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Irrational approximations and their applications to partial differential equations in exterior domains

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Abstract A family of orthogonal systems of irrational functions on the semi-infinite interval is introduced. The proposed orthogonal systems are based on Jacobi polynomials through an irrational coordinate transform. This family of orthogonal systems offers great flexibility to match a wide range of asymptotic behaviors at infinity. Approximation errors by the basic orthogonal projection and various other orthogonal projections related to partial differential equations in unbounded domains are established. As an example of applications, a Galerkin approximation using the proposed irrational functions to an exterior problem is analyzed and implemented. Numerical results in agreement with our theoretical estimates are presented.

Keywords rational and irrational functions · spectral method · semi-infinite interval · exterior problems

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1. Introduction

Many physical applications, especially in fluid dynamics and electro-magnetics, require solving PDEs on unbounded domains. How to efficiently and accurately solve PDEs on unbounded domains is a very challenging problem since it involves an extra degree of difficulty in addition to all other difficulties in bounded domains.

Numerical methods for unbounded domains can be essentially classified into three categories: (a) Truncate a unbounded domain to an appropriate bounded domain; (b) map a unbounded domain to a bounded domain via a (necessarily) singular mapping; and (c) treat the unbounded domain directly. The first approach is a popular choice with (lower-order) finite difference or finite element methods. The second and third approaches are more appropriate for (high-order) spectral methods.

The mapping technique has been used by Grosch and Orszag [14] and others (cf. [8]), and analyzed in [16, 19]. The main advantage of using a mapped approach is that existing results and codes on finite intervals can be used. But its main disadvantage is that the transformed equations may become too complicated to handle. Thus, direct approaches are often preferred.

The most straightforward direct approaches are based on the classical orthogonal polynomials/functions in unbounded intervals, namely the Hermite or Laguerre polynomials/functions (see, for instance, [8, 12, 13, 17, 20, 27, 30–32]). Rational approximation through classical orthogonal polynomials in finite intervals were proposed in [6, 7, 9]. This technique has been generalized and analyzed recently in [15, 21]. The rational systems introduced in [15, 21] are designed for specific weighted spaces so that their applications are somewhat limited.

We propose in this paper a new family of orthogonal systems in semi-infinite interval consisting of rational/irrational functions of the following type:

$$I_l^{(\gamma,\delta)}(r) := \frac{1}{r^\gamma} J_l^{(\alpha,0)}\left(1-\frac{2}{r^\delta}\right), \tag{1.1}$$

where $J_l^{(\alpha,0)}(r)$ is the Jacobi polynomial of degree l with index $(\alpha, 0)$. The parameter γ is chosen to match, as closely as possible, the asymptotic behavior of the function to be approximated; the parameter $\delta > 0$ is a mapping parameter which will affect the accuracy of the approximation in a way which will be made clear in Section 5; α is determined in such a way that $\{I_k^{(\gamma,\delta)}(r)\}$ form an orthogonal system in $L^2_{\omega_\sigma}(\Lambda)$, where σ is another parameter, $\Lambda = (1, \infty)$ and $\omega_\sigma = r^\sigma$. This latter condition requires that $\alpha = \frac{1}{\delta}(2\gamma - \delta - \sigma - 1)$. Hence, α is *not* a free parameter. Therefore, the proposed family of orthogonal systems $\{I_k^{(\gamma,\delta)}(r)\}$ is very general and includes in particular many special cases already studied in the literature. The great flexibility afforded by the free parameters γ, δ (and σ) allows us to design suitable approximations for a large class of partial differential equations. We note that a similar approach was used in [28, 29] to analyze mapped Legendre and Jacobi orthoganal systems with quite general but regular mappings in a finite interval.

The paper is organized as follows. In the next section, we introduce the proposed irrational orthogonal systems and establish basic results on the orthogonal projection in $L^2_{\omega_\sigma}(\Lambda)$. In Section 3, we study several orthogonal projections in non-uniformly weighted Sobolev spaces related to certain partial differential equations in unbounded domains. The results in Sections 2 and 3 are based on the Sturm–Liouville operators associated with the proposed irrational orthogonal systems. In Section 4, we use recent results on the Jacobi approximations to derive error estimates in more explicit norms directly related to the function and its derivatives. The results in Sections 2, 3 and 4 are derived under the condition $\sigma < 2\gamma - 1$. In Section 5, we establish corresponding results for the special but important case $\sigma = 2\gamma - 1$. In Section 6, we present an application to a model exterior problem (6.1) with implementation details and numerical results. Some concluding remarks are given in the final section.

2. Orthogonal projection in $L^2_{\omega_\sigma}(\Lambda)$

In this section, we introduce an orthogonal system of irrational functions on the semi-infinite interval Λ, induced by the Jacobi polynomials, and establish a basic approximation result which will play an important role in the forthcoming sections.

Let us first recall some basic properties of Jacobi polynomials $J_l^{(\alpha,\beta)}(x)$ which are defined by

$$(1-x)^\alpha(1+x)^\beta J_l^{(\alpha,\beta)}(x) = \frac{(-1)^l}{2^l l!}\frac{d^l}{dx^l}((1-x)^{l+\alpha}(1+x)^{l+\beta}),$$

and satisfy the Sturm–Liouville equation

$$\frac{d}{dx}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}v(x)\right) + \lambda_l^{(\alpha,\beta)}(1-x)^\alpha(1+x)^\beta v(x) = 0, \quad -1 < x < 1, \tag{2.1}$$

with

$$\lambda_l^{(\alpha,\beta)} = l(l+\alpha+\beta+1).$$

For $\alpha, \beta > -1$, we have (see Askey [2])

$$J_l^{(\alpha,\beta)}(1) = \frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}, \tag{2.2}$$

$$J_l^{(\alpha,\beta)}(-1) = (-1)^l\frac{\Gamma(l+\beta+1)}{l!\Gamma(\beta+1)}. \tag{2.3}$$

The Jacobi polynomials fulfill the recurrence relation

$$J_{l,1}^{(\alpha,\beta)}(x) \equiv \frac{d}{dx}J_l^{(\alpha,\beta)}(x) = \frac{1}{2}(l+\alpha+\beta+1)J_{l-1}^{(\alpha+1,\beta+1)}(x), \quad l \geq 1. \tag{2.4}$$

Therefore, we derive from (2.2)–(2.4) that

$$J_{l,1}^{(\alpha,\beta)}(1) = \frac{1}{2}(l+\alpha+\beta+1)\frac{\Gamma(l+\alpha+1)}{(l-1)!\Gamma(\alpha+2)}, \quad l \geq 1, \tag{2.5}$$

$$J_{l,1}^{(\alpha,\beta)}(-1) = \frac{1}{2}(-1)^{l+1}(l+\alpha+\beta+1)\frac{\Gamma(l+\beta+1)}{(l-1)!\Gamma(\beta+2)}, \quad l \geq 1. \tag{2.6}$$

Let $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. For $\alpha, \beta > -1$, the Jacobi polynomials satisfy the following orthogonal relation:

$$\int_{-1}^{1} J_l^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x)\chi^{(\alpha,\beta)}(x)dx = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \quad l, m \geq 0, \tag{2.7}$$

where $\delta_{l,m}$ is the Kronecker function and

$$\gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}, \qquad l \geq 0. \tag{2.8}$$

For given $\delta > 0$, we set $r = (\frac{2}{1-x})^{\frac{1}{\delta}}$ which maps $x \in (-1, 1)$ to $r \in (1, \infty)$. Then,

$$x = 1 - \frac{2}{r^\delta}, \quad 1 - x = \frac{2}{r^\delta}, \quad 1 + x = \frac{2}{r^\delta}(r^\delta - 1) \tag{2.9}$$

and

$$\frac{dx}{dr} = \frac{2\delta}{r^{\delta+1}} = 2\delta\left(\frac{1-x}{2}\right)^{\frac{\delta+1}{\delta}}, \quad \frac{dr}{dx} = \frac{1}{2\delta}r^{\delta+1} = \frac{1}{2\delta}\left(\frac{2}{1-x}\right)^{\frac{\delta+1}{\delta}}. \tag{2.10}$$

For any real number γ, the irrational function $I_l^{(\gamma,\delta)}(r)$ is defined by

$$I_l^{(\gamma,\delta)}(r) = \frac{1}{r^\gamma} J_l^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right).$$

According to (2.1), (2.9) and (2.10), $I_l^{(\gamma,\delta)}(r)$ satisfies the Sturm–Liouville equation

$$\frac{1}{\delta^2}\frac{d}{dr}(r^{1-\alpha\delta-\delta}(r^\delta - 1))\frac{d}{dr}(r^\gamma v(r))) + \lambda_l^{(\alpha)} r^{\gamma-\alpha\delta-\delta-1} v(r) = 0, \quad r \in \Lambda, \tag{2.11}$$

with $\lambda_l^{(\alpha)} = l(l+\alpha+1)$. For simplicity, we denote $\lambda_l^{(\alpha)}$ by λ_l in the sequel. Furthermore, (2.2) and (2.3) imply that

$$\lim_{r\to\infty} r^\gamma I_l^{(\gamma,\delta)}(r) = \frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}, \quad l \geq 0, \tag{2.12}$$

$$I_l^{(\gamma,\delta)}(1) = (-1)^l, \qquad l \geq 0. \tag{2.13}$$

Differentiating $I_l^{(\gamma,\delta)}(r)$ and using (2.9) and (2.10) yield that

$$\frac{d}{dr} I_l^{(\gamma,\delta)}(r) = -\frac{\gamma}{r^{\gamma+1}} J_l^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right) + \frac{2\delta}{r^{\gamma+\delta+1}} J_{l,1}^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right).$$

Thanks to (2.2), (2.3), (2.5) and (2.6), we find that

$$\lim_{r\to\infty} r^{\gamma+1}\frac{d}{dr} I_l^{(\gamma,\delta)}(r) = -\gamma\frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}, \quad l \geq 1, \tag{2.14}$$

$$\frac{d}{dr} I_l^{(\gamma,\delta)}(1) = (-1)^{l+1}(\delta l(l+\alpha+1) + \gamma), \quad l \geq 1. \tag{2.15}$$

Let $\omega_\sigma(r) = r^\sigma$ be a weight function, where σ is related to the particular problem that we are interested in. Let γ be a real number which is determined by the asymptotic behavior of function that we want to approximate. The condition that $\{I_l^{(\gamma,\delta)}(x)\}$ form an orthogonal system in $L^2_{\omega_\sigma}(\Lambda)$ leads to

$$\alpha = \frac{1}{\delta}(2\gamma - \delta - \sigma - 1). \tag{2.16}$$

In particular, we have $\alpha > -1$ for $\sigma < 2\gamma - 1$. In this case, a simple calculation with (2.7)–(2.10) and (2.16) leads to

$$\begin{aligned}\int_\Lambda I_l^{(\gamma,\delta)}(r) I_m^{(\gamma,\delta)}(r)\omega_\sigma(r)dr &= \frac{1}{\delta} 2^{\frac{-2\gamma+\sigma+1}{\delta}} \int_{-1}^{1} J_l^{(\alpha,0)}(x) J_m^{(\alpha,0)}(x)(1-x)^\alpha dx \\ &= c_l^{(\gamma,\delta,\sigma)}\delta_{l,m}, \qquad l, m \geq 0\end{aligned} \tag{2.17}$$

where

$$c_l = c_l^{(\gamma,\delta,\sigma)} = \frac{1}{2\delta l + 2\gamma - \sigma - 1}, \qquad l \geq 0. \tag{2.18}$$

We now prove the completeness of the set of $I_l^{(\gamma,\delta)}(r)$ in $L^2_{\omega_\sigma}(\Lambda)$, which is defined by

$$L^2_{\omega_\sigma}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\omega_\sigma} < \infty\}$$

with the associated inner product and norm

$$(u, v)_{\omega_\sigma} = \int_\Lambda u(r)v(r)\omega_\sigma(r)dr, \qquad \|v\|_{\omega_\sigma} = (v, v)^{\frac{1}{2}}_{\omega_\sigma}.$$

For any $v \in L^2_{\omega_\sigma}(\Lambda)$, we set $u(x) = \left(\frac{2}{1-x}\right)^{\frac{\gamma}{\delta}} v\left(\left(\frac{2}{1-x}\right)^{\frac{1}{\delta}}\right)$. Then by (2.9), (2.10) and (2.16),

$$\int_{-1}^{1} u^2(x)(1-x)^\alpha dx = 2^{\alpha+1}\delta \int_\Lambda v^2(r) r^\sigma dr < \infty.$$

Therefore $u \in L^2_{(1-x)^\alpha}(-1, 1)$. According to the completeness of the set of Jacobi polynomials, we have

$$u(x) = \sum_{l=0}^{\infty} a_l J_l^{(\alpha,0)}(x).$$

By the definition of $I_l^{(\gamma,\delta)}(r)$, we obtain

$$v(r) = \sum_{l=0}^{\infty} a_l I_l^{(\gamma,\delta)}(r).$$

This implies the completeness of the set of $I_l^{(\gamma,\delta)}(r)$ in $L^2_{\omega_\sigma}(\Lambda)$.

