(y)ydy\Big| \le 2||v||_{L^2}^2$$

and by (2.10),

$$\|v\|_{L^2}^2 \le \sqrt{2}\|v\|_H\Big(\int_\Omega v^2(x)\frac{dx}{1+x}\Big)^{1/2} \le C\|v\|_H\|v\|_V.$$

In fact, we have established

$$\sup_x \left(|v(x)|^2 \frac{\omega(x)}{\dot{\omega}(x)} \right) \leq C\|v\|_H \|v\|_V. \qquad (4.4)$$

Proceed now to the second term in the right-hand side of (4.3). Since

$$\left| \int_\Omega u(x)v(x)\phi(x)\dot{\omega}(x)dx \right| \leq C\|u\|_H \|v\|_{L^2} \sup_x \left(\phi^2(x) \frac{\dot{\omega}^2(x)}{\omega(x)} \right),$$

the proof will be complete if

$$\sup_x \left(\phi^2(x)(1-x)^{-3}(1+x)^{-1} \right) \leq C\|\phi\|_V^2. \qquad (4.5)$$

This holds true due to the trace theorem which follows. □

Proposition 4.2. *Let $u \in V$. Then, $\frac{u^2(x)}{(1-x)^3} \in W^{1,1}(\Omega) \subset C(\bar{\Omega})$ and $\frac{u^2(x)}{(1-x)^3}|_{x=1} = 0$. Moreover, $\frac{u^2(x)}{(1+x)} \in W^{1,1}(\Omega) \subset C(\bar{\Omega})$ and $\frac{u^2(x)}{(1+x)}|_{x=-1} = 0$.*

Proof. Note that $u \in V$ implies $\frac{u_x(x)}{1-x} \in L^2$. Thanks to the Hardy inequality (2.7), we also have $\frac{u(x)}{(1-x)^2} \in L^2$. Hence, $\frac{u(x)u_x(x)}{(1-x)^3} \in L^1$ and $\partial_x(\frac{u^2}{(1-x)^3}) \in L^1$. Therefore, $\frac{u^2}{(1-x)^3} \in W^{1,1}(\Omega) \subset C(\bar{\Omega})$. Since $\frac{u^2}{(1-x)^3}\frac{1}{1-x} \in L^1$ and $\frac{1}{1-x}$ is not integrable at $x = 1$, one concludes that $\frac{u^2}{(1-x)^3}|_{x=1} = 0$. The second result can be proved in a similar manner. □

4.2. **Fixed-point argument.** Introduce, for a given $T > 0$, a Banach space

$$E = C([0,T], H) \cap L^2(0,T;V) \qquad (4.6)$$

with the norm

$$\|u\|_E^2 = \sup_{t\in[0,T]} \left(\|u(t)\|_H^2 + \frac{1}{3}\int_0^t \|u(s)\|_V^2 ds \right).$$

We intend to prove that if T is small enough, then

$$u(t) \to \mathcal{T}(u(t)) = W(t)u_0 - \frac{1}{2}\int_0^t W(t-s)(u^2)_x ds \qquad (4.7)$$

is a contraction in E.

Theorem 4.3. *For u_0 in H, there exists T depending only on $\|u_0\|_H$ such that* (4.7) *possesses a unique solution in E.*

Proof. First step: Set $R = 2\|u_0\|_H$. We first prove that if T is small enough, then $\mathcal{T}$ maps the ball of radius R of E into itself.

For $u \in E$, denote $M_0 = \sup_{t\in[0,T]}(\|u(t)\|_H)$ and $M_1 = \|u(t)\|_{L^2(0,T);V)}$. Hence $M_0 \leq R$ and $M_1 \leq R$. Thanks to (3.3),

$$\|\mathcal{T}(u(t))\|_E^2 \leq \|u_0\|_H^2 + 3\|(u^2)_x\|_{L^2(0,T,V')}^2. \tag{4.8}$$

Thanks to (4.2), we find that for T sufficiently small,

$$\|\mathcal{T}(u(t))\|_E^2 \leq \|u_0\|_H^2 + C\int_0^T \|u\|_H^3\|u\|_V ds \leq \frac{R^2}{4} + C\sqrt{T}M_0^3 M_1 \leq R^2.$$

Therefore, $\mathcal{T}$ maps the ball of radius R in E into itself.

Second step: We now prove that $\mathcal{T}$ is a contraction mapping. Using once again (3.3) and (4.2),

$$\begin{aligned}\|\mathcal{T}(u(t)) - \mathcal{T}(v(t))\|_E^2 &\leq \frac{C}{4m}\int_0^T \|u-v\|_H^2\|u+v\|_H\|u+v\|_V ds\\ &\leq C\sqrt{T}R^2 \sup_t \|(u-v)(t)\|_H^2.\end{aligned}$$

This concludes the proof. □

4.3. **A global result.**

Theorem 4.4. *Assume that u_0 is in $H \cap L^2$. Then there exists a unique global solution of the KdV equation in $C([0,+\infty);H)\cap L^2((0,+\infty);V)$.*

Moreover, if $\|u_0\|_{L^2}$ is sufficiently small, the solution decays towards 0 in H exponentially.

