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PROJECTION METHODS FOR TIME-DEPENDENT NAVIER-STOKES EQUATIONS

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Abstract—The classical projection method and its variants have been widely used in practice because of their efficiency. However, to the author's knowledge, rigorous error analyses for those schemes are still not available. We consider in this paper two projection schemes in semi-discretized form for the Navier-Stokes equations, and for these schemes we provide error estimates for the velocity as well as for the pressure.

1. INTRODUCTION

We consider the time dependent Navier-Stokes equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \ \forall\,(\vec{x},t) \ \text{ in } \ Q = \Omega\times[0,T], \\ \text{div } u = 0, \ \forall\,(\vec{x},t) \ \text{ in } \ Q, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in R^d (d=2 or 3) with a sufficiently smooth boundary Γ. The initial condition is: $u(\cdot,t) = u_0$, and for the sake of simplicity, we consider the homogeneous boundary condition: $u(t)|_\Gamma = 0, \ \forall\, t\in[0,T]$.

In (1.1), the velocity u and the pressure p are coupled together by the incompressibility condition "div$u = 0$," which makes the equations difficult to solve numerically. In the late 60's, Chorin [1] and Temam [2] constructed the so called projection method (or fractional step method) which decoupled the velocity and the pressure. A semi-discretized version of the projection method can be written as follows:

$$\begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+1} + (u^n\cdot\nabla)\tilde{u}^{n+1} = f(t_{n+1}), \\ \tilde{u}^{n+1}|_\Gamma = 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \nabla\phi^{n+1} = 0, \\ \text{div } u^{n+1} = 0, \\ u^{n+1}\cdot\vec{n}|_\Gamma = 0, \end{cases} \tag{1.3}$$

where k is the size of time step, $t_{n+1} = (n+1)k$ and $\vec{n}$ is the outward normal.

Applying the divergence operator to (1.3), one can easily verify that (1.3) is equivalent to

$$\Delta\phi^{n+1} = \frac{1}{k}\text{div}\tilde{u}^{n+1}, \ \frac{\partial\phi^{n+1}}{\partial\vec{n}}|_\Gamma = 0; \ u^{n+1} = \tilde{u}^{n+1} - k\nabla\phi^{n+1}. \tag{1.3$'$}$$

Hence, the velocity and the pressure in (1.2)–(1.3) are totally decoupled. (1.2)–(1.3) is known as a projection scheme because u^{n+1} is actually the projection of $\tilde{u}^{n+1}$ in $L^2(\Omega)^d$ onto the space $H = \{u\in L^2(\Omega)^d : \text{div } u = 0, \ u\cdot\vec{n}|_\Gamma = 0\}$.

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One immediately notices that ϕ^{n+1} in (1.3') is subject to the homogeneous Neumann boundary condition which is not necessarily satisfied by the exact pressure. Therefore, it is by no means obvious that (1.2)–(1.3) provides a consistent approximation to (1.1). Nevertheless, Chorin [1] and Temam [2] were still able to prove the convergence of $\tilde{u}^{n+1}$ and u^{n+1} towards $u(t_{n+1})$ in appropriate norms. The scheme (or its variants) has been widely used in practice because of its efficiency, and it is believed that the scheme provides some sort of first order approximations to (1.1). Several higher order schemes of projection type have been proposed as well (see [3,4,5]).

Chorin [1] proved that the rate of convergence of the scheme, with a finite difference space discretization, was at least $O(k^{\frac{1}{4}})$. The author and Temam [6] proposed a more complicated fractional step scheme, providing a better pressure approximation, and proved its rate of convergence to be at least $O(k^{\frac{1}{2}})$. Orszag *et al.* [4] analyzed the scheme applied to a one-dimensional linear model, i.e., *the two-dimensional Stokes equations with Dirichlet boundary condition in one direction and periodical boundary condition in the other.* They used normal mode analyses to show that for this simple model case, the rate of convergence of the scheme was $O(k)$. However, to the author's knowledge, there is still no rigorous error analysis confirming the first order accuracy for fully nonlinear equations (1.1) with Dirichlet boundary condition.

Using the classical energy method, we have derived precise error estimates for (1.2)–(1.3) (see Theorem 1 below). We have also analyzed an improved projection scheme and obtained precise error estimates for it as well. Our results are summarized in the next section. For detailed proofs, we refer to Shen [7].

2. MAIN RESULTS

To simplify the presentation, we assume that the data $\{u_0, f, \Omega\}$ and (in some case) the solution $\{u, p\}$ are sufficiently smooth (we refer to Shen [7] for detailed assumptions).

To classify the precision of time discretization schemes, we use the following:

DEFINITION. *$\{f_k\}$ is a weakly order α approximation of f in X over [0,T] if $\exists c$ depending on T and f such that*

$$k\sum_{n=0}^{T/k} \|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \le ck^{2\alpha};$$

$\{f_k\}$ is a strongly order α approximation of f in X over [0,T] if $\exists c$ depending on T and f such that

$$\max_{0\le n\le T/k} \|f_k(t_n^{(k)}) - f(t_n^{(k)})\|_X^2 \le ck^{2\alpha},$$

where $t_n^{(k)} = nk$.