Thus, for any $v \in L^2_{\omega_\sigma}(\Lambda)$ and $\sigma < 2\gamma - 1$, we write

$$v(r) = \sum_{l=0}^{\infty} \hat{v}_l^{(\gamma,\delta,\sigma)} I_l^{(\gamma,\delta)}(r)$$

with

$$\hat{v}_l = \hat{v}_l^{(\gamma,\delta,\sigma)} = \frac{1}{c_l} \int_\Lambda v(r) I_l^{(\gamma,\delta)}(r) \omega_\sigma(r) dr, \quad l \geq 0. \tag{2.19}$$

We also introduce the weight function

$$\eta(r) = \eta^{(\gamma,\delta,\sigma)}(r) = r^{2-2\gamma+\sigma}(r^\delta - 1)$$

and define a second set of functions

$$K_l^{(\gamma,\delta)}(r) = \frac{d}{dr}\left(r^\gamma I_l^{(\gamma,\delta)}(r)\right).$$

Due to (2.9) and (2.10), we have

$$K_l^{(\gamma,\delta)}(r) = \frac{d}{dr} J_l^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right) = \frac{2\delta}{r^{\delta+1}} J_{l,1}^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right). \tag{2.20}$$

Thanks to (2.16), we can rewrite (2.11) as

$$\frac{1}{\delta^2} r^\gamma \frac{d}{dr}\left(\eta(r) K_l^{(\gamma,\delta)}(r)\right) + \lambda_l \omega_\sigma(r) I_l^{(\gamma,\delta)}(r) = 0. \tag{2.21}$$

Multiplying the above by $I_m^{(\gamma,\delta)}(r)$, integrating the result by parts, and using (2.17) and the fact that

$$\begin{aligned} r^\gamma \eta(r) K_l^{(\gamma,\delta)}(r) I_m^{(\gamma,\delta)}(r) &= 2\delta r^{1-2\gamma+\sigma-\delta}(r^\delta - 1) J_{l,1}^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right) \\ &\quad \times J_m^{(\alpha,0)}\left(1 - \frac{2}{r^\delta}\right) \to 0, \text{ as } r \to \infty, \end{aligned}$$

we obtain

$$\int_\Lambda K_l^{(\gamma,\delta)}(r) K_m^{(\gamma,\delta)}(r) \eta(r) dr = \delta^2 c_l \lambda_l \delta_{l,m}. \tag{2.22}$$

Therefore the set of $K_l^{(\gamma,\delta)}(r)$ is a mutually orthogonal system associated with the weight $\eta(r)$.

We now prove the completeness of the set of $K_l^{(\gamma,\delta)}(r)$ in $L^2_\eta(\Lambda)$, which is defined as usual with the inner product $(u, v)_\eta$ and the norm $\|v\|_\eta$. For any $v \in L^2_\eta(\Lambda)$, we set $u(x) = \left(\frac{2}{1-x}\right)^{\frac{\delta+1}{\delta}} v\left(\left(\frac{2}{1-x}\right)^{\frac{1}{\delta}}\right)$. Then by (2.9), (2.10) and (2.16),

$$\int_{-1}^{1} u^2(x)(1-x)^{\alpha+1}(1+x) dx = 2^{\alpha+3}\delta \int_\Lambda v^2(r)\eta(r) dr < \infty.$$

Therefore $u \in L^2_{(1-x)^{\alpha+1}(1+x)}(-1,1)$. According to the completeness of the set of Jacobi polynomials, we use (2.4) to obtain that

$$u(x) = \sum_{l=0}^{\infty} b_l J_{l,0}^{(\alpha+1,1)}(x) = 2\sum_{l=1}^{\infty} \frac{b_{l-1}}{l+\alpha+1} J_{l,1}^{(\alpha,0)}(x).$$

Using (2.20), we obtain

$$v(r) = \frac{1}{\delta}\sum_{l=1}^{\infty} \frac{b_{l-1}}{l+\alpha+1} K_l^{(\gamma,\delta,\sigma)}(r).$$

This implies the completeness of the set of $K_l^{(\gamma,\delta)}(r)$ in $L^2_\eta(\Lambda)$.

We are now in position to study the orthogonal irrational approximations. Let L be any positive integer and

$$Q_L = \text{span}\left\{I_0^{(\gamma,\delta)}, I_1^{(\gamma,\delta)}, \ldots, I_L^{(\gamma,\delta)}\right\}.$$

We define the orthogonal projection $P_{L,\sigma} : L^2_{\omega_\sigma}(\Lambda) \to Q_L$ by

$$(P_{L,\sigma}v - v, \phi)_{\omega_\sigma} = 0, \qquad \forall \phi \in Q_L. \tag{2.23}$$

In order to describe the approximation result, we introduce the operator $A = A_{\gamma,\delta,\sigma}$ such that

$$Av = -\frac{1}{\delta^2} r^{\gamma-\sigma} \frac{d}{dr}\left(\eta(r)\frac{d}{dr}(r^\gamma v)\right). \tag{2.24}$$

Accordingly, we define the space $D(A^0) = L^2_{\omega_\sigma}(\Lambda)$ and

$$D(A) = \left\{v | v, Av \in L^2_{\omega_\sigma}(\Lambda), r^\gamma |v(r)| \text{ is bounded as } r \to \infty\right\},$$

with the norm $\|v\|_{D(A)} = (\|Av\|^2_{\omega_\sigma} + \|v\|^2_{\omega_\sigma})^{\frac{1}{2}}$. For any integer $\mu \geq 2$, we define $D(A^\mu)$ and its norm by induction.

Remark 2.1. We have from (2.21) and (2.24) that

$$(A\,u, v)_{\omega_\sigma} = \sum_{l=0}^{\infty} \hat{u}_l \hat{v}_l \lambda_l c_l = (A\,v, u)_{\omega_\sigma}, \quad \forall u, v \in D(A),$$

$$(Av, v)_{\omega_\sigma} \geq 0, \qquad \forall v \in D(A).$$

Moreover, $(Av, v) = 0$ implies $v(r) \equiv \frac{\hat{v}_0}{r^\gamma}$. Hence A is a non-negative, self-adjoint operator in $L^2_{\omega_\sigma}(\Lambda)$. Consequently, there exists an operator $A^{\frac{1}{2}}$ such that

$$(A\,u, v)_{\omega_\sigma} = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)_{\omega_\sigma}. \tag{2.25}$$

Thus, we can define the space $D(A^\mu)$ for any $\mu \geq 0$ and its norm by space interpolation. In particular, $\|v\|_{D(A^{\frac{1}{2}})} = ((A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)^2_{\omega_\sigma} + \|v\|^2_{\omega_\sigma})^{\frac{1}{2}}$.

Remark 2.2. By integration by parts,

$$
\begin{aligned}
(A\,v, v)_{\omega_\sigma} &= \sum\nolimits_{l=0}^{\infty}\sum\nolimits_{m=0}^{\infty} c_l c_m \left(A\, I_l^{(\gamma,\delta)}, I_m^{(\gamma,\delta)}\right)_{\omega_\sigma} \\
&= \frac{1}{\delta^2}\sum\nolimits_{l=0}^{\infty}\sum\nolimits_{m=0}^{\infty} c_l c_m \left(\frac{d}{dr}(r^\gamma I_l), \frac{d}{dr}(r^\gamma I_m)\right)_{L^2_\eta(\Lambda)} \\
&= \frac{1}{\delta^2}\left\|\frac{d}{dr}(r^\gamma v)\right\|^2_{L^2_\eta(\Lambda)} \geq 0, \qquad \forall v \in D(A). \qquad (2.26)
\end{aligned}
$$

In the sequel, we denote by c a generic positive constant independent of any function and L.

Theorem 2.1. If $\sigma < 2\gamma - 1$ and (2.16) holds, then, for any $v \in D(A^{\frac{\mu}{2}})$ and $0 \leq s \leq \mu$,

$$
\|P_{L,\sigma}v - v\|_{D(A^{\frac{s}{2}})} \leq cL^{s-\mu}\|v\|_{D(A^{\frac{\mu}{2}})}.
$$

Proof. We first consider the case of even integer $\mu = 2m$ and $s = 0$. Thanks to the definition of $K_l^{(\gamma,\delta)}$, (2.20), $v \in D(A^m)$ and $\sigma < 2\gamma - 1$, we have that for all $0 \leq j \leq m-1$,

$$
r^\gamma A^j v(r)\frac{d}{dr}\big(r^\gamma I_l^{(\gamma,\delta)}(r)\big)\eta(r) \to 0, \quad \text{as } r \to \infty.
$$

Hence by (2.19), (2.21) and integration by parts successively, we obtain that

$$
\begin{aligned}
\hat{v}_l &= \frac{1}{c_l\lambda_l}\int_\Lambda v(r) A\, I_l^{(\gamma,\delta)}(r)\omega_\sigma(r)dr \\
&= \frac{1}{\delta^2 c_l\lambda_l}\int_\Lambda \frac{d}{dr}(r^\gamma v(r))\frac{d}{dr}\left(r^\gamma I_l^{(\gamma,\delta)}(r)\right)\eta(r)dr \\
&= \frac{1}{c_l\lambda_l}\int_\Lambda Av(r)I_l^{(\gamma,\delta)}(r)\omega_\sigma(r)dr. \\
&= \frac{1}{c_l\lambda_l^m}\int_\Lambda A^m v(r)I_l^{(\gamma,\delta)}(r)\omega_\sigma(r)dr. \qquad (2.27)
\end{aligned}
$$

Consequently by (2.17), (2.18) and (2.27),

$$
\begin{aligned}
\|P_{L,\sigma}v - v\|^2_{\omega_\sigma} &= \sum_{l=L+1}^{\infty} \hat{v}_l^2 \left\|I_l^{(\gamma,\delta)}\right\|^2_{\omega_\sigma} \\
&\leq c\lambda_{L+1}^{-2m}\sum_{l=L+1}^{\infty}\left(\frac{\int_\Lambda A^m v(r)I_l^{(\gamma,\delta)}(r)\omega_\sigma(r)dr}{\|I_l^{(\gamma,\delta)}\|^2_{\omega_\sigma}}\right)^2\left\|I_l^{(\gamma,\delta)}\right\|^2_{\omega_\sigma} \\
&\leq cL^{-4m}\|A^m v\|^2_{\omega_\sigma} \leq cL^{-2\mu}\|v\|^2_{D(A^{\frac{\mu}{2}})}. \qquad (2.28)
\end{aligned}
$$

Next, let $\mu = 2m+1$ and $s = 0$. Integrating the right side of (2.27) and using (2.21) again, we deduce that

$$\begin{aligned}\hat{v}_l &= -\frac{1}{\delta^2 c_l \lambda_l^{m+1}} \int_\Lambda r^\gamma A^m v(r) \frac{d}{dr}\left(\eta(r)\frac{d}{dr}\left(r^\gamma I_l^{(\gamma,\delta)}(r)\right)\right) dr \\ &= \frac{1}{\delta^2 c_l \lambda_l^{m+1}} \int_\Lambda \frac{d}{dr}(r^\gamma A^m v(r)) K_l^{(\gamma,\delta)}(r)\eta(r) dr. \end{aligned} \tag{2.29}$$

By (2.25) and (2.26),

$$\left\| \frac{d}{dr}(r^\gamma A^m v)\right\|^2_{L^2_\eta(\Lambda)} = \delta^2 (A(A^m v), A^m v)_{\omega_\sigma} \leq \delta^2 \|v\|^2_{D(A^{\frac{\mu}{2}})}.$$

Thus $\frac{d}{dr}(r^\gamma A^m v) \in L^2_{\eta(\Lambda)}$. Therefore we can write

$$\frac{d}{dr}(r^\gamma A^m v(r)) = \sum_{l=0}^{\infty} \hat{a}_l K_l^{(\gamma,\delta)}(r), \tag{2.30}$$

This with (2.29) implies that

$$\hat{a}_l = \delta^2 c_l \lambda_l^{m+1} \hat{v}_l \left\| K_l^{(\gamma,\delta)}\right\|^{-2}_{L^2_\eta(\Lambda)}.$$

The above with (2.17), (2.22) and (2.30) leads to

$$\begin{aligned}\|P_{L,\sigma}v - v\|^2_{\omega_\sigma} &= \sum_{l=L+1}^{\infty} \hat{v}_l^2 \left\| I_l^{(\gamma,\delta)}\right\|^2_{\omega_\sigma} = \sum_{l=L+1}^{\infty} \frac{\hat{a}_l^2}{\delta^4 c_l^2 \lambda_l^{2m+2}} \left\| I_l^{(\gamma,\delta)}\right\|^2_{\omega_\sigma} \left\| K_l^{(\gamma,\delta)}\right\|^4_{L^2_\eta(\Lambda)} \\ &\leq \frac{c}{\delta^2} \sum_{l=L+1}^{\infty} \frac{\hat{a}_l^2}{\lambda_l^{2m+1}} \left\| K_l^{(\gamma,\delta)}\right\|^2_{L^2_\eta(\Lambda)} \leq \frac{c}{\delta^2} \lambda_{L+1}^{-2m-1} \left\| \frac{d}{dr}(r^\gamma A^m v)\right\|^2_{L^2_\eta(\Lambda)} \\ &= \frac{c}{\delta^2} L^{-2\mu} \left\| \frac{d}{dr}(r^\gamma A^m v)\right\|^2_{L^2_\eta(\Lambda)} \leq c L^{-2\mu} \|v\|^2_{D(A^{\frac{\mu}{2}})}. \end{aligned} \tag{2.31}$$

This fact leads to the desired result for $\mu = 2m+1$ and $s = 0$.