Proof. Thanks to Theorem 4.3, there exists a unique solution which is local in time. This solution belongs almost everywhere in t to $V \subset L^2$. Moreover, due to (1.5), the L^2-norm of the solution remains bounded along the trajectories. Therefore, we just need to prove an upper bound for the H-norm of a trajectory to get the global solution. Similarly as in deriving (3.4), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 + \frac{1}{3}\|u\|_V^2 \leq -\int_\Omega u^2 u_x \omega dx = \frac{1}{3}\int_\Omega u^3\dot{\omega} dx. \tag{4.9}$$

Then, thanks to the Cauchy-Schwarz inequality and (4.5) , the expression on the right-hand side can be bounded by

$$c\|u\|_{L^2}\|u\|_H \sup_x\left(|u(x)|(1+x)^{-1/2}(1-x)^{-3/2}\right) \leq c\|u\|_{L^2}\|u\|_V\|u\|_H$$

$$\leq c\|u_0\|_{L^2}\|u\|_V\|u\|_H \leq \frac{1}{6}\|u\|_V^2 + C\|u_0\|_{L^2}^2\|u\|_H^2,$$

and the conclusion follows promptly by applying the Gronwall lemma. Furthermore, if $\|u_0\|_{L^2}^2$ is sufficiently small, it can be seen that $u(t)$ decays exponentially towards 0 in H, since $V \hookrightarrow H$ and

$$\frac{d}{dt}\|u\|_H^2 + \frac{1}{3}\|u\|_V^2 \leq 2C\|u_0\|_{L^2}^2\|u\|_H^2. \tag{4.10}$$

Remark 4.1. This theorem leads us to ask two more questions: First, is it true that any solution of this equation converges to 0 as time goes to infinity? The answer is negative since the example in the next section indicates that such a result is only valid for small initial data. Second, can we prove that the solution becomes more regular for smooth initial data? The next theorem will provide a result in this direction, namely, the well posedness of the equation in $D(A)$.

Denote X as the set of functions u such that Au belongs to V.

Theorem 4.5. *Assume that u_0 is in $D(A)\cap L^2$. Then there exists a unique global solution of the KdV equation in $C^1([0,+\infty);H)\cap C([0,+\infty);D(A))\cap L^2((0,+\infty);X)$.*

Proof. First of all, observe that if u_0 is in $D(A)$, then $u_t(0) = -Au(0) - u(0)u_x(0)$ is in H. Moreover $v = u_t$ is a solution of the abstract equation

$$\begin{aligned} &v_t + Av = -(vu)_x, \\ &v(0) = -Au(0) - u(0)u_x(0). \end{aligned} \tag{4.11}$$

This equation can be solved locally in time by the same methods as above. To prove the persistence of u in $D(A)$, one just needs to prove the persistence of v in H, since $Au = -v - uu_x$, and uu_x belongs to H if u is in V. Actually, one derives from (4.4) that

$$\|uu_x\|_H \leq \|u\|_V \sup_x(|u|\sqrt{\frac{\omega}{\dot{\omega}}}) \leq C\|u\|_V^{3/2}\|u\|_H^{1/2}.$$

Applying (3.4) and (4.2) to the equation (4.11) leads to

$$\frac{d}{dt}\|v\|_H^2 + \frac{1}{3}\|v\|_V^2 \leq C\|u\|_H^2\|v\|_H\|v\|_V.$$

Since we have an upper bound on the H norm of u, to get a bound on the H norm of v is then straightforward. □

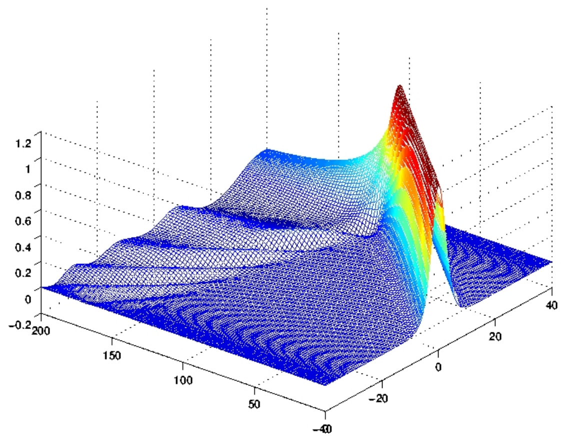

FIGURE 3. Evolution of the KdV solution for $t \in [0, 200]$

We now present a numerical result for the KdV equation (1.1) in the interval $(-40, 40)$ with the initial condition an exact soliton solution of the KdV equation (1.1) with $\beta = 0$.

In Figure 3, the solution of the KdV equation is plotted for $t \in [0, 200]$. As expected, the soliton does not change shape before it hits the right boundary, but it starts to decay as soon as the solution reaches the right boundary.