For the classical projection scheme (1.2)–(1.3), we have:

THEOREM 1. *Both $\tilde{u}^{n+1}$ and u^{n+1} are weakly first order approximations to $u(t_{n+1})$ in $L^2(\Omega)^d$, and ϕ^{n+1} as well as $(I - k\nu\Delta)\phi^{n+1}$ are weakly order $\frac{1}{2}$ approximations to $p(t_{n+1})$ in $L^2(\Omega)/R$. Namely, $\exists c$ depending only on $\{\nu, u_0, f, \Omega, T\}$ such that*

$$k\sum_{n=0}^{T/k-1} \left\{\|e^{n+1}\|_{L^2}^2 + \|\tilde{e}^{n+1}\|_{L^2}^2\right\} \le ck^2,$$

$$k\sum_{n=0}^{T/k-1} \left\{\|\phi^{n+1} - p(t_{n+1})\|_{L^2/R}^2 + \|(I - k\nu\Delta)\phi^{n+1} - p(t_{n+1})\|_{L^2/R}^2\right\} \le ck,$$

where $e^{n+1} = u(t_{n+1}) - u^{n+1}$, $\tilde{e}^{n+1} = u(t_{n+1}) - \tilde{u}^{n+1}$.

Notice that despite the incompatible Neumann boundary condition, ϕ^{n+1} is still an approximation to $p(t_{n+1})$ as accurate as $(I - k\nu\Delta)\phi^{n+1}$. Notice also that the error estimates in Theorem 1 for the velocity as well as for the pressure are most likely of optimal order. Hence, in order to get comparable accuracy as a conventional coupled scheme, which is strongly first order for the velocity, we have to modify the scheme. In the process of the proof (see [7]), we notice that the lack of pressure gradient term in (1.2) prevents us from obtaining strongly first order error estimate. Therefore, a natural modification is the following:

$$\begin{cases} \frac{1}{k}(\tilde{u}^{n+1} - u^n) - \nu\Delta\tilde{u}^{n+1} + (u^n \cdot \nabla)\tilde{u}^{n+1} + \nabla\phi^n = f(t_{n+1}), \\ \tilde{u}^{n+1}|_\Gamma = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \frac{1}{k}(u^{n+1} - \tilde{u}^{n+1}) + \alpha\nabla(\phi^{n+1} - \phi^n) = 0, \\ \text{div } u^{n+1} = 0, \\ u^{n+1} \cdot \vec{n}|_\Gamma = 0, \end{cases} \tag{2.2}$$

where α can be any constant ≥ 1 and ϕ^0 is arbitrary.

It is easy to see that this scheme is numerically as efficient as (1.2)–(1.3) and we expect that it provides better approximations. In fact, we can prove the following improved error estimates.

THEOREM 2. *Both $\tilde{u}^{n+1}$ and u^{n+1} are strongly first order approximations to $u(t_{n+1})$ in $L^2(\Omega)^d$, as well as weakly first order approximations to $u(t_{n+1})$ in $H^1(\Omega)^d$. ϕ^{n+1}, as well as $\phi^{n+1} - k\nu\Delta(\phi^{n+1} - \phi^n)$, are weakly first order approximations to $p(t_{n+1})$ in $L^2(\Omega)/R$. Namely,*

$$\|e^{N+1}\|_{L^2}^2 + \|\tilde{e}^{N+1}\|_{L^2}^2 + k\nu\sum_{n=0}^{N}\left\{\|\tilde{e}^{n+1}\|_{H^1}^2 + \|e^{n+1}\|_{H^1}^2\right\} \leq Mk^2, \ \forall\, 0 \leq N \leq T/k - 1,$$

$$k\nu \sum_{n=0}^{T/k-1}\left\{\|\phi^{n+1} - p(t_{n+1})\|_{L^2(\Omega)/R}^2 + \|\phi^{n+1} - k\nu\Delta(\phi^{n+1} - \phi^n) - p(t_{n+1})\|_{L^2(\Omega)/R}^2\right\} \leq Mk.$$

REMARKS. We derive from (2.2) that

$$\frac{\partial\phi^{n+1}}{\partial\vec{n}}|_\Gamma = \frac{\partial\phi^n}{\partial\vec{n}}|_\Gamma = \cdots = \frac{\partial\phi^0}{\partial\vec{n}}|_\Gamma,$$

which is again not satisfied by the exact pressure. However, we were still able to prove that ϕ^{n+1} was a weakly first order approximation to the pressure despite the incompatible Neumann boundary condition.

It is interesting to observe that the choice of α (as long as $\alpha \geq 1$) and ϕ^0 does not affect the precision of the first order scheme (2.1)–(2.2).

The results in Theorem 2 indicate that the scheme (2.1)–(2.2) has the same order of accuracy as a conventional coupled scheme. We recall that the scheme (2.1)–(2.2) is much easier to implement and it consumes less CPU time than a conventional coupled scheme.

In a further coming paper [8], we shall study several second order projection schemes and provide error estimates for them as well.

REFERENCES

1. A.J. Chorin, On the convergence of discrete approximations to the Navier-Stokes equations, *Math. Comp.* **23**, 341–353 (1969).
2. R. Temam, Sur l'approximation de la solution des equations de Navier-Stokes par la méthode des pas fractionnaires II, *Arch. Rat. Mech. Anal.* **33**, 377–385 (1969).
3. J. Kim and P. Moin, Application of a fractional-step method to incompressible Navier-Stokes equations, *J. Comp. Phys.* **59**, 308–323 (1985).
4. S.A. Orszag, M. Israeli and M. Deville, Boundary conditions for incompressible flows, *J. Scient. Comp.* **1** (1986).
5. J. Van Kan, A second-order accurate pressure-correction scheme for viscous incompressible flow, *J. Sci. Stat. Comp.* **7**, 870–891 (1986).
6. J. Shen and R. Temam, A new fractional scheme for the approximation of incompressible flows, *Mat. Aplic. Comp.* **8**, 3–22 (1989).
7. J. Shen, On error estimates of projection methods for Navier-Stokes equations: First order schemes, *Siam J. Numer. Anal.* (to appear).
8. J. Shen, On error estimates of projection methods for Navier-Stokes equations: Second order schemes (in preparation).