The result with any $\mu \geq 0$ and $s = 0$ follows from the previous statements and space interpolation. Finally, applying the result to $A^{\frac{s}{2}}(P_{L,\sigma}v - v)$, we obtain the result for $0 \leq s \leq \mu$. □

Remark 2.3. Theorem 2.1 implies many interesting results. For instance, we have from Theorem 2.1 with $s = 0$ that

$$\int_\Lambda (P_{L,\sigma}v - v)^2 r^\sigma dr \leq c L^{-2\mu} \|v\|^2_{D(A^{\frac{\mu}{2}})}. \tag{2.32}$$

Next, we derive from (2.25), (2.26) and Theorem 2.1 with $s = 1$ that

$$\int_\Lambda \left(\frac{d}{dr}(r^\gamma(P_{L,\sigma}v - v))\right)^2 \eta(r) dr \leq \|P_{L,\sigma}v - v\|^2_{D(A^{\frac{1}{2}})} \leq c L^{2-2\mu} \|v\|^2_{D(A^{\frac{\mu}{2}})}.$$

In particular, for $\gamma = 0$ and $\sigma < -1$,

$$\int_\Lambda \left(\frac{d}{dr}(P_{L,\sigma}v - v)\right)^2 r^{2+\sigma}(r^\delta - 1)dr \le cL^{2-2\mu}\|v\|^2_{D(A^{\frac{\mu}{2}})}. \tag{2.33}$$

On the other hand, we may approximate the function v by $\xi_L := r^\gamma P_{L,\sigma}(\frac{v}{r^\gamma})$. In this case, we have from (2.32) and (2.33) that

$$\int_\Lambda \left(\frac{d}{dr}(\xi_L - v)\right)^2 \eta(r)dr + L^2\int_\Lambda (\xi_L - v)^2 r^{\sigma-2\gamma}dr \le cL^{2-2\mu}\left\|\frac{v}{r^\gamma}\right\|^2_{D(A^{\frac{\mu}{2}})}.$$

3. Orthogonal projections in $H^1_{\sigma,\lambda}(\Lambda)$

In numerical analysis of spectral methods for partial differential equations in unbounded domains, we need to measure numerical errors in certain weighted spaces related to underlying problems. Various weighted Sobolev spaces were investigated in [1, 4] and [24], in which the weight functions for the derivative $\frac{d^k v}{dr^k}$ are the same as for the function itself, i.e., r^α, or the standard Jacobi weight functions $r^{\alpha+k}$. This framework works well in some situations, but it is not the most appropriate in some practical cases. For instance, consider the Poisson-type equation outside a unit sphere (in the spherical coordinates),

$$-\Delta u(r,\lambda,\theta) + \mu u(r,\lambda,\theta) = f, \quad r > 1.$$

For simplicity, we assume $u(1,\lambda,\theta) = 0$ and $r^2 u(r)\frac{du(r)}{dr} \to 0, \quad$ as $r \to \infty$. By expanding the solution (resp. the forcing function) in spherical harmonic functions $Y_{l,m}(\lambda,\theta)$, with the coefficients $c_{l,m}(r)$ (resp. $f_{l,m}(r)$) (cf. [10]), we obtain a system of ordinary differential equations with the unknown functions $c_{l,m}(r)$:

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_{l,m}}{dr}(r)\right) + \frac{l(l+1)}{r^2}c_{l,m}(r) + \mu c_{l,m}(r) = f_{l,m}(r), \quad l \ge 0.$$

By multiplying the above equation by $r^2 v(r)$ and integrating by parts, we deduce that

$$\left(\frac{dc_{l,m}}{dr}, \frac{dv}{dr}\right)_{\omega_2} + l(l+1)(c_{l,m}, v)_{\omega_0} + \mu(c_{l,m}, v)_{\omega_2} = (f_{l,m}, v)_{\omega_2}, \quad l \ge 0.$$

Taking $v = c_{l,m}$, we find that $\frac{dc_{l,m}}{dr} \in L^2_{\omega_2}(\Lambda)$, while $c_{l,m} \in L^2_{\omega_2}(\Lambda)$ for $\mu > 0$, and $c_{l,m} \in L^2_{\omega_0}(\Lambda)$ for $\mu = 0$ and $l \ne 0$. Therefore, we have to study the orthogonal approximation in non-uniformly weighted Sobolev space.

Now, let

$$H^1_{\sigma,\lambda}(\Lambda) = \{v | v \text{ is measurable on } \Lambda \text{ and } \|v\|_{1,\sigma,\lambda} < \infty\}$$

where

$$\|v\|_{1,\sigma,\lambda} = \left(\left\|\frac{dv}{dr}\right\|^2_{\omega_{\sigma+\lambda}} + \|v\|^2_{\omega_\sigma}\right)^{\frac{1}{2}}.$$

Let $b, d, \theta \geq 0$. We define the corresponding bilinear form

$$a_{\sigma,\lambda,\theta}(u,v) = \left(\frac{du}{dr}, \frac{dv}{dr}\right)_{\omega_{\sigma+\lambda}} + b\,(u,v)_{\omega_\sigma} + d(u,v)_{\omega_{\sigma-\theta}}. \tag{3.1}$$

and the orthogonal projection $P^1_{L,\sigma,\lambda,\theta} : H^1_{\sigma,\lambda}(\Lambda) \to Q_L$ by

$$a_{\sigma,\lambda,\theta}\left(P^1_{L,\sigma,\lambda,\theta}v - v, \phi\right) = 0, \qquad \forall \phi \in Q_L. \tag{3.2}$$

In order to estimate $\|P^1_{L,\sigma,\lambda,\theta}v - v\|_{1,\sigma,\lambda}$, we need the following generalized Poincaré inequality.

Lemma 3.1. For any $v \in H^1_{\mu,2}(\Lambda)$ with $v(1) = 0$ and $\mu \neq -1$,

$$\|v\|_{\omega_\mu} \leq \frac{2}{|\mu+1|}\left\|\frac{dv}{dr}\right\|_{\omega_{\mu+2}}.$$

Proof. Since $\mu \neq -1$, we have that for $r_1, r_2 \geq 1$,

$$\begin{aligned}
|v(r_2) - v(r_1)| &\leq \left(\int_{r_1}^{r_2} \rho^{-\mu-2}d\rho\right)^{\frac{1}{2}} \left(\int_{r_1}^{r_2}\left(\frac{dv}{d\rho}(\rho)\right)^2 \rho^{\mu+2}d\rho\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{|\mu+1|}\left|r_2^{-\mu-1} - r_1^{-\mu-1}\right|\right)^{\frac{1}{2}} \left\|\frac{dv}{d\rho}\right\|_{\omega_{\mu+2}}.
\end{aligned}$$

Thus, $v(r)$ is continuous. Also, due to $v \in L^2_{\omega_\mu}(\Lambda)$, we have $r^{\mu+1}v^2(r) \to 0$ a.e. as $r \to \infty$. This fact, along with the continuity of v, implies that $r^{\mu+1}v^2(r) \to 0$ as $r \to \infty$. Therefore, for any $r \in \Lambda$,

$$\begin{aligned}
v^2(r)r^{\mu+1} &= \int_1^r \frac{d}{d\rho}(v^2(\rho)\rho^{\mu+1})d\rho \\
&= 2\int_1^r v(\rho)\frac{dv}{d\rho}(\rho)\rho^{\mu+1}d\rho + (\mu+1)\int_1^r v^2(\rho)\rho^\mu d\rho.
\end{aligned}$$

Letting $r \to \infty$ and using the Cauchy inequality, we obtain

$$|\mu+1|\|v\|^2_{\omega_\mu} \leq 2\|v\|_{\omega_\mu}\|\frac{dv}{dr}\|_{\omega_{\mu+2}}$$

which completes the proof. □

The above lemma is similar to the inequality (9.9.10) of [23] where the integrals are over the interval $(0, \infty)$. However Lemma 3.1 cannot be derived from (9.9.10) of [23].

In the forthcoming discussions, we denote by $\mathbb{P}_L$ the set of all algebraic polynomials of degree at most L, and ${}_0\mathbb{P}_L = \{v \in \mathbb{P}_L | v(1) = 0\}$.

Theorem 3.1. Let $\sigma < 2\gamma - 1,\ \lambda \leq 2$ and α be given by (2.16). If $v \in H^1_{\sigma,\lambda}(\Lambda)$ and $r^\delta v, r^{\delta+1}\frac{dv}{dr} \in D(A^{\frac{\mu-1}{2}})$ with $\mu \geq 1$, we have

$$\left\| P^1_{L,\sigma,\lambda,\theta} v - v \right\|_{1,\sigma,\lambda} \leq cL^{1-\mu} \left(\left\| r^{\delta+1} \frac{dv}{dr} \right\|_{D(A^{\frac{\mu-1}{2}})} + \| r^\delta v \|_{D(A^{\frac{\mu-1}{2}})} \right).$$

Proof. By the definition of $P^1_{L,\sigma,\lambda,\theta}$ and the projection theorem,

$$\left\| P^1_{L,\sigma,\lambda,\theta} v - v \right\|^2_{1,\sigma,\lambda} + \left\| P^1_{L,\sigma,\lambda,\theta} v - v \right\|^2_{\omega_{\sigma-\theta}} \leq \| \phi - v \|^2_{1,\sigma,\lambda} + \| \phi - v \|^2_{\omega_{\sigma-\theta}}, \quad \forall \phi \in Q_L. \tag{3.3}$$

Let

$$u(r) = \int_1^r \rho^{\delta-\gamma+1} \frac{d}{d\rho}(\rho^\gamma v(\rho)) d\rho.$$

Then,

$$\frac{du}{dr}(r) = r^{\delta-\gamma+1} \frac{d}{dr}(r^\gamma v(r)) = r^{\delta+1} \frac{dv}{dr}(r) + \gamma r^\delta v(r). \tag{3.4}$$

Hence,

$$v(r) = \frac{1}{r^\gamma} \left(\int_1^r \rho^{\gamma-\delta-1} \frac{du}{d\rho}(\rho) d\rho + v(1) \right).$$

We now take

$$\phi(r) = \frac{1}{r^\gamma} \left(\int_1^r \rho^{\gamma-\delta-1} P_{L-1,\sigma} \frac{du}{d\rho}(\rho) d\rho + v(1) \right). \tag{3.5}$$

By the definition of $P_{L-1,\sigma}$, there exists $q_{L-1} \in \mathbb{P}_{L-1}$ such that

$$P_{L-1,\sigma} \frac{du}{d\rho}(\rho) = \frac{1}{\rho^\gamma} q_{L-1} \left(1 - \frac{2}{\rho^\delta} \right). \tag{3.6}$$

The above with (2.10) implies that

$$\begin{aligned} \phi(r) &= \frac{1}{r^\gamma} \left(\int_1^r \rho^{-\delta-1} q_{L-1} \left(1 - \frac{2}{\rho^\delta} \right) d\rho + v(1) \right) \\ &= \frac{1}{2\delta r^\gamma} \left(\int_{-1}^{1-\frac{2}{r^\delta}} q_{L-1}(x) dx + 2\delta v(1) \right) \in Q_L. \end{aligned}$$

Obviously

$$\phi(r) - v(r) = \frac{1}{r^\gamma} \left(\int_1^r \rho^{\gamma-\delta-1} \left(P_{L-1,\sigma} \frac{du}{d\rho}(\rho) - \frac{du}{d\rho}(\rho) \right) d\rho \right). \tag{3.7}$$

By virtue of Lemma 3.1 with $\mu = \sigma - 2\gamma$, Theorem 2.1 and (3.4), we have that

$$\begin{aligned}
\|\phi - v\|^2_{\omega_{\sigma-\theta}} &\le \|\phi - v\|^2_{\omega_\sigma} \\
&= \int_\Lambda r^{\sigma-2\gamma} \left(\int_1^r \rho^{\gamma-\delta-1} \left(P_{L-1,\sigma} \frac{du}{d\rho}(\rho) - \frac{du}{d\rho}(\rho) \right) d\rho \right)^2 dr \\
&\le c \int_\Lambda r^{\sigma-2\delta} \left(P_{L-1,\sigma} \frac{du}{dr}(r) - \frac{du}{dr}(r) \right)^2 dr \\
&\le c \left\| P_{L-1,\sigma} \frac{du}{dr} - \frac{du}{dr} \right\|^2_{\omega_\sigma} \\
&\le cL^{2-2\mu} \left\| \frac{du}{dr} \right\|^2_{D\left(A^{\frac{\mu-1}{2}}\right)} \\
&\le cL^{2-2\mu} \left(\left\| r^{\delta+1} \frac{dv}{dr} \right\|^2_{D\left(A^{\frac{\mu-1}{2}}\right)} + \|r^\delta v\|^2_{D\left(A^{\frac{\mu-1}{2}}\right)} \right). \qquad (3.8)
\end{aligned}$$

Next, differentiating (3.7) yields

$$\frac{d}{dr}(\phi(r) - v(r)) = \frac{-\gamma}{r}(\phi(r) - v(r)) + \frac{1}{r^{\delta+1}} \left(P_{L-1,\delta} \frac{du}{dr}(r) - \frac{du}{dr}(r) \right).$$

Due to $\lambda \le 2$, we verify that

$$\left\| \frac{d}{dr}(\phi - v) \right\|^2_{\omega_{\sigma+\lambda}} \le 2\gamma^2 \|\phi - v\|^2_{\omega_\sigma} + 2 \left\| P_{L-1,\sigma} \frac{du}{dr} - \frac{du}{dr} \right\|_{\omega_{\sigma-2\delta}}.$$

Clearly, the right side of the above inequality is also bounded by the right side of (3.8). This completes the proof. □

We also need to study another orthogonal projection related to the homogeneous boundary condition at $r = 1$. We set

$$\begin{aligned}
{}_0H^1_{\sigma,\lambda}(\Lambda) &= \{v \in H^1_{\sigma,\lambda}(\Lambda) \mid v(1) = 0\}, \\
{}_0Q_L &= \{v \mid v \in Q_L \text{ and } v(1) = 0\},
\end{aligned}$$

and define the orthogonal projection ${}_0P^1_{L,\sigma,\lambda,\theta} : {}_0H^1_{\sigma,\lambda}(\Lambda) \to_0 Q_L$ by

$$a_{\sigma,\lambda,\theta}({}_0P^1_{L,\sigma,\lambda,\theta} v - v, \phi) = 0, \quad \forall \phi \in {}_0Q_L. \qquad (3.9)$$

Theorem 3.2. Let $\sigma < 2\gamma - 1$, $\lambda \le 2$ and α be given by (2.16). If $v \in {}_0H^1_{\sigma,\lambda}(\Lambda)$ and $r^\delta v, r^{\delta+1} \frac{dv}{dr} \in D\left(A^{\frac{\mu-1}{2}}\right)$ with $\mu \ge 1$, then

$$\|{}_0P^1_{L,\sigma,\lambda,\theta} v - v\|_{1,\sigma,\lambda} \le cL^{1-\mu} \left(\left\| r^{\delta+1} \frac{dv}{dr} \right\|_{D\left(A^{\frac{\mu-1}{2}}\right)} + \|r^\delta v\|_{D\left(A^{\frac{\mu-1}{2}}\right)} \right).$$

Proof. The proof is essentially the same as that of Theorem 3.1, except that we now take

$$\phi(r) = \frac{1}{r^\gamma} \int_1^r \rho^{\gamma-\delta-1} P_{L-1,\sigma} \frac{du}{d\rho}(\rho) d\rho.$$

$\square$

4. Projection errors in non-uniformly weighted spaces

We studied several orthogonal irrational projections in Sections 2 and 3 where the approximation results are described with the operator A. Although those results are sharp, but they are not always the most suitable from the numerical analysis point of view. In this section, we derive error estimates in certain non-uniformly weighted spaces, which are directly related to the function and its derivatives. Indeed, the proposed irrational functions are induced by the Jacobi polynomials, and so there is a close relation between the Jacobi approximation and the irrational approximation. Thus, we can derive error estimates by using recent results on the Jacobi approximation in [22].