On the left of Figure 4 is the evolution of the weighted L^2-norm of the KdV solution. One can see that after the soliton hits the right boundary, the weighted L^2-norm of the solution starts to decay exponentially as proved in Theorem 4.4. For the sake of comparison, the evolution of the standard L^2-norm of the KdV solution is presented in the right of Figure 4. We observe that the L^2-norm is conserved (as expected) before the soliton hits the right boundary and also decays exponentially after that.

4.4. **A nontrivial stationary solution.**

Proposition 4.6. *There exists $\alpha > 0$ and a positive solution u^* in V of $u_{xxx} + uu_x = 0$ that satisfies $||u^*||_{L^2} = \sqrt{2\alpha}$.*

Proof. Define α to be

$$\alpha = \frac{3}{2} \left(\int_0^1 \frac{dt}{\sqrt{t}\sqrt{1-t^2}} \right)^4 \simeq 7.86617266 \tag{4.12}$$

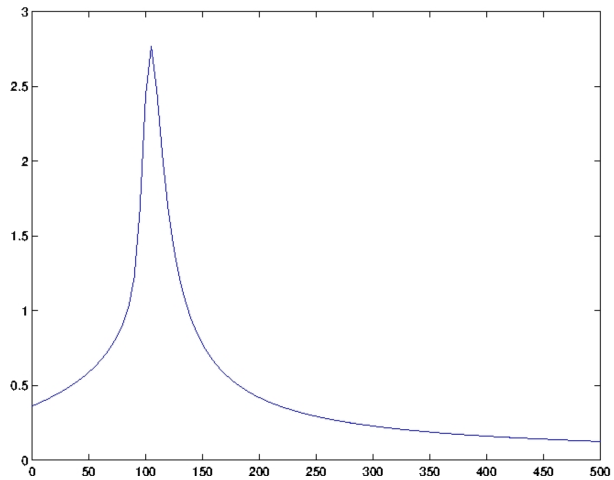

FIGURE 4. Left: decay in the weighted L^2-norm; Right: decay in the standard L^2-norm

and let u be the solution of the ordinary differential equation

$$u_{xx}(x) + \frac{u^2(x)}{2} = \alpha, \tag{4.13}$$

supplemented with initial data $u(0) = \sqrt{6\alpha}$, $\dot{u}(0) = 1$. Then u is the unique solution of a Cauchy problem defined on $(-y, y)$; moreover, u is an even function. We shall prove that u is positive on $(-1, 1)$, and that u satisfies $u(1) = \dot{u}(1) = 0$. Multiply (4.13) by u_x and integrate from 0 to x (notice that $\frac{u_x^2(0)}{2} + \frac{u^3(0)}{6} - \alpha u(0) = 0$) to get

$$\frac{u_x^2(x)}{2} + \frac{u^3(x)}{6} - \alpha u(x) = 0. \tag{4.14}$$

This implies that $\Phi(X) = \frac{u^3(x)}{6} - \alpha u(x) \leq 0$. Drawing the graph of the function $\frac{X^3}{6} - \alpha X$, we see that, since $u(0) = \sqrt{6\alpha}$, $u(x)$ is trapped in the interval where $\Phi(X) \leq 0$, that is, $[0, \sqrt{6\alpha}]$. Therefore, the mapping $x \to u(x)$ decays on $[0, z]$, where $z > 0$ is the first time where $u(z) = 0$. We will prove that in fact $z = 1$ and $u(1) = u_x(1) = 0$. Actually,

$$-\int_{\sqrt{6\alpha}}^{0} \frac{du}{\sqrt{u}\sqrt{2\alpha - \frac{u^3}{6}}} = z, \tag{4.15}$$

so the very definition of α implies that $z = 1$. Thus, $u(x)$ for $x \in \Omega$ is a nontrivial stationary solution for the KdV equation (1.1) with (1.3)-(1.4). □

Remark 4.2. This result indicates that not all solutions of the KdV equation (1.1) with (1.3)-(1.4) will converge to 0 when t goes to infinity.

Remark 4.3. Going back to the proof above, substituting $3^4\alpha$ for α, one can construct an even function v that solves the ODE $v_{xxx} + vv_x = 0$ and that satisfies $v(\frac{1}{3}) = v_x(\frac{1}{3}) = 0$. One can then extend v to another function w that is $\frac{2}{3}$-periodic and that matches with v in $(-\frac{1}{3}, \frac{1}{3})$. We then have a "three bumps" nonzero positive stationary solution to our problem. This technique can be extended to construct multi-bumps stationary solutions.

Remark 4.4. One may wonder, for instance, whether there is a solution of the KdV equation with u_0 in H (or even $D(A)$) belonging to $D(A^2)$. The answer is negative since our one bump stationary solution is such that $u_x^{(6)}(1) \neq 0$ (by differentiating three times the ODE).

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