Let $L^2_{\chi^{(\alpha,\beta)}}(-1,1)$ be the weighted space of square integrable functions on the interval $|x| < 1$. For any integer $\mu \geq 0$, we define

$$H^\mu_{\chi^{(\alpha,\beta)},B}(-1,1) = \{v \mid \ v \text{ is measurable for } |x| < 1 \text{ and } \|v\|_{\mu,\chi^{(\alpha,\beta)},B} < \infty\},$$

equipped with the semi-norm and norm as follows,

$$|v|_{\mu,\chi^{(\alpha,\beta)},B} = \left\| \frac{d^\mu v}{dx^\mu} \right\|_{L^2_{\chi^{(\alpha+\mu,\beta+\mu)}}}, \qquad \|v\|_{\mu,\chi^{(\alpha,\beta)},B} = \left(\sum_{k=0}^{\mu} |v|^2_{k,\chi^{(\alpha,\beta)},B} \right)^{\frac{1}{2}}.$$

Furthermore let

$$H^\mu_{\chi^{(\alpha,\beta)},*}(-1,1) = \left\{ v \mid \frac{dv}{dx} \in H^{\mu-1}_{\chi^{(\alpha,\beta)},B}(-1,1) \right\}$$

and

$$|v|_{\mu,\chi^{(\alpha,\beta)},*} = \left\| \frac{d^\mu v}{dx^\mu} \right\|_{L^2_{\chi^{(\alpha+\mu-1,\beta+\mu-1)}}}, \qquad \|v\|_{\mu,\chi^{(\alpha,\beta)},*} = \left(\sum_{k=0}^{\mu} |v|^2_{k,\chi^{(\alpha,\beta)},*} \right)^{\frac{1}{2}}.$$

For any $\mu > 0$, the spaces $H^\mu_{\chi^{(\alpha,\beta)},B}(-1,1)$, $H^\mu_{\chi^{(\alpha,\beta)},*}(-1,1)$ and their norms are defined by space interpolation as usual.

We define the orthogonal projection $P^*_{L,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(-1,1) \to \mathbb{P}_L$ by

$$\int_{-1}^{1} (P^*_{L,\alpha,\beta} v(x) - v(x)) \phi(x) \chi^{(\alpha,\beta)}(x) dx = 0, \qquad \forall \phi \in \mathbb{P}_L. \tag{4.1}$$

According to Theorem 2.1 in [22], we know that for any $v \in H^{\mu}_{\chi^{(\alpha,\beta)},B}(-1,1)$, $\alpha, \beta > -1$, integer μ and $0 \le s \le \mu$, we have

$$\left\| P^{*}_{L,\alpha,\beta} v - v \right\|_{s,\chi^{(\alpha,\beta)},B} \le c L^{s-\mu} |v|_{\mu,\chi^{(\alpha,\beta)},B}.^{1} \tag{4.2}$$

Next, for $\alpha, \beta, \gamma, \delta > -1$, we set $H^{0}_{\alpha,\beta,\gamma,\delta}(-1,1) = L^{2}_{\chi^{(\gamma,\delta)}}(-1,1)$ and define

$$H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) = \left\{ v \mid v \in L^{2}_{\chi^{(\gamma,\delta)}}(-1,1) \text{ and } \frac{dv}{dx} \in L^{2}_{\chi^{(\alpha,\beta)}}(-1,1) \right\}$$

equipped with the norm

$$\|v\|_{1,\alpha,\beta,\gamma,\delta} = \left(\left\| \frac{dv}{dx} \right\|^{2}_{\chi^{(\alpha,\beta)}} + \|v\|^{2}_{\chi^{(\gamma,\delta)}} \right)^{\frac{1}{2}}.$$

For $0 \le \mu \le 1$, we define the space $H^{\mu}_{\alpha,\beta,\gamma,\delta}(-1,1)$ and its norm $\|v\|_{\mu,\alpha,\beta,\gamma,\delta}$ by space interpolation. We also set ${}_{0}H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) = \{v \in H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) | v(1) = 0\}$. Let us denote

$$a^{*}_{\alpha,\beta,\gamma,\delta}(u,v) = \int_{-1}^{1} \frac{du}{dx}(x) \frac{dv}{dx}(x) \chi^{(\alpha,\beta)}(x) dx + \int_{-1}^{1} u(x) v(x) \chi^{(\gamma,\delta)}(x) dx, \tag{4.3}$$

and define the orthogonal projection $P^{*,1}_{L,\alpha,\beta,\gamma,\delta} : H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) \to \mathbb{P}_{L}$ by

$$a^{*}_{\alpha,\beta,\gamma,\delta}\left(P^{*,1}_{L,\alpha,\beta,\gamma,\delta} v - v, \phi \right) = 0, \quad \forall \phi \in \mathbb{P}_{L}.$$

Similarly, we define the orthogonal projection ${}_{0}P^{*,1}_{L,\alpha,\beta,\gamma,\delta} : {}_{0}H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) \to {}_{0}\mathbb{P}_{L}$ by

$$a^{*}_{\alpha,\beta,\gamma,\delta}\left({}_{0}P^{*,1}_{L,\alpha,\beta,\gamma,\delta} v - v, \phi \right) = 0, \quad \forall \phi \in {}_{0}\mathbb{P}_{L}.$$

Theorem 3.1 in [22] asserts that if $\alpha \le \gamma + 2$ and $\beta \le \delta + 2$, then, for any $v \in H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) \bigcap H^{\mu}_{\chi^{(\alpha,\beta)},*}(-1,1)$ with integer $\mu \ge 1$, we have

$$\left\| P^{*,1}_{L,\alpha,\beta,\gamma,\delta} v - v \right\|_{1,\alpha,\beta,\gamma,\delta} \le c L^{1-\mu} |v|_{\mu,\chi^{(\alpha,\beta)},*}. \tag{4.4}$$

We also know from Theorem 3.2 in [22] that if $\alpha \le \gamma + 2$, $\beta \le 0$ and $\delta \ge 0$, then for any $v \in_{0} H^{1}_{\alpha,\beta,\gamma,\delta}(-1,1) \bigcap H^{\mu}_{\chi^{(\alpha,\beta)}}, *(-1,1)$ and integer $\mu \ge 1$,

$$\left\| {}_{0}P^{*,1}_{L,\alpha,\beta,\gamma,\delta} v - v \right\|_{1,\alpha,\beta,\gamma,\delta} \le c L^{1-\mu} |v|_{\mu,\chi^{(\alpha,\beta)},*}. \tag{4.5}$$

We now study again the orthogonal irrational projections $P_{L,\sigma}$, $P^{1}_{L,\sigma,\lambda,\theta}$ and ${}_{0}P^{1}_{L,\sigma,\lambda,\theta}$, defined, respectively, by (2.23), (3.2) and (3.9), by using non-uniformly weighted spaces as specified below.

For any integer $\mu \ge 0$, we set

$$B_{\mu,\delta,\sigma,\theta}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\mu,\delta,\sigma,\theta} < \infty\}$$

[1] The estimate (4.2) improves Theorem 2.3 of Guo [18]. Moreover, the dependence of c on α and β are described explicitly in [22]. Noted that Funaro [11] obtained a similar result for integer $\mu \ge 0$ and $s = 0$, while Babuška and Guo [3] derived the result for real $\alpha = \beta$ and $0 \le s \le \mu$.

where

$$\|v\|_{B_{\mu,\delta,\sigma,\theta}} = \left(\sum_{k=1}^{\mu} \left\| r^{k-\frac{\theta}{2}} (r^\delta - 1)^{\frac{\mu}{2}} \frac{d^k v}{dr^k} \right\|_{\omega_\sigma}^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

The spaces $B_{\mu,\delta,\sigma,\theta}(\Lambda)$ with real $\mu \geq 0$ and their norms are defined by space interpolation.

Theorem 4.1. If $\sigma < 2\gamma - 1$ and α is given by (2.16), then, for any $v \in B_{\mu,\delta,\sigma,0}(\Lambda)$ and $\mu \geq 0$,

$$\|P_{L,\sigma} v - v\|_{\omega_\sigma} \leq c L^{-\mu} \|v\|_{B_{\mu,\delta,\sigma,0}}.$$

Proof. Let $u(r) = r^\gamma v(r)$, $u_L(r) = r^\gamma P_{L,\sigma} v(r)$, $w(x) = u((\frac{2}{1-x})^{\frac{1}{\delta}})$ and $w_L(x) = u_L((\frac{2}{1-x})^{\frac{1}{\delta}}) \in \mathbb{P}_L$. For any $\phi \in Q_L$, let ψ be defined by $r^\gamma \phi(r) = \psi(1 - \frac{2}{r^\delta})$. It is clear that $\psi \in \mathbb{P}_L$. By the definition of $P_{L,\sigma}$,

$$\int_\Lambda (u_L(r) - u(r)) \psi \left(1 - \frac{2}{r^\delta}\right) r^{\sigma - 2\gamma} dr = 0, \quad \forall \psi \in \mathbb{P}_L.$$

By using (2.9), (2.10) and (2.16), we can rewrite the above as

$$\int_{-1}^{1} (w_L(x) - w(x)) \psi(x) (1-x)^\alpha dx = 0, \quad \forall \psi \in \mathbb{P}_L.$$

Thus,

$$w_L(x) = P^*_{L,\alpha,0} w(x).$$

Moreover, by (4.2), for any integer $\mu \geq 0$,

$$\|w_L - w\|_{L^2_{\chi^{(\alpha,0)}}(-1,1)} \leq c L^{-\mu} \left\| \frac{d^\mu w}{dx^\mu} \right\|_{L^2_{\chi^{(\alpha+\mu,\mu)}}(-1,1)}. \tag{4.7}$$

By induction with (2.10), it can be checked that

$$\frac{d^\mu w}{dx^\mu}(x) = \sum_{k=1}^{\mu} \frac{d^k u}{dr^k}(r) r^{\mu\delta + k} q_k(r) \tag{4.8}$$

where $q_k(r)$ are some irrational functions which are uniformly bounded on Λ. Therefore, a direct calculation leads to

$$\left\| \frac{d^\mu w}{dx^\mu} \right\|^2_{L^2_{\chi^{(\alpha+\mu,\mu)}}} \leq c \sum_{k=1}^{\mu} \left\| r^{k-\gamma} (r^\delta - 1)^{\frac{\mu}{2}} \frac{d^k u}{dr^k} \right\|^2_{\omega_\sigma} \leq c\|u\|^2_{B_{\mu,\delta,\sigma,2\gamma}} \leq c\|v\|^2_{B_{\mu,\delta,\sigma,0}}.$$

Thus, the desired result with integer $\mu \geq 0$ follows from the above estimate, (4.7) and the fact that

$$\|P_{L,\sigma} v - v\|_{\omega_\sigma} = \|u_L - u\|_{\omega_{\sigma - 2\gamma}} = \|w_L - w\|_{L^2_{\chi^{(\alpha,0)}}(-1,1)}.$$

Finally, the conclusion for any $\mu \geq 0$ follows from space interpolation. □

Theorem 4.2. If $\sigma < 2\gamma - 1$, $\lambda \le 2$, $\theta \ge 0$ and (2.16) holds, then for any $v \in B_{\mu-1,\delta,\sigma,0}(\Lambda)$ and $\mu \ge 1$,

$$\left\| P^1_{L,\sigma,\lambda,\theta} v - v \right\|_{1,\sigma,\lambda} \le c L^{1-\mu} \|v\|_{B_{\mu-1,\delta,\sigma,0}}.$$

Proof. Let $u(r) = r^\gamma v(r)$, $w(x) = u((\frac{2}{1-x})^{\frac{1}{\delta}})$, $w_L(x) = P^{*,1}_{\alpha+2,0,\alpha,0} w(x)$ and $\psi_L(r) = w_L(1 - \frac{2}{r^\delta})$. Then,

$$\phi(r) = \frac{1}{r^\gamma} \psi_L(r) = \frac{1}{r^\gamma} w_L \left(1 - \frac{2}{r^\delta}\right) \in Q_L.$$

Thanks to (3.3) and the fact that $\theta \ge 0$, we only need to estimate $\|\phi - v\|_{1,\sigma,\lambda}$.

By virtue of (2.9), (2.10), (2.16) and (4.4), for any integer $\mu \ge 1$,

$$\begin{aligned} \|\phi - v\|^2_{\omega_\sigma} = \|\psi_L - u\|^2_{\omega_{\sigma-2\gamma}} &= \frac{1}{\delta} 2^{-\alpha-1} \int_{-1}^{1} (w_L(x) - w(x))^2 (1-x)^\alpha dx \\ &\le c L^{2-2\mu} \left\| \frac{d^\mu w}{dx^\mu} \right\|^2_{L^2_{\chi^{(\alpha+\mu+1,\mu-1)}}}. \end{aligned} \tag{4.9}$$

On the other hand,

$$\frac{d}{dr}(\phi(r) - v(r)) = \frac{-\gamma}{r^{\gamma+1}}(\psi_L(r) - u(r)) + \frac{1}{r^\gamma}\frac{d}{dr}(\psi_L(r) - u(r)).$$

Hence

$$\left\| \frac{d}{dr}(\phi - v) \right\|^2_{\omega_{\sigma+\lambda}} \le 2\gamma^2 \|\psi_L - u\|^2_{\omega_{\sigma+\lambda-2\gamma-2}} + 2 \left\| \frac{d}{dr}(\psi_L - u) \right\|^2_{\omega_{\sigma+\lambda-2\gamma}}. \tag{4.10}$$

Clearly,

$$\|\psi_L - u\|^2_{\omega_{\sigma+\lambda-2\gamma-2}} \le \|\psi_L - u\|^2_{\omega_{\sigma-2\gamma}}. \tag{4.11}$$

Moreover, by (2.9), (2.10), (2.16) and (4.4),

$$\begin{aligned} \left\| \frac{d}{dr}(\psi_L - u) \right\|^2_{\omega_{\sigma+\lambda-2\gamma}} &= \int_\Lambda r^{\sigma+\lambda-2\gamma} \left(\frac{d}{dr}(\psi_L(r) - u(r)) \right)^2 dr \\ &= \delta 2^{\frac{\sigma+\lambda-2\gamma-1}{\delta}} \int_{-1}^{1} \left(\frac{d}{dx}(w_L(x) - w(x)) \right)^2 (1-x)^{\frac{2\gamma-\sigma-\lambda+\delta+1}{\delta}} dx \\ &= \delta 2^{\frac{\sigma+\lambda-2\gamma-1}{\delta}} \int_{-1}^{1} \left(\frac{d}{dx}(w_L(x) - w(x)) \right)^2 (1-x)^{\alpha+2+\frac{2-\lambda}{\delta}} dx \\ &\le c \left\| \frac{d}{dx}(w_L - w) \right\|^2_{L^2_{\chi^{(\alpha+2,0)}}} \\ &\le c L^{2-2\mu} \left\| \frac{d^\mu w}{dx^\mu} \right\|^2_{L^2_{\chi^{(\alpha+\mu+1,\mu-1)}}}. \end{aligned} \tag{4.12}$$

By virtue of (4.8),

$$\left\|\frac{d^\mu w}{dx^\mu}\right\|_{L^2_{\chi^{(\alpha+\mu+1,\mu-1)}}} \le c\sum_{k=1}^{\mu}\left\|r^{k-\gamma}(r^\delta-1)^{\frac{\mu-1}{2}}\frac{d^k u}{dr^k}\right\|^2_{\omega_\sigma} = c\|v\|^2_{B_{\mu-1,\delta,\sigma,0}}. \tag{4.13}$$

The combination of (3.3) and (4.9)– (4.13) implies the desired result for any integer $\mu \ge 1$. The result for any $\mu \ge 1$ follows from space interpolation. □

In the special case $b = 0$ in (3.1), we may obtain a slightly better result.

Theorem 4.3. Let $b = 0$ in (3.1). If $\sigma < 2\gamma - 1, \lambda \le 2-\theta, \theta \ge 0$ and (2.16) holds, then, for any $v \in B_{\mu-1,\delta,\sigma,\theta}(\Lambda)$ and $\mu \ge 1$,

$$\left\|\frac{d}{dr}(P^1_{L,\sigma,\lambda,\theta}v - v)\right\|_{\omega_{\sigma+\lambda}} + \left\|P^1_{L,\sigma,\lambda,\theta}v - v\right\|_{\omega_{\sigma-\theta}} \le cL^{1-\mu}\|v\|_{B_{\mu-1,\delta,\sigma,\theta}}.$$

Proof. We use the same notations as in the proof of the last theorem. For any integer $\mu \ge 1$,

$$\begin{aligned}\|\phi - v\|^2_{\omega_{\sigma-\theta}} = \|\psi_L - u\|^2_{\omega_{\sigma-2\gamma-\theta}} &= \int_\Lambda r^{\sigma-2\gamma-\theta}(\psi_L(r)-u(r))^2dr\\ &= \frac{1}{\delta}2^{-\alpha-1-\frac{\theta}{\delta}}\int_{-1}^{1}(w_L(x)-w(x))^2(1-x)^{\alpha+\frac{\theta}{\delta}}dx\\ &\le cL^{2-2\mu}\left\|\frac{d^\mu w}{dx^\mu}\right\|^2_{L^2_{\chi^{(\alpha+\frac{\theta}{\delta}+\mu+1,\mu-1)}}}.\end{aligned} \tag{4.14}$$

Since $\lambda \le 2-\theta$, we have

$$\|\psi_L - u\|^2_{\omega_{\sigma+\lambda-2\gamma-2}} \le \|\psi_L - u\|^2_{\omega_{\sigma-2\gamma-\theta}}. \tag{4.15}$$

On the other hand, (4.10) is still valid. Therefore, we derive from (2.9), (2.10), (2.16) and (4.4) that

$$\begin{aligned}\left\|\frac{d}{dr}(\psi_L - u)\right\|^2_{\omega_{\sigma+\lambda-2\gamma}} &\le \left\|\frac{d}{dr}(\psi_L - u)\right\|^2_{\omega_{\sigma-2\gamma-\theta+2}}\\ &\le c\int_{-1}^{1}\left(\frac{d}{dx}(w_L(x)-w(x))\right)^2(1-x)^{\frac{2\gamma-\sigma+\delta+\theta-1}{\delta}}dx\\ &\le cL^{2-2\mu}\left\|\frac{d^\mu w}{dx^\mu}\right\|^2_{L^2_{\chi^{(\alpha+\frac{\theta}{\delta}+\mu+1,\mu-1)}}}.\end{aligned} \tag{4.16}$$

Moreover, a direct calculation with (4.8) shows that

$$\left\|\frac{d^\mu w}{dx^\mu}\right\|^2_{L^2_{\chi^{(\alpha+\frac{\theta}{\delta}+\mu+1,\mu-1)}}} \le c\sum_{k=1}^{\mu}\left\|r^{k-\frac{\theta}{2}}(r^\delta-1)^{\frac{\mu-1}{2}}\frac{d^k v}{dr^k}\right\|^2_{\omega_\sigma}. \tag{4.17}$$

The combination of projection theorem with (4.10) and (4.14)–(4.17) leads to the desired result with integer $\mu \ge 1$. The result for any $\mu \ge 1$ comes from space interpolation. □

Note that the proofs of Theorems 4.2 and 4.3 are based on the estimate (4.4). By similar arguments, we can use (4.5) to prove the following results:

Theorem 4.4. If $\sigma < 2\gamma - 1,\ \lambda \leq 2,\ \theta \geq 0$ and (2.16) holds, then, for any $v \in {}_0H^1_{\sigma,\lambda}(\Lambda) \bigcap B_{\mu-1,\delta,\sigma,0}(\Lambda)$ with $\mu \geq 1$,

$$\left\| {}_0P^1_{L,\sigma,\lambda,\theta} v - v \right\|_{1,\sigma,\lambda} \leq cL^{1-\mu} \|v\|_{B_{\mu-1,\delta,\sigma,0}}.$$

Theorem 4.5. Let $b = 0$ in (3.1). If $\sigma < 2\gamma - 1,\ \lambda \leq 2 - \theta,\ \theta \geq 0$ and (2.16) holds, then, for any $v \in B_{\mu-1,\delta,\sigma,\theta}(\Lambda)$ with $v(1) = 0$ and $\mu \geq 1$,

$$\left\| \frac{d}{dr}({}_0P^1_{L,\sigma,\lambda,\theta} v - v) \right\|_{\omega_{\sigma+\lambda}} + \left\| {}_0P^1_{L,\sigma,\lambda,\theta} v - v \right\|_{\omega_{\sigma-\theta}} \leq cL^{1-\mu} \|v\|_{B_{\mu-1,\delta,\sigma,\theta}}.$$

5. The special case: $\sigma = 2\gamma - 1$

In this section, we deal with a special, but important case: $\sigma = 2\gamma - 1$. We introduce the basis functions

$$\widetilde{I}^{(\gamma,\delta)}_l(r) = \frac{1}{r^{\gamma+\delta}} J^{(1,0)}_l\left(1 - \frac{2}{r^\delta}\right).$$

By (2.7)–(2.10),

$$\left(\widetilde{I}^{(\gamma,\delta)}_l, \widetilde{I}^{(\gamma,\delta)}_m\right)_{\omega_\sigma} = \frac{1}{2\delta(l+1)}\delta_{l,m}. \tag{5.1}$$

Moreover, by (2.1),

$$\frac{1}{\delta^2}\frac{d}{dr}(r^{1-2\delta}(r^\delta - 1)\frac{d}{dr}\left(r^{\gamma+\delta}\widetilde{I}^{(\gamma,\delta)}_l(r)\right) + l(l+2)r^{\gamma-\delta-1}\widetilde{I}^{(\gamma,\delta)}_l(r) = 0. \tag{5.2}$$

Now, let us denote

$$\widetilde{Q}_L = \mathrm{span}\{\widetilde{I}^{(\gamma,\delta)}_0, \widetilde{I}^{(\gamma,\delta)}_1, \ldots, \widetilde{I}^{(\gamma,\delta)}_L\},$$

and define the orthogonal projection $\widetilde{P}_{L,\sigma} : L^2_{\omega_\sigma}(\Lambda) \to \widetilde{Q}_L$ by

$$(\widetilde{P}_{L,\sigma} v - v, \phi)_{\omega_\sigma} = 0, \quad \forall \phi \in \widetilde{Q}_L. \tag{5.3}$$

Next, we set $\widetilde{\eta}(r) = r^{1-2\delta}(r^\delta - 1)$, and define the operator

$$\widetilde{A}v = -\frac{1}{\delta^2}r^{\gamma+\delta-\sigma}\frac{d}{dr}\left(\widetilde{\eta}(r)\frac{d}{dr}(r^{\gamma+\delta}v(r))\right) \tag{5.4}$$

with

$$D(\widetilde{A}) = \{v \mid v,\ \widetilde{A}v \in L^2_{\omega_\sigma}(\Lambda), r^{\gamma+\delta}v(r) \text{ is bounded and } r^{\gamma+\delta}\widetilde{\eta}(r)v(r)\frac{d}{dr}(r^{\gamma+\delta}v(r)) \to 0 \text{ as } r \to \infty\}.$$

We set $D(\widetilde{A}^0) = L^2_{\omega_\sigma}(\Lambda)$. For any integer $\mu > 0$, $D(\widetilde{A^\mu})$ is defined by induction. By (5.2), (5.4) and an argument as in Remarks 2.1 and 2.2, we know that $\widetilde{A}$ is a non-negative, self-adjoint operator in $L^2_{\omega_\sigma}(\Lambda)$, and there exists $\widetilde{A}^{\frac{1}{2}}$ such that

$$(\widetilde{A}v, v)_{\omega_\sigma} = (\widetilde{A}^{\frac{1}{2}}v, \widetilde{A}^{\frac{1}{2}}v)_{\omega_\sigma} = \frac{1}{\delta^2}\left\|\frac{d}{dr}(r^{\gamma+\delta}v)\right\|^2_{L^2_{\widetilde{\eta}}(\Lambda)}. \tag{5.5}$$

So we could define the space $D(\widetilde{A}^{\mu})$ and its norm for any $\mu \geq 0$ by space interpolation. In particular, $\|v\|^2_{D(\widetilde{A}^{\frac{1}{2}})} = ((\widetilde{A}v, v)_{\omega_\sigma} + \|v\|^2_{\omega_\sigma})^{\frac{1}{2}}$.

Theorem 5.1. If $\sigma = 2\gamma - 1$, then for any $v \in D(\widetilde{A}^{\frac{\mu}{2}})$ and $\mu \geq s \geq 0$,

$$\|\widetilde{P}_{L,\sigma}v - v\|_{D(\widetilde{A}^{\frac{s}{2}})} \leq cL^{s-\mu}\|v\|_{D(\widetilde{A}^{\frac{\mu}{2}})}.$$

Proof. Let m be any positive integer. For $v \in D(\widetilde{A}^{m})$ and $j \leq m-1$,

$$r^{\gamma+\delta}\widetilde{\eta}(r)\widetilde{A}^{j}v(r)\frac{d}{dr}\left(r^{\gamma+\delta}\widetilde{l}_l^{(\gamma,\delta)}(r)\right) \to 0, \qquad \text{as} \quad r \to \infty.$$

Thus, following the same line as in the first part of proof of Theorem 2.1, we can use (5.2) and (5.4) to obtain that for any even integer $\mu = 2m$ and $s = 0$,

$$\|\widetilde{P}_{L,\sigma}v - v\|^2_{\omega_\sigma} \leq cL^{-2\mu}\|v\|^2_{D(\widetilde{A}^{\frac{\mu}{2}})}.$$

Also, using the same argument as in the derivation of (2.3), we can use (5.2), (5.4) and (5.5) to deduce that for $\mu = 2m+1$ and $s = 0$,

$$\|\widetilde{P}_{L,\sigma}v - v\|^2_{\omega_\sigma} \leq cL^{-2\mu}\left\|\frac{d}{dr}(r^{\gamma+\delta}\widetilde{A}^{m}v)\right\|^2_{L^2_{\widetilde{\eta}}(\Lambda)} = cL^{-2\mu}\|v\|_{D(\widetilde{A}^{\frac{\mu}{2}})}.$$

Finally, the desired result follows from the above results and space interpolation. $\square$

We now turn to the orthogonal projection $\widetilde{P}^1_{L,\sigma,\lambda,\theta} : H^1_{\sigma,\lambda}(\Lambda) \to \widetilde{Q}_L$ defined by

$$a_{\sigma,\lambda,\theta}\left(\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v, \phi\right) = 0, \quad \forall\phi \in \widetilde{Q}_L. \tag{5.6}$$

Theorem 5.2. Let $\sigma = 2\gamma - 1$, $\lambda \leq 2$ and $\theta \geq 0$. If $v \in H^1_{\sigma,\lambda}(\Lambda)$ and $r^{\delta}v, r^{\delta+1}\frac{dv}{dr} \in D(\widetilde{A}^{\frac{\mu-1}{2}})$ with $\mu \geq 1$, then

$$\|\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v\|_{1,\sigma,\lambda} \leq cL^{1-\mu}\left(\left\|r^{\delta+1}\frac{dv}{dr}\right\|_{D\left(\widetilde{A}^{\frac{\mu-1}{2}}\right)} + \|r^{\delta}v\|_{D\left(\widetilde{A}^{\frac{\mu-1}{2}}\right)}\right).$$

Proof. The proof is essentially the same as that of Theorem 3.1, except that we now take $u(r)$ and $\phi(r)$ as follows:

$$u(r) = \int_1^r \rho^{-\gamma+1}\frac{d}{d\rho}(\rho^{\gamma+\delta}v(\rho))d\rho,$$

$$\phi(r) = \frac{1}{r^{\gamma+\delta}}\left(\int_1^r \rho^{\gamma-1}\widetilde{P}_{L-1,\sigma}\frac{du}{d\rho}(\rho)d\rho + v(1)\right) \in \widetilde{Q}_L.$$

$\square$

To account for the homogeneous boundary condition at $r = 1$, we set ${}_0\widetilde{Q}_L = \{v \in \widetilde{Q}_L \mid v(1) = 0\}$ and define the orthogonal projection ${}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta} : {}_0H^1_{\sigma,\lambda}(\Lambda) \to {}_0\widetilde{Q}_L$ by

$$a_{\sigma,\lambda,\theta}\left({}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v, \phi\right) = 0, \quad \forall\phi \in {}_0\widetilde{Q}_L. \tag{5.7}$$

Theorem 5.3. Let $\sigma = 2\gamma - 1$, $\lambda \le 2$ and $\theta \ge 0$. If $v \in {}_0H^1_{\sigma,\lambda}(\Lambda)$ and $r^\delta v, r^{\delta+1}\frac{dv}{dr} \in D(\widetilde{A}^{\frac{\mu-1}{2}})$ with $\mu \ge 1$, then

$$\|{}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v\|_{1,\sigma,\lambda} \le cL^{1-\mu}\left(\left\|r^{\delta+1}\frac{dv}{dr}\right\|_{D\left(\widetilde{A}^{\frac{\mu-1}{2}}\right)} + \|r^\delta v\|_{D\left(\widetilde{A}^{\frac{\mu-1}{2}}\right)}\right).$$

Proof. Once again, the proof is essentially the same as that of the last theorem, except that we take $u(r)$ as above and

$$\phi(r) = \frac{1}{r^{\gamma+\delta}}\int_1^r \rho^{\gamma-1}\widetilde{P}_{L-1,\sigma}\frac{du}{d\rho}(\rho)d\rho \in_0 \widetilde{Q}_L.$$

□

As in Section 4, we may also use recent results on the Jacobi approximation [22] to bound the norms $\|\widetilde{P}_{L,\sigma}v - v\|_{\omega_\sigma}$, $\|\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v\|_{1,\sigma,\lambda}$ and $\|{}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta}v - v\|_{1,\sigma,\lambda}$. For this purpose, we set $\widetilde{\mathbb{P}}_L = \text{span}\{\widetilde{J}_l^{(-1,0)} : l = 1, 2, \cdots, L\}$ with

$$\widetilde{J}_l^{(-1,0)}(x) = (1-x)J_{l-1}^{(1,0)}(x), \quad l = 1, 2, \dots \quad . \tag{5.8}$$

According to (2.7)–(2.10),

$$\int_{-1}^1 \widetilde{J}_l^{(-1,0)}(x)\widetilde{J}_m^{(-1,0)}(x)\chi^{(-1,0)}(x)dx = \gamma_{l-1}^{(1,0)}\delta_{l,m} = \frac{2}{l}\delta_{l,m}, \quad l, m \ge 1.$$

For any $v \in L^2_{\chi^{(-1,0)}}(-1,1)$, we have

$$v(x) = \sum_{l=1}^{\infty} v_l^*\widetilde{J}_l^{(-1,0)}(x), \qquad P^*_{L,-1,0}v = \sum_{l=1}^{L} v_l^*\widetilde{J}_l^{(-1,0)}(x) \tag{5.9}$$

with

$$v_l^* = \frac{l}{2}\int_{-1}^1 v(x)\widetilde{J}_l^{(-1,0)}(x)\chi^{(-1,0)}(x)dx = \frac{l}{2}\int_{-1}^1 \frac{v(x)}{1-x}J_{l-1}^{(1,0)}(x)\chi^{(1,0)}(x)dx.$$

Thus, $\{v_l^* : l \ge 1\}$ are exactly the same as the coefficients of the expansion of function $\frac{v(x)}{1-x}$ in terms of $J_{l-1}^{(1,0)}(x)$, $l \ge 1$. So by (4.1), we have $P^*_{L,-1,0}v = (1-x)P^*_{L,1,0}(\frac{v}{1-x})$. Therefore, we derive from (4.2) that for any $\mu \ge 0$,

$$\|P^*_{L,-1,0}v - v\|_{\chi^{(-1,0)}} = \left\|P^*_{L,1,0}\left(\frac{v}{1-x}\right) - \frac{v}{1-x}\right\|_{\chi^{(1,0)}} \le cL^{-\mu}\left|\frac{v}{1-x}\right|_{\mu,\chi^{(1,0)},B}. \tag{5.10}$$

Next, we introduce another orthogonal projection related to ${}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta}$ which is defined by (5.7). Let

$$H^\mu_{0,\alpha,\beta,\gamma,\sigma}(-1,1) = \{v \in H^\mu_{\alpha,\beta,\gamma,\delta}(-1,1) \mid v(1) = v(-1) = 0\},$$

$$\mathbb{P}^0_L = \{v \in \mathbb{P}_L \mid v(1) = v(-1) = 0\},$$

and define the orthogonal projection $P^{*,1,0}_{L,\alpha,\beta,\gamma,\delta} : H^1_{0,\alpha,\beta,\gamma,\delta}(-1,1) \to \mathbb{P}^0_L$ by

$$a^*_{\alpha,\beta,\gamma,\delta}\left(P^{*,1,0}_{L,\alpha,\beta,\gamma,\delta}v - v, \phi\right) = 0, \quad \forall\phi \in \mathbb{P}^0_L.$$

According to Theorem 3.6 in [22], for any $v \in H^1_{0,1,0,-1,0}(-1,1) \bigcap H^{\mu}_{\chi^{(1-\varepsilon,0)},*}(-1,1)$ with integer $\mu \geq 1$ and $0 < \varepsilon < 2$, we have

$$\left\| \frac{d}{dx}(P^{*,1,0}_{L,1,0,-1,0}v - v)\right\|_{\chi^{(1,0)}} + \left\| P^{*,1,0}_{L,1,0,-1,0}v - v\right\|_{\chi^{(-1,0)}} \leq cL^{1-\mu}|v|_{\mu,\chi^{(1-\varepsilon,0)},*}{}^{2} \qquad (5.11)$$

We now turn to the upper bound of $\| \widetilde{P}_{L,\sigma}v - v\|_{\omega_\sigma}$. Let $B_{\mu,\delta,\sigma,\theta}(\Lambda)$ be the same as in (4.6).

Theorem 5.4. If $\sigma = 2\gamma - 1$, then for any $v \in L^2_{\omega_\sigma}(\Lambda) \bigcap B_{\mu-1,\delta,\sigma,-2\delta}(\Lambda)$ and $\mu \geq 1$,

$$\| \widetilde{P}_{L,\sigma}v - v\|_{\omega_\sigma} \leq cL^{-\mu}\|v\|_{B_{\mu-1,\delta,\sigma,-2\delta}}.$$

Proof. Let $u(r) = r^{\gamma}v(r)$, $u_L(r) = r^{\gamma}\widetilde{P}_{L,\sigma}v$, $w(x) = u((\frac{2}{1-x})^{\frac{1}{\delta}})$ and $w_L(x) = u_L((\frac{2}{1-x})^{\frac{1}{\delta}}) \in \widetilde{\mathbb{P}}_L$. For any $\phi \in \widetilde{Q}_L$, we define ψ by $r^{\gamma}\phi(r) = \psi(1 - \frac{2}{r^{\delta}})$. Then $\psi \in \widetilde{\mathbb{P}}_L$. By the definition of $\widetilde{P}_{L,\sigma}$ in (5.3), we have that

$$\int_{\Lambda}(u_L(r) - u(r))\psi\left(1 - \frac{2}{r^{\delta}}\right)r^{\sigma-2\gamma}dr = 0, \quad \forall\psi \in \widetilde{\mathbb{P}}_L,$$

or equivalently,

$$\int_{-1}^{1}(w_L(x) - w(x))\psi(x)(1-x)^{-1}dx = 0, \quad \forall\psi \in \widetilde{\mathbb{P}}_L.$$

Thus, $w_L(x) = P^{*}_{L,-1,0}w$, and by (5.10), we have for integer $\mu \geq 0$,

$$\|w_L - w\|_{L^2_{\chi^{(-1,0)}}} \leq cL^{-\mu}\left\| \frac{d^{\mu}}{dx^{\mu}}\left(\frac{w}{1-x}\right)\right\|_{L^2_{\chi^{(\mu,\mu-1)}}}.$$

Since $\frac{w(x)}{1-x} = \frac{1}{2}r^{\delta}u(r)$, we can use an induction argument to show that

$$\frac{d^{\mu}}{dx^{\mu}}\left(\frac{w}{1-x}\right) = \sum_{k=1}^{\mu} r^{\mu\delta+\delta+k}\widetilde{q}_k(r)\frac{d^k u}{dr^k}(r) \qquad (5.12)$$

where $\widetilde{q}_k(r)$ are some irrational functions which are uniformly bounded on Λ. Therefore, a direct calculation with (2.9) and (2.10) yields that

$$\begin{aligned}\sqrt{2\delta}\| \widetilde{P}_{L,\sigma}v - v\|_{\omega_\sigma} = \|w_L - w\|_{L^2_{\chi^{(-1,0)}}} &\leq cL^{-\mu}\sum_{k=1}^{\mu}\|r^{k+\delta-\frac{1}{2}\sigma-\frac{1}{2}}(r^{\delta}-1)^{\frac{\mu-1}{2}}\frac{d^k u}{dr^k}\|_{\omega_\sigma} \\ &= cL^{-\mu}\sum_{k=1}^{\mu}\|r^{k+\delta-\gamma}(r^{\delta}-1)^{\frac{\mu-1}{2}}\frac{d^k u}{dr^k}\|_{\omega_\sigma} \\ &\leq cL^{-\mu}\sum_{k=1}^{\mu}\|r^{k+\delta}(r^{\delta}-1)^{\frac{\mu-1}{2}}\frac{d^k v}{dr^k}\|_{\omega_\sigma}\end{aligned}$$

The conclusion follows from the above and space interpolation. □

[2] Bernardi et al. [5] and Guo [19] also studied the Jacobi approximation with $\alpha = 1, \gamma = -1$, and $\beta = \delta = 0$. But the estimate (5.11) significantly improves those results.

Theorem 5.5. If $\sigma = 2\gamma - 1,\ \lambda \le 2$ and $\theta \ge 0$, then, for any $v \in {}_0H^1_{\sigma,\lambda}(\Lambda) \bigcap B_{\mu-1,\delta,\sigma,-\epsilon\delta}(\Lambda)$ with $\mu \ge 1$ and $0 < \epsilon < 2$,

$$\left\| {}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta} v - v \right\|_{1,\sigma,\lambda} \le cL^{1-\mu} \|v\|_{B_{\mu-1,\delta,\sigma,-\epsilon\delta}}.$$

Proof. By the definition of ${}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta}$ (see (5.7)),

$$\left\| {}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta} v - v \right\|_{1,\sigma,\lambda} \le \|\phi - v\|_{1,\sigma,\lambda} + \|\phi - v\|_{\omega_{\sigma-\theta}}, \quad \forall \phi \in {}_0\widetilde{Q}_L. \tag{5.13}$$

By space interpolation, we only need to consider the case where $\mu \ge 1$ is an integer. The key point is to choose a suitable $\phi \in {}_0\widetilde{Q}_L$. For this purpose, let $u(r) = r^\gamma v(r)$ and $w(x) = u((\frac{2}{1-x})^{\frac{1}{\delta}})$. Due to $v \in L^2_{\omega_{2\gamma-1}}(\Lambda)$, we assert that $w(1) = \lim_{r\to\infty} r^\gamma v(r) = 0$. Therefore we can take $w_L(x) = P^{*,1,0}_{L,1,0,-1,0} w(x)$ and $\psi_L(r) = w_L(1 - \frac{2}{r^\delta})$. Since $w_L \in \mathbb{P}^0_L$, we have

$$\phi(r) := \frac{1}{r^\gamma}\psi_L(r) = \frac{1}{r^\gamma} w_L\left(1 - \frac{2}{r^\delta}\right) \in {}_0\widetilde{Q}_L. \tag{5.14}$$

By virtue of (2.9), (2.10) and (5.11), for any $\epsilon > 0$,

$$\begin{aligned}\|\phi - v\|^2_{\omega_\sigma} &= \|\psi_L - u\|^2_{\omega_{\sigma-2\gamma}} = \frac{1}{\delta}\int_{-1}^{1} (w_L(x) - w(x))^2 (1-x)^{-1} dx \\ &\le cL^{2-2\mu} \left\| \frac{d^\mu w}{dx^\mu} \right\|^2_{L^2_{\chi^{(\mu-\epsilon,\mu-1)}}}. \end{aligned} \tag{5.15}$$

Note that (4.10) and (4.11) are also valid for u, v, ϕ and ψ_L defined in the above. Moreover, by (5.11),

$$\begin{aligned}\left\| \frac{d}{dr}(\psi_L - u) \right\|^2_{\omega_{\sigma+\lambda-2\gamma}} &= \delta 2^{\frac{\lambda-2}{\delta}} \int_{-1}^{1} \left(\frac{d}{dx}(w_L(x) - w(x))\right)^2 (1-x)^{1+\frac{2-\lambda}{\delta}} dx \\ &\le c \left\| \frac{d}{dx}(w_L(x) - w(x)) \right\|^2_{L^2_{\chi^{(1,0)}}} \le cL^{2-2\mu} \left\| \frac{d^\mu w}{dx^\mu} \right\|_{L^2_{\chi^{(\mu-\epsilon,\mu-1)}}} \end{aligned} \tag{5.16}$$

On the other hand, a direct calculation with (4.8) yields

$$\left\| \frac{d^\mu w}{dx^\mu} \right\|_{L^2_{\chi^{(\mu-\epsilon,\mu-1)}}} \le c \sum_{k=1}^{\mu} \left\| r^{k+\frac{1}{2}\epsilon\delta} (r^\delta - 1)^{\frac{\mu-1}{2}} \frac{d^k v}{dr^k} \right\|_{\omega_\sigma}. \tag{5.17}$$

The combination of (4.10), (4.11), (5.13) and (5.15)– (5.17) leads to the desired result. □

We end this section with an improved result for a special case.

Theorem 5.6. If $b = 0$ in (3.1), $\sigma = 2\gamma - 1,\ \lambda \le 2 - \theta$ and $\theta \ge 0$, then, for any $v \in {}_0H^1_{\sigma,\lambda}(\Lambda) \bigcap B_{\mu-1,\delta,\sigma,\theta-\epsilon\delta}(\Lambda)$ with $\mu \ge 0$ and $0 < \epsilon < 2$,

$$\left\| \frac{d}{dr}({}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta} v - v) \right\|_{\omega_{\sigma+\lambda}} + \left\| {}_0\widetilde{P}^1_{L,\sigma,\lambda,\theta} v - v \right\|_{\omega_{\sigma-\theta}} \le cL^{1-\mu} \|v\|_{B_{\mu-1,\delta,\sigma,\theta-\epsilon\delta}}.$$

Proof. The proof is similar to that of Theorems 5.5. □

6. Applications to an exterior problem

To illustrate how to use the proposed irrational functions to approximate PDEs in exterior domains, we consider again the one-dimensional equation induced from the multi-dimensional Poisson-type equation outside a unit disk ($n = 2$) or a unit sphere ($n = 3$):

$$-\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{du}{dr}(r)\right)+\frac{d}{r^2}u(r)+b\,u(r)=f(r),\quad b,d\geq 0,\ r\in\Lambda. \tag{6.1}$$

For simplicity, we assume that $u(1) = 0$, and

$$r^{n-1}u(r)\frac{du}{dr}(r)\to 0,\qquad \text{as } r\to\infty. \tag{6.2}$$

Then a proper variational formulation for (6.1) is: Find $u \in {}_0H^1_{\sigma,\lambda}(\Lambda)$ such that

$$\left(\frac{du}{dr},\frac{dv}{dr}\right)_{\omega_{n-1}}+d(u,v)_{\omega_{n-3}}+b\,(u,v)_{\omega_{n-1}}=(f,v)_{\omega_{n-1}},\ \forall v\in{}_0H^1_{\sigma,\lambda}(\Lambda) \tag{6.3}$$

where σ and λ are parameters to be determined depending on whether $b > 0$ or $b = 0$. Indeed, taking $u = v$ in (6.3), we find that

$$\left\|\frac{du}{dr}\right\|^2_{\omega_{n-1}}+d\|u\|^2_{\omega_{n-3}}+b\,\|u\|^2_{\omega_{n-1}}=(f,v)_{\omega_{n-1}}. \tag{6.4}$$

Hence, if $b > 0$, the dominate terms at the left hand side of (6.4) are $\|\frac{du}{dr}\|^2_{\omega_{n-1}}$ and $\|u\|^2_{\omega_{n-1}}$. It indicates that we should take $\sigma = n - 1$ and $\lambda = 0$. On the other hand, if $b = 0$, the dominate terms are $\|\frac{du}{dr}\|^2_{\omega_{n-1}}$ and $\|u\|^2_{\omega_{n-3}}$. This implies that we should take $\sigma = n - 3$ and $\lambda = 2$.

The above choice of σ and λ also ensures the validity of (6.2). Indeed, $\frac{du}{dr} \in L^2_{\omega_{n-1}}(\Lambda)$ implies $r^{\frac{n}{2}}\frac{du}{dr}(r) \to 0$ as $r \to \infty$, a.e. Next, $u \in L^2_{\omega_{n-1}}(\Lambda)$ (or $L^2_{\omega_{n-3}}(\Lambda)$) leads to that $r^{\frac{n}{2}-1}u(r) \to 0$ as $r \to \infty$, a.e. Thus, (6.2) is valid.

It remains to determine the parameter γ. If $b > 0$ and $f \in L^2_{\omega_{n-1}}(\Lambda)$, then we derive from (6.4) that $\|u\|_{\omega_{n-1}} \leq \frac{1}{b}\|f\|_{\omega_{n-1}}$ which implies that $u(r) = o(r^{-\frac{n}{2}})$ as $r \to \infty$, a.e. In this case, we can take $\gamma = \frac{n}{2}$. On the other hand, if $b = 0, d > 0$ and $f \in L^2_{\omega_{n+1}}(\Lambda)$, then we have from (6.4) that $\|u\|_{\omega_{n-3}} \leq \frac{1}{d}\|f\|_{\omega_{n+1}}$ which implies $u(r) = o(r^{1-\frac{n}{2}})$ as $r \to \infty$, a.e. In this case, we can take $\gamma = \frac{n}{2} - 1$. Therefore, in both cases, $\sigma = 2\gamma - 1$ which is the special case considered in Section 5.

6.1. Galerkin approximation using irrational functions

Let us denote

$${}_0\tilde{Q}_L=\operatorname{span}\left\{\tilde{I}_k^{(\gamma,\delta)}:k=0,1,\cdots,L\right\}\cap{}_0H^1_{\sigma,\lambda}(\Lambda).$$

The Galerkin approximation for (6.1) in ${}_0\tilde{Q}_L$ is: Find $u_L \in {}_0\tilde{Q}_L$ such that

$$(\partial_r u_L, \partial_r v_L)_{\omega_{n-1}} + b(u_L, v_L)_{\omega_{n-1}} + d(u_L, v_L)_{\omega_{n-3}} = (f, v_L)_{\omega_{n-1}}, \ \forall v_L \in {}_0\tilde{Q}_L. \tag{6.5}$$

By the definition of ${}_0\tilde{P}^1_{L,\sigma,\lambda,\theta}$ in (5.7), we have $u_L = {}_0\tilde{P}^1_{L,n-1,0,2}u$ if $b > 0$, and $u_L = {}_0\tilde{P}^1_{L,n-3,2,0}u$ for $b = 0$. Hence, direct applications of Theorems 5.3 and 5.5 lead to the following error estimates:

Theorem 6.1. Given $b > 0$, $d \geq 0$. Let $\gamma = \frac{n}{2}$ and $\sigma = n-1$. If $r^\delta u, r^{\delta+1}\frac{du}{dr} \in D(\tilde{A}^{\frac{\mu-1}{2}})$ and $u \in {}_0H^1_{n-1,0}(\Lambda)$, then, for any $\mu \geq 1$, $\delta > 0$ and $0 < \varepsilon < 2$, we have

$$\|u - u_L\|_{1,n-1,0} \leq cL^{1-\mu}\left(\left\|r^{\delta+1}\frac{du}{dr}\right\|_{D\left(\tilde{A}^{\frac{\mu-1}{2}}\right)} + \|r^\delta u\|_{D\left(\tilde{A}^{\frac{\mu-1}{2}}\right)}\right).$$

Moreover, if $v \in {}_0H^1_{n-1,0}(\Lambda) \cap B_{\mu-1,\delta,n-1,-\varepsilon\delta}(\Lambda)$, then

$$\|u - u_L\|_{1,n-1,0} \leq cL^{1-\mu}\|u\|_{B_{\mu-1,\delta,n-1,-\varepsilon\delta}}.$$

Similarly, a direct application of Theorem 5.6 leads to the following conclusion.

Theorem 6.2. Given $b = 0$, $d \geq 0$. Let $\gamma = \frac{n}{2} - 1$ and $\sigma = n-3$. If $u \in {}_0H^1_{n-3,2}(\Lambda) \cap B_{\mu-1,\delta,n-3,-\varepsilon\delta}(\Lambda)$ with $\mu \geq 1$, $\delta > 0$ and $0 < \varepsilon < 2$, then

$$\|\partial_r(u - u_L)\|_{\omega_{n-1}} + \|u - u_L\|_{\omega_{n-3}} \leq cL^{1-\mu}\|u\|_{B_{\mu-1,\delta,n-3,-\varepsilon\delta}}.$$

Remark 6.1. Although the above results are only proved for the case $b \geq 0$, the numerical algorithm presented below is also applicable to the case $b < 0$. The error analysis of the proposed irrational approximation for the latter case is still an open problem.

6.2. Implementation details

For the sake of simplicity, we shall only discuss in detail the case $b \neq 0$. The case $b = 0$ can be treated in a similar fashion. Hence, we fix $\sigma = n-1$ and $\gamma = \frac{n}{2}$ below.

For practical implementations, it is convenient to use Legendre polynomials. Let $L_k(x)$ be the Legendre polynomial of degree k. We denote $R_k(r) = R_k^{(\delta)}(r) := L_k(1 - \frac{2}{r^\delta})$. Thanks to the identity (see [2, 18])

$$(1-x)J_k^{(1,0)}(x) = L_k(x) - L_{k+1}(x), \tag{6.6}$$

we find

$$\tilde{I}_k^{(\frac{n}{2},\delta)}(r) = \frac{1}{r^{\frac{n}{2}+\delta}}J_k^{(1,0)}\left(1 - \frac{2}{r^\delta}\right) = \frac{1}{2r^{\frac{n}{2}}}(R_k(r) - R_{k+1}(r)). \tag{6.7}$$

As demonstrated in [25, 26], it is better to use compact combinations of orthogonal functions satisfying the underlying boundary conditions as basis functions. To this end, we set

$$\phi_k(r) = \phi_k^{(\delta)}(r) := \left(\tilde{I}_k^{(\frac{n}{2},\delta)}(r) + \tilde{I}_{k+1}^{(\frac{n}{2},\delta)}(r)\right) = \frac{1}{2}r^{-\frac{n}{2}}(R_k(r) - R_{k+2}(r)). \tag{6.8}$$

Then, we have $\phi_k(1) = 0$ and ${}_0\tilde{Q}_L = \text{span}\{\phi_k : \ k = 0, 1, \cdots, L-1\}$.
Let us denote

$$m_{kj} = (\phi_j, \phi_k)_{\omega_{n-1}}, \quad M = (m_{kj})_{k,j=0,1,\cdots,L-1},$$

$$s_{kj} = (\partial_r\phi_j, \partial_r\phi_k)_{\omega_{n-1}}, \quad S = (s_{kj})_{k,j=0,1,\cdots,L-1},$$

$$h_{kj} = (\phi_j, \phi_k)_{\omega_{n-3}}, \quad H = (h_{kj})_{k,j=0,1,\cdots,L-1},$$

$$\tilde{f}_k = (f, \phi_k)_{\omega_{n-1}}, \quad \bar{\boldsymbol{f}} = (\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{L-1})^T,$$

$$u_L(r) = \sum_{k=0}^{L-2} \tilde{u}_k\phi_k(r), \quad \bar{\boldsymbol{u}} = (\tilde{u}_0, \tilde{u}_1, \cdots, \tilde{u}_{L-1})^T. \tag{6.9}$$

Then, (6.5) is equivalent to the linear system:

$$(bM + dH + S)\bar{\boldsymbol{u}} = \bar{\boldsymbol{f}}. \tag{6.10}$$

By construction, we have (see (5.1))

$$\int_1^\infty \tilde{I}_k^{(\frac{n}{2},\delta)}(r)\tilde{I}_j^{(\frac{n}{2},\delta)}(r)r^{n-1}dr = \frac{1}{2\delta(k+1)}\delta_{kj}. \tag{6.11}$$

Therefore, we have

$$\begin{aligned} m_{jk} = m_{kj} &= (\tilde{I}_k^{(\frac{n}{2},\delta)} + \tilde{I}_{k+1}^{(\frac{n}{2},\delta)}, \tilde{I}_j^{(\frac{n}{2},\delta)} + \tilde{I}_{j+1}^{(\frac{n}{2},\delta)})_{\omega_{n-1}} \\ &= \begin{cases} \frac{1}{2\delta(k+1)}, & j = k-1, \\ \frac{1}{2\delta(k+1)} + \frac{1}{2\delta(k+2)}, & j = k, \\ \frac{1}{2\delta(k+2)}, & j = k+1, \\ 0, & |j-k| > 1. \end{cases} \end{aligned} \tag{6.12}$$

Next, we derive by using the transform $x = 1 - \frac{2}{r^\delta}$ that

$$\begin{aligned} h_{jk} = h_{kj} &= \left(\tilde{I}_k^{(\frac{n}{2},\delta)} + \tilde{I}_{k+1}^{(\frac{n}{2},\delta)}, \tilde{I}_j^{(\frac{n}{2},\delta)} + \tilde{I}_{j+1}^{(\frac{n}{2},\delta)}\right)_{\omega_{n-3}} \\ &= \frac{1}{4}\int_1^\infty (R_k(r) - R_{k+2}(r))(R_j(r) - R_{j+2}(r))r^{-3}dr \\ &= \frac{1}{8\delta}\int_{-1}^1 (L_k(x) - L_{k+2}(x))(L_j(x) - L_{j+2}(x))\left(\frac{1-x}{2}\right)^{\frac{2-\delta}{\delta}} dx. \end{aligned} \tag{6.13}$$

Due to the identity $\partial_x L_k(x) - \partial_x L_{k+2}(x) = -(2k+3)L_{k+1}(x)$, we find that

$$\partial_r\phi_k(r) = -\frac{n}{4}r^{-\frac{n}{2}-1}(R_k(r) - R_{k+2}(r)) - \delta r^{-\frac{n}{2}-\delta-1}(2k+3)R_{k+1}(r). \tag{6.14}$$

Using the above and the transform $x = 1 - \frac{2}{r^\delta}$ again, we get

$$\begin{aligned} s_{jk} = s_{kj} &= \frac{1}{4}\int_1^\infty \Big(\frac{n}{2} r^{-\frac{n}{2}-1}(R_k(r) - R_{k+2}(r)) + 2\delta r^{-\frac{n}{2}-\delta-1}(2k+3)R_{k+1}(r)\Big) \\ &\quad \times \Big(\frac{n}{2} r^{-\frac{n}{2}-1}(R_j(r) - R_{j+2}(r)) + 2\delta r^{-\frac{n}{2}-\delta-1}(2j+3)R_{j+1}(r)\Big) r^{n-1} dr \\ &= \frac{1}{8\delta}\int_{-1}^1 \Big(\frac{n}{2}(L_k(x) - L_{k+2}(x)) + \delta(2k+3)(1-x)L_{k+1}(x)\Big) \\ &\quad \times \Big(\frac{n}{2}(L_j(x) - L_{j+2}(x)) + \delta(2j+3)(1-x)L_{j+1}(x)\Big)\left(\frac{1-x}{2}\right)^{\frac{2-\delta}{\delta}} dx. \end{aligned} \tag{6.15}$$

Thanks to the orthogonal properties of the Legendre polynomials, one can easily check that H and S are sparse matrices if $\delta = \frac{1}{2},\ 1,\ 2$. More precisely, we have

$$\begin{aligned} &\text{if } \delta = \frac{1}{2} : h_{kj} = s_{kj} = 0,\ |k-j| > 5; \\ &\text{if } \delta = 1 : h_{kj} = s_{kj} = 0,\ |k-j| > 3; \\ &\text{if } \delta = 2 : h_{kj} = 0,\ k \neq j, j \pm 2;\quad s_{kj} = 0,\ |k-j| > 2. \end{aligned} \tag{6.16}$$

The nonzero elements of these matrices can be either determined analytically, although little tedious, using the properties of Legendre polynomials, or numerically using an appropriate Legendre–Gauss type quadrature. Thus, the linear system (6.10) can be efficiently solved in these special but generally suitable cases. We note that for all other δ, the matrices H and S are full.

6.3. Numerical results

We now present some numerical experiments using the above method to solve (6.1) with $b = d = 1$ and $n = 3$. Three illustrative examples involving three typical decaying behaviors are considered. All the errors are measured in the discrete $L^2_{\omega_2}$-norm.

Example 1. $u(r) = \sin(2r - 2)\, e^{-r}$.

Here, the function decays exponentially at infinity. So Theorem 6.1 predicts an exponential convergence rate. On the left of Figure 1, we plot the $L^2_{\omega_2}$ errors against the square root of the number of modes with $\delta = 0.5$, 1 and $\delta = 2$. The figure indicates that the $L^2_{\omega_2}$ error converges exponentially like $exp(-c\sqrt{L})$.

Example 2. $u(r) = \frac{\sin(2r-2)}{r^h}$.

The second example decays algebraically at infinity with a persistent oscillation. For this solution, Theorem 6.1 predicts an algebraic convergence rate. More precisely, for this particular solution, one can check that $\|u\|_{B_{\mu-1,\delta,2,-\varepsilon\delta}}$ is bounded if

$$\mu - 1 < \frac{2h - 3 - \varepsilon\delta}{2 + \delta}. \tag{6.17}$$

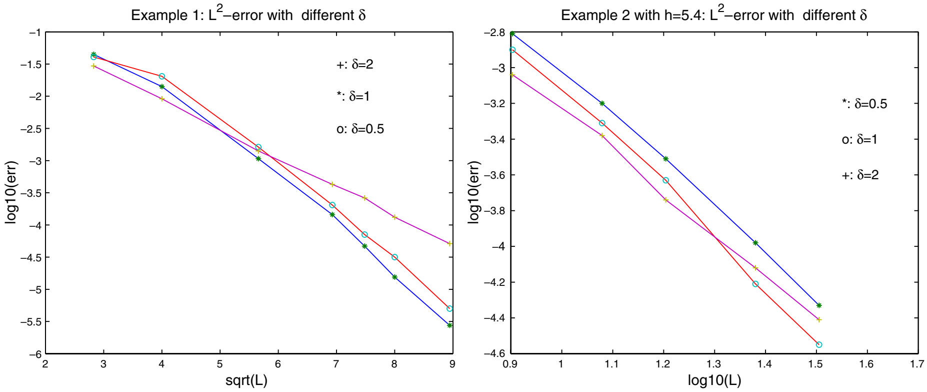

Figure 1 *Left*: Example 1; *Right*: Example 2.

Thus, Theorem 6.1 predicts that the convergence rate in $L^2_{\omega_2}$ is at least of order $\frac{2h-3-\varepsilon\delta}{2+\delta}$ for any $\varepsilon > 0$. On the right of Figure 1, we plot the $L^2_{\omega_2}$ errors vs. the number of modes in the log–log scale. The straight lines in the log–log scale indicate algebraic convergence rates as predicted by Theorem 6.1. Note that there is no significant difference between the results with different δ.

Example 3. $u(r) = \frac{r-1}{r^h}$.

The solution of this example monotonically decays at infinity algebraically. For general δ and h, Theorem 6.1 predicts that the convergence rate is algebraic. More precisely, one can check that the convergence rate in $L^2_{\omega_2}$ is at least of order

$$\frac{2h-5-\varepsilon\delta}{\delta}. \tag{6.18}$$

Note that for large h, the convergence rate is essentially inversely proportional to δ. On the other hand, there are special pairs of (δ, h) for which the solution $u(r) = \frac{r-1}{r^h} \in {}_0\tilde{Q}_L$ (when L is sufficiently large) so that our irrational approximation produces the exact solution. These cases are:

$$(\delta, h) = \left(\frac{1}{2}, \frac{k}{2}\right), k = 6, 7, 8, \cdots;$$

$$(\delta, h) = \left(1, k + \frac{1}{2}\right), k = 3, 4, 5, \cdots. \tag{6.19}$$

Thus, our irrational approximation is extremely accurate and efficient for solutions which are linear combinations of the monotonically decaying functions with above parameters.

In Figure 2, we plot the $L^2_{\omega_2}$ errors vs. the number of modes in the log–log scale with different δ for $h = 3.4$ (left) and $h = 5.4$ (right). We observe that the convergence rates are indeed algebraic and the observed convergence rates are essentially in agreement with (6.18) and (6.19).

However, for small δ, it takes more modes for the approximation error to be in the asymptotic range. This fact is reflected in the figures as multiple slopes for small L

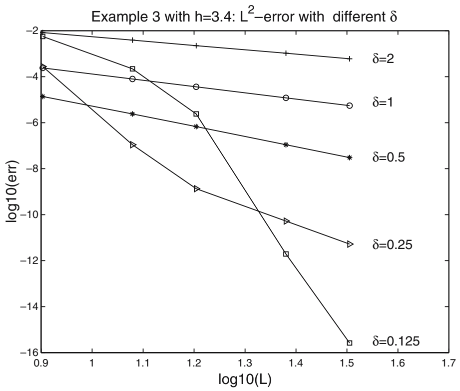

Figure 2 Example 3: *Left*, $h = 3.4$; *right*: $h = 5.4$.

when δ is small, and can be explained by the distribution of the mapped collocation points through the transform $x = 1 - \frac{2}{r^\delta}$, see Figure 3 where we show the distribution of mapped collocation points ($\{r_i\}$ in log scale) with fixed $L = 32$ and different δ (left), and with fixed $\delta = 0.125$ and different L (right). It is clear that for smaller δ, a larger number of points are needed for the approximation error to be in the asymptotic range.

In summary, a good choice of δ depends on a number of factors:

- The results in Section 5 indicate that for a large class of functions, the convergence rate of the irrational approximation increases as δ decreases. However, it is not advisable to take δ too small since for very small δ, the mapped collocation points (through the transform $x = 1 - \frac{2}{r^\delta}$), especially when L is small, could be over stretched (see Figure 3). In general, it is better to choose a 'smaller' δ when the solution decays 'slowly' at infinity, and a 'larger' δ when the solution

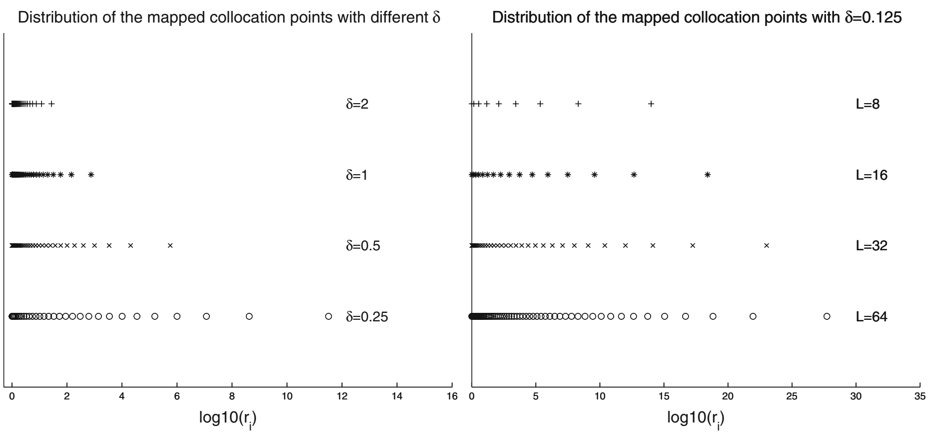

Figure 3 Distribution of mapped collocation points.

decays ‘rapidly’ at infinity. For the three types of problems considered above, an appropriate range for δ seems to be $[\frac{1}{8}, 1]$.

- For the sake of numerical efficiency, it is advisable to choose δ such that the resulting linear system is sparse. For the model equation 6.1, such choices are $\delta = \frac{1}{2}, 1, 2$.
- For the special case $\sigma = 2\gamma - 1$ considered here, one should also take into account the asymptotic behavior of the solution (or of the forcing function f) when choosing δ. In fact, the basis functions $\{\tilde{I}_k^{(\gamma,\delta)}(r)\}$ dictate that the approximation u_L converges to 0 at the infinity at least as fast as $\frac{1}{r^{\gamma+\delta}}$. Hence, in addition to other considerations, it is better to choose δ such that $r^{\gamma+\delta}u(r)$ remains bounded as $r \to \infty$.

7. Concluding remarks

We introduced in this paper a family of orthogonal systems consisting of irrational functions on the semi-infinite interval. This family of orthogonal systems offers great flexibility to match a wide range of asymptotic behaviors at infinity. We have established error estimates for various orthogonal projections, which will play essential roles, as demonstrated in this paper, in numerical analysis of the irrational approximations to partial differential equations in exterior domains. We have also presented illustrative numerical experiments which agree well with our theoretical analysis and demonstrate the effectiveness of this approach.

We have only studied the orthogonal projections and related Galerkin approximations in this paper, the interpolation operators and related pseudospectral approximations are subjects of future studies. Also, we have concentrated in matching the asymptotic behaviors at infinity, so we did not consider the most general cases where the PDEs have singular or degenerated coefficients at $r = 1$. However, these cases can be treated by introducing the following family of orthogonal systems:

$$I_k^{(\gamma,\delta)}(r) := \frac{1}{r^\gamma} J_k^{(\alpha,\beta)}\left(1 - \frac{2}{r^\delta}\right), \tag{7.1}$$

where the set of parameters γ, α, β can be chosen to match the solution behaviors at both ends of the interval. Similar orthogonal systems on the whole line can also be constructed. These issues will be addressed in forthcoming papers.

In short, we believe that this new family of irrational orthogonal systems will become an increasingly useful tool for numerical approximation of PDEs in unbounded domains.